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Micro-seismic waveform modeling using integral equation methods

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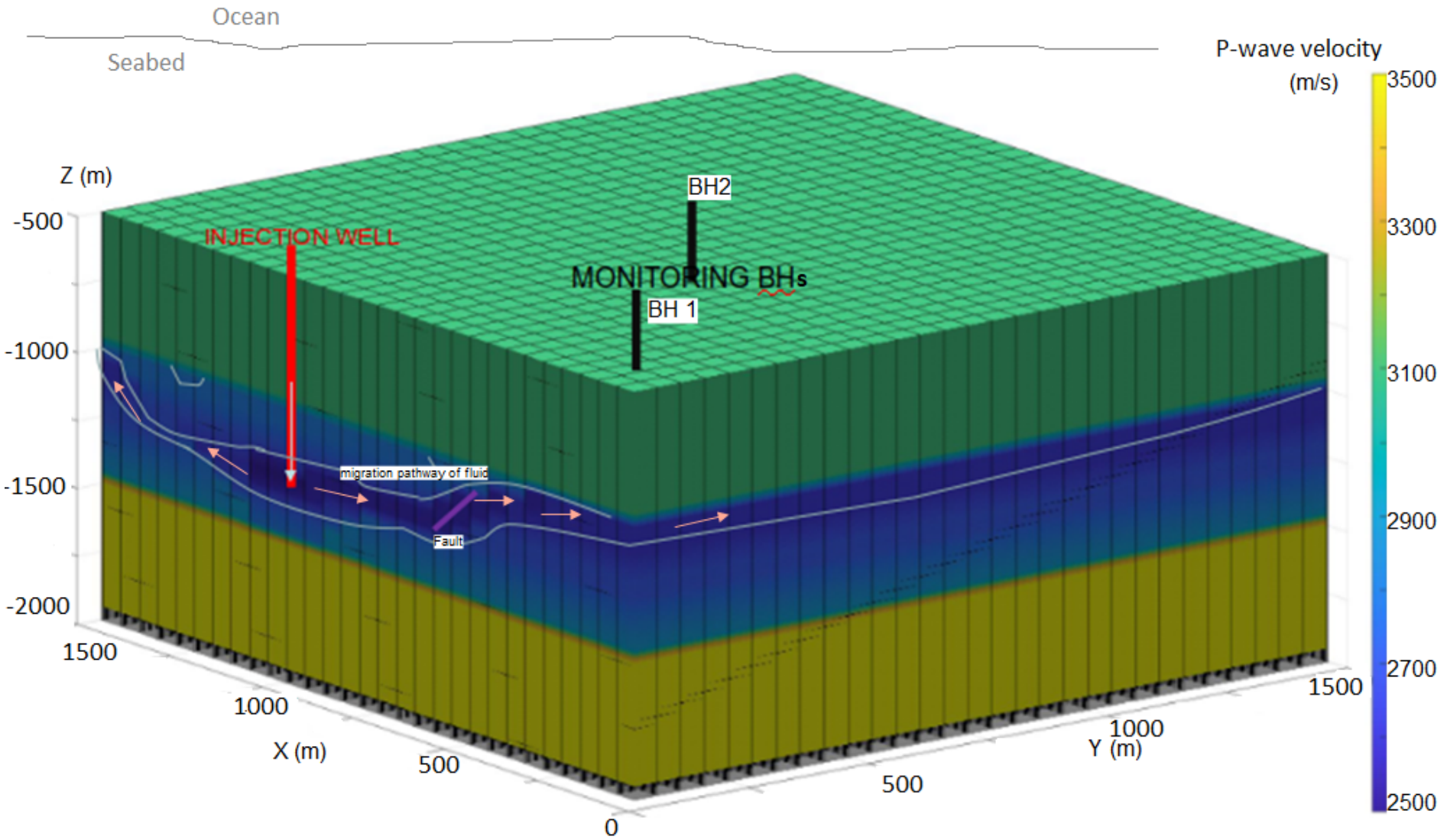
Motivation

- To develop an efficient integral equation method for seismic modeling.
- To demonstrate the elastic wave propagation in a microseismic event.

Outline

- Microseismic event
- Moment tensor source
- Integral equation approach
- Wavefield in homogeneous-isotropic medium
- Wavefield in heterogeneous-anisotropic medium
- 3D Wavefield in VTI medium

Microseismic event scenario



Elastic wave equation

- The elastic wave equation in the frequency domain

$$\frac{\partial}{\partial x_j} C_{ijkl}(\mathbf{x}) \frac{\partial}{\partial x_l} u_k(\mathbf{x}, \omega) + \rho(\mathbf{x}) \omega^2 u_i(\mathbf{x}, \omega) = -f_i(\mathbf{x}, \omega)$$

C_{ijkl} is the stiffness tensor, u_i is the elastic displacement component, ω is the circular frequency, ρ is the density of the medium and f_i is the external force vector.

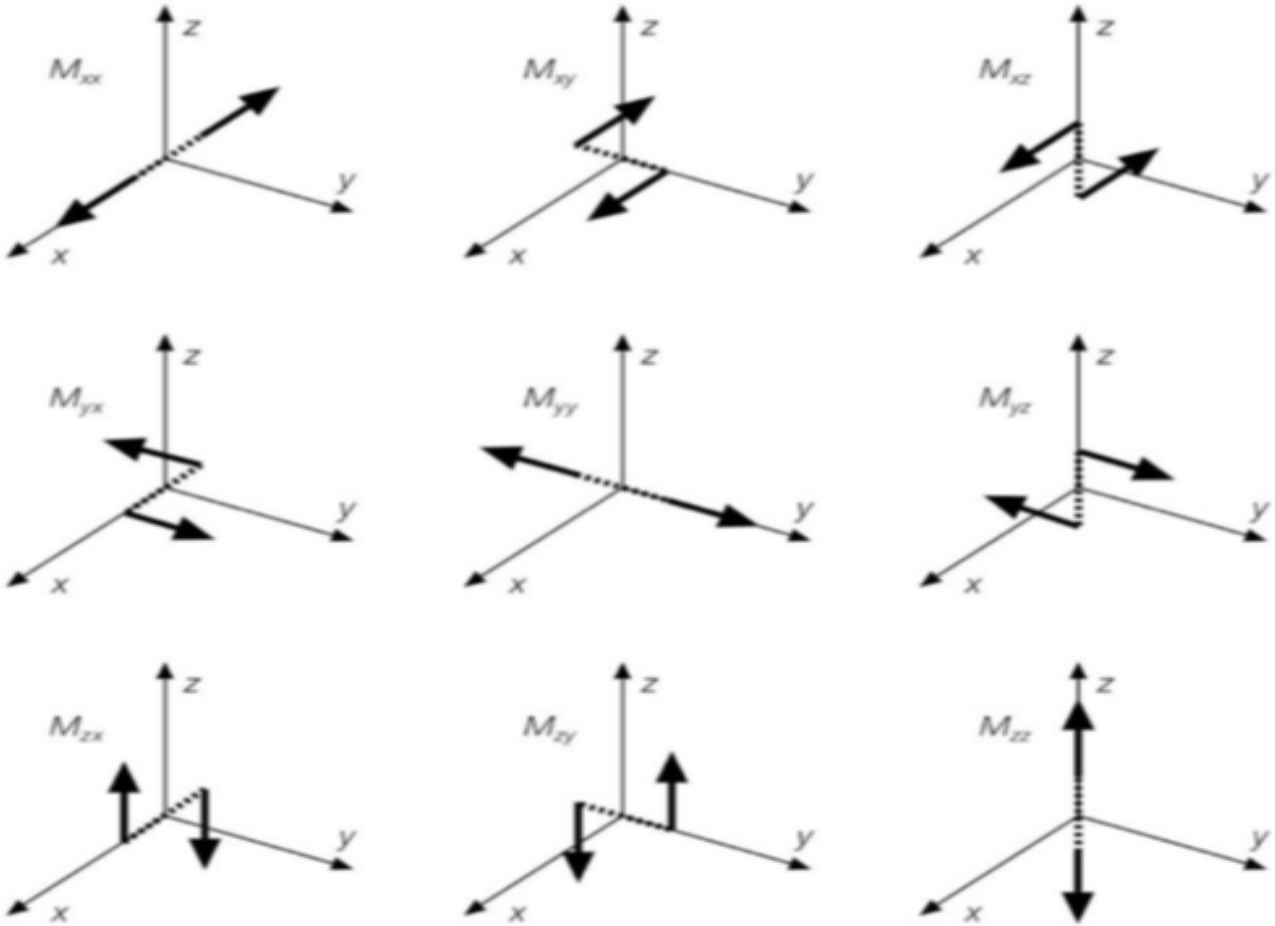
Mathematical models of microseismic sources

- Equivalent body force density (Aki and Richards, 1983):

$$f_i(\mathbf{x}) = -M_{ij}\frac{\partial}{\partial x_j}\delta(\mathbf{x}-\mathbf{s})e^{i\omega t_0}$$

where M_{ij} is a moment tensor component, $\mathbf{s}$ represents the positioning of the microseismic event and t_0 may be referred to as the rupture time.

Moment tensor



Seismic moment tensor components represented as force couples (Stein and Wysession, 2003)

Integral equation approach

- Stiffness tensor in the medium

$$C_{ijkl}(\mathbf{x}) = C^{(0)}_{ijkl} + \Delta C_{ijkl}(\mathbf{x})$$

- Density in the medium

$$\rho(\mathbf{x}) = \rho^{(0)} + \Delta\rho(\mathbf{x})$$

Modified Green's function

- Elastic Green's function for the homogeneous reference medium (Jakobsen et al., 2020):

$$G_{ij}^{(0)} = G_{ij}^{(0)}(P) + G_{ij}^{(0)}(S)$$

- First order spatial derivative

$$H_{ijk}^{(0)} = \frac{1}{2}(G_{ij,k}^{(0)} + G_{ik,j}^{(0)})$$

- Second order spatial derivative

$$\Gamma_{ijkl}^{(0)} = \frac{1}{4}(G_{ij,kl}^{(0)} + G_{kj,il}^{(0)} + G_{il,kj}^{(0)} + G_{kl,ij}^{(0)})$$

Two coupled integral equations

- Linear superposition of background and scattered fields:

$$u_i(\mathbf{x}) = u_i^{(0)}(\mathbf{x}) + \int d\mathbf{x}' H_{ijk}^{(0)}(\mathbf{x}-\mathbf{x}')\Delta C_{jklm}(\mathbf{x}')\epsilon_{lm}(\mathbf{x}') + \omega^2 \int d\mathbf{x}' G_{ij}^{(0)}(\mathbf{x}-\mathbf{x}')\Delta\rho(\mathbf{x}')u_j(\mathbf{x}')$$

- Equation for the strain field:

$$\epsilon_{ij}(\mathbf{x}) = \epsilon_{ij}^{(0)}(\mathbf{x}) + \int d\mathbf{x}' \Gamma_{ijkl}^{(0)}(\mathbf{x}-\mathbf{x}')\Delta C_{jklm}(\mathbf{x}')\epsilon_{lm}(\mathbf{x}') + \omega^2 \int d\mathbf{x}' H_{ijk}^{(0)}(\mathbf{x}-\mathbf{x}')\Delta\rho(\mathbf{x}')u_k(\mathbf{x}')$$

Elastic displacement due to a moment tensor source

- The particle displacement field at the receiver position **r** due to a moment tensor source at position **s** is given by

$$u_i^{(0)}(\mathbf{r}) = H_{ijk}^{(0)}(\mathbf{r}, \mathbf{s}) M_{jk} e^{i\omega t_0}$$

where M_{jk} denotes one of the 9 moment tensor components and t_0 represents the timing of the microseismic event.

Elastic displacement due to a moment tensor source

- Using Voigt notation, we can write

$$u_i^{(0)}(\mathbf{r}) = H_{iJ}^{(0)}(\mathbf{r}, \mathbf{s}) M_J e^{i\omega t_0}$$

where M_J denotes one of the 6 independent components of the symmetric moment tensor. $i = 1, 2, 3$ and $I, J, K = 1, 2, ..., 6$

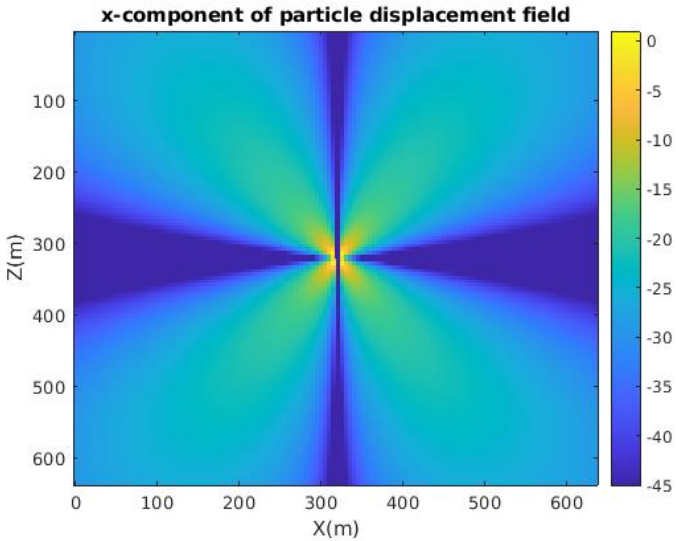
A homogeneous model

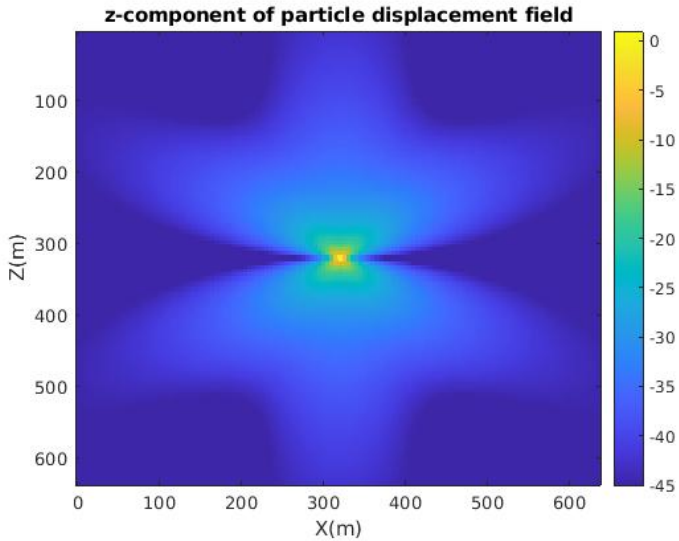
- (640m by 640m by 640m) model with source located at the center of the model.
- Background medium is homogeneous isotropic. Actual medium is homogeneous VTI.
- Density normalized stiffness (in GPa) matrix for the VTI medium (Schoenberg and Helbig, 1997) and density $= 2300 kg/m^3$

$$\mathbf{c_{ij}} = \begin{bmatrix} 10 & 4 & 2.5 & 0 & 0 & 0 \\ & 10 & 2.5 & 0 & 0 & 0 \\ & & 6 & 0 & 0 & 0 \\ & & & 2 & 0 & 0 \\ & & & & 2 & 0 \\ & & & & & 3 \end{bmatrix}$$

Displacement field in isotropic (background) medium

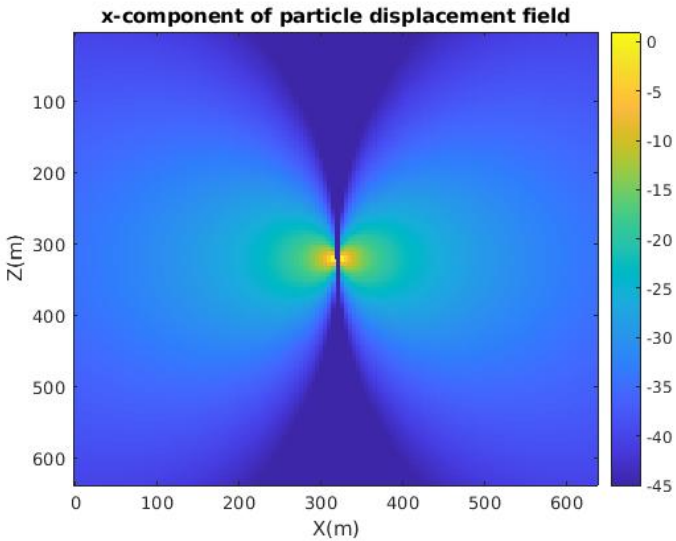
- Elastic displacement components in XZ plane passing through source position, when only M_{33} is non zero

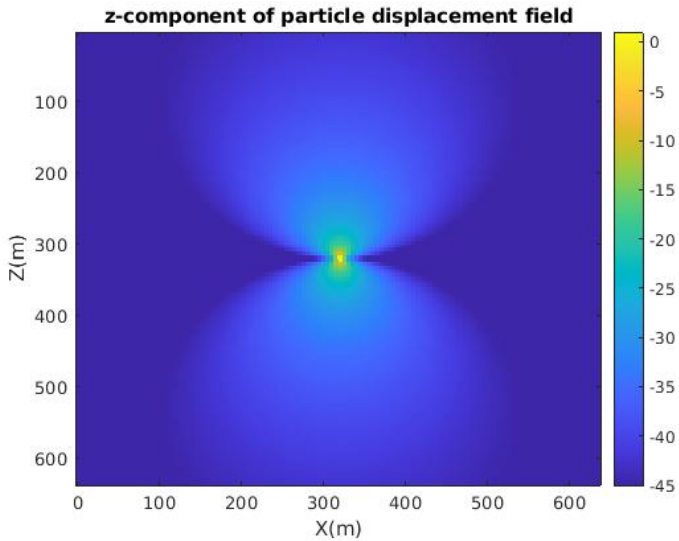




Displacement field in isotropic (background) medium

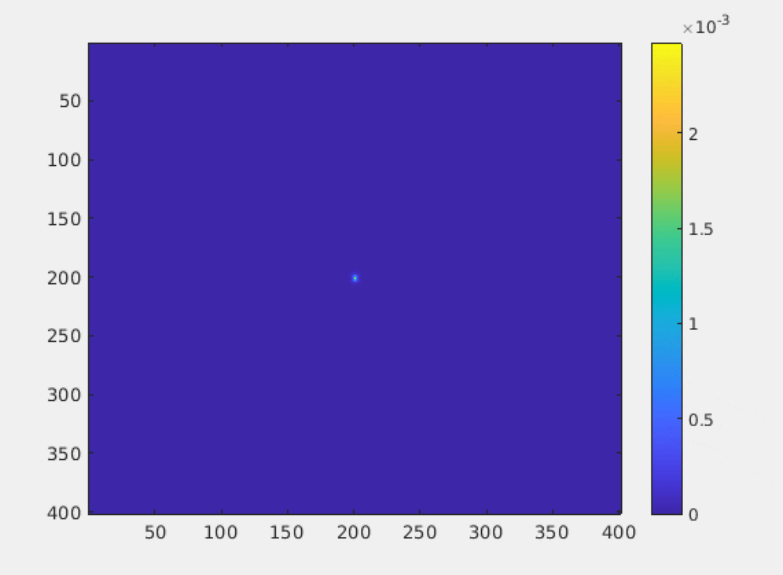
- Elastic displacement components in XZ plane when only M_{11} and M_{33} are non zero

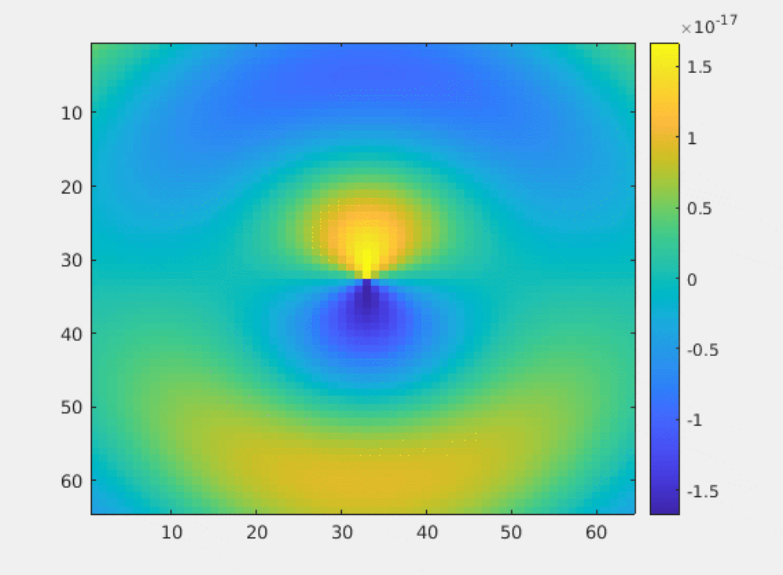




Elastic wave propagation

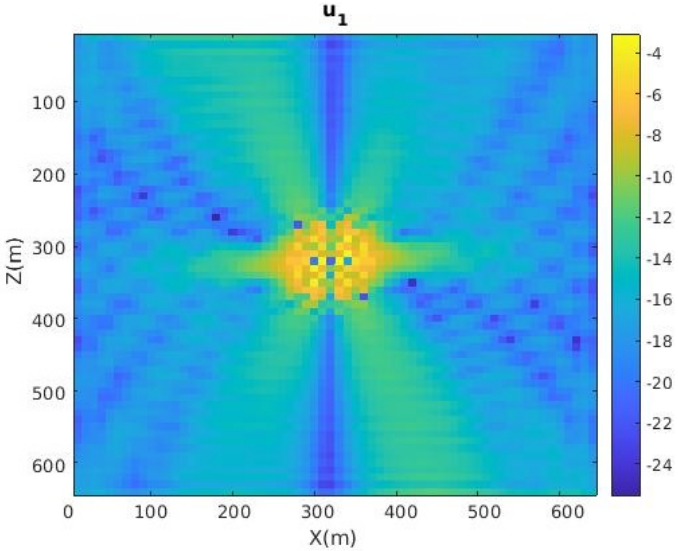
- Comparison of wavefront generated by a vectorial point source and a moment tensor two-dipole source:

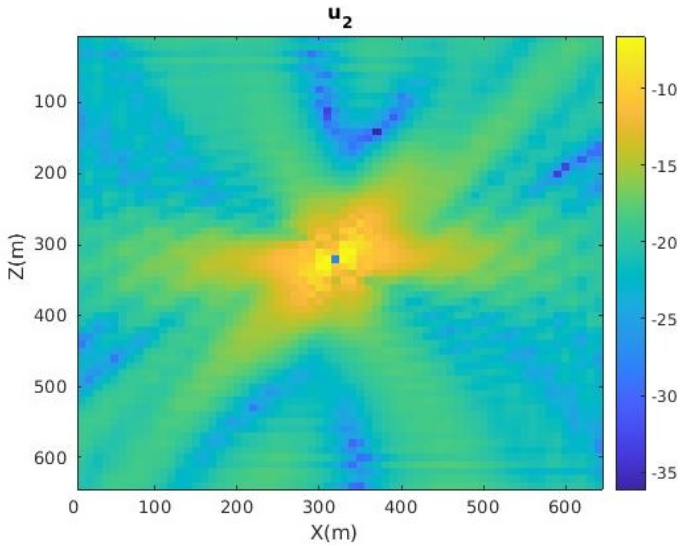


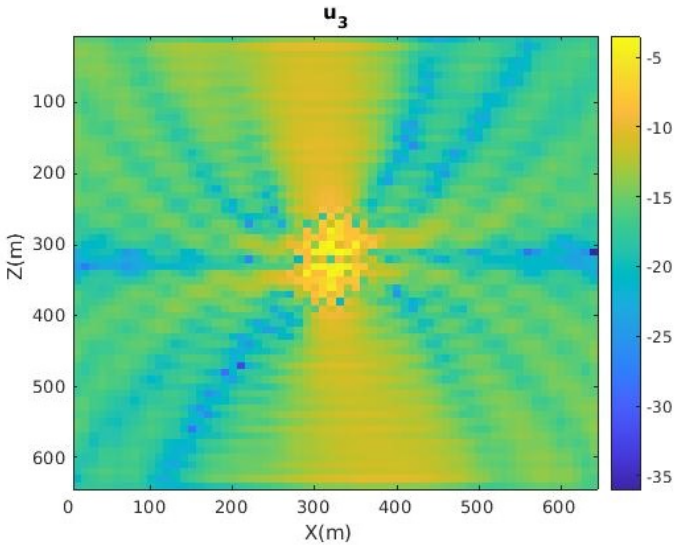


Displacement field in VTI (actual)medium

- Wavefield due to double-dipole moment tensor source in XZ-plane







Fast integral equation formulation

- In hybrid tensor-operator notation,

$$u_i = \mathcal{F}^{-1}\left[\mathcal{F}(H^{(0)}_{iJ}) \circ \mathcal{F}(M_J)\right] + \mathcal{F}^{-1}\left[\mathcal{F}(H^{(0)}_{iJ}) \circ \mathcal{F}(\Delta C_{JK}\epsilon_K)\right] + \omega^2 \mathcal{F}^{-1}\left[\mathcal{F}(G^{(0)}_{ij}) \circ \mathcal{F}(\Delta \rho u_j)\right]$$

$$\epsilon_I = \mathcal{F}^{-1}\left[\mathcal{F}(\Gamma^{(0)}_{IJ}) \circ \mathcal{F}(M_J)\right] + \mathcal{F}^{-1}\left[\mathcal{F}(\Gamma^{(0)}_{IJ}) \circ \mathcal{F}(\Delta C_{JK}\epsilon_K)\right] + \omega^2 \mathcal{F}^{-1}\left[\mathcal{F}(H^{(0)}_{Ij}) \circ \mathcal{F}(\Delta \rho u_j)\right]$$

- where, $(\mathcal{F})$ and $(\mathcal{F}^{-1})$ are 3-dimensional Fourier and inverse Fourier transform respectively.

Strain calculation (no density contrast)

- Richardson iteration

$$\epsilon_K^{(i)} = \epsilon_K^{(0)} + \Gamma_{KL}^{(0)} \Delta C_{LM} \epsilon_M^{(i-1)}$$

$$K, L, M = 1, 2, ..., 6$$

i is the iteration number.

$$r_K^{(i)} = \epsilon_K^{(0)} + \Gamma_{KL}^{(0)} \Delta C_{LM} \epsilon_M^{(i-1)} - \epsilon_K^{(i-1)}$$

$$\epsilon_K^{(i)} = \epsilon_K^{(i-1)} + \gamma r_K^{(i)}$$

γ is the relaxation parameter.

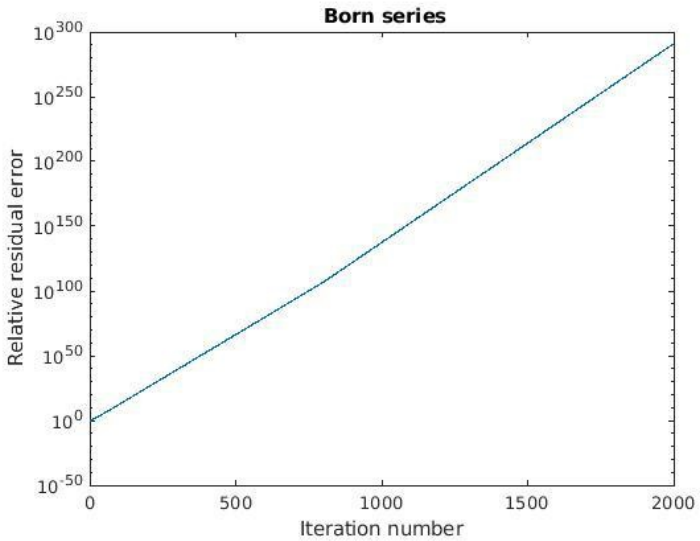
- Rewriting strain equation in the form mentioned below to apply Krylov subspace method

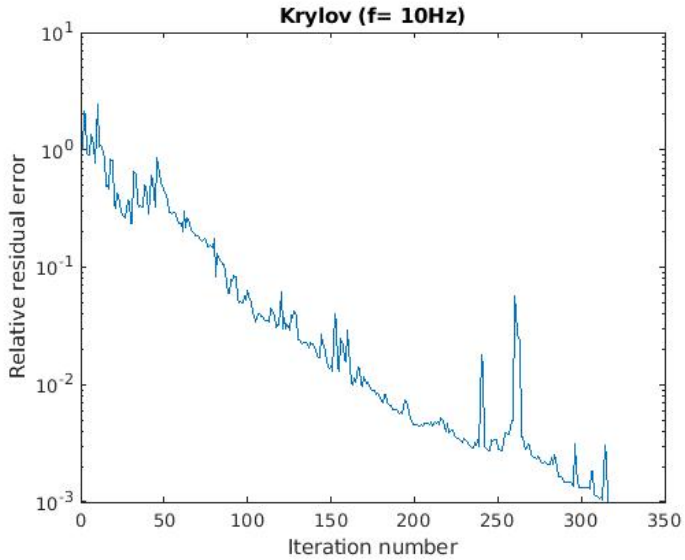
$$\boldsymbol{Ax} = \boldsymbol{b}$$

Convergence analysis for methods

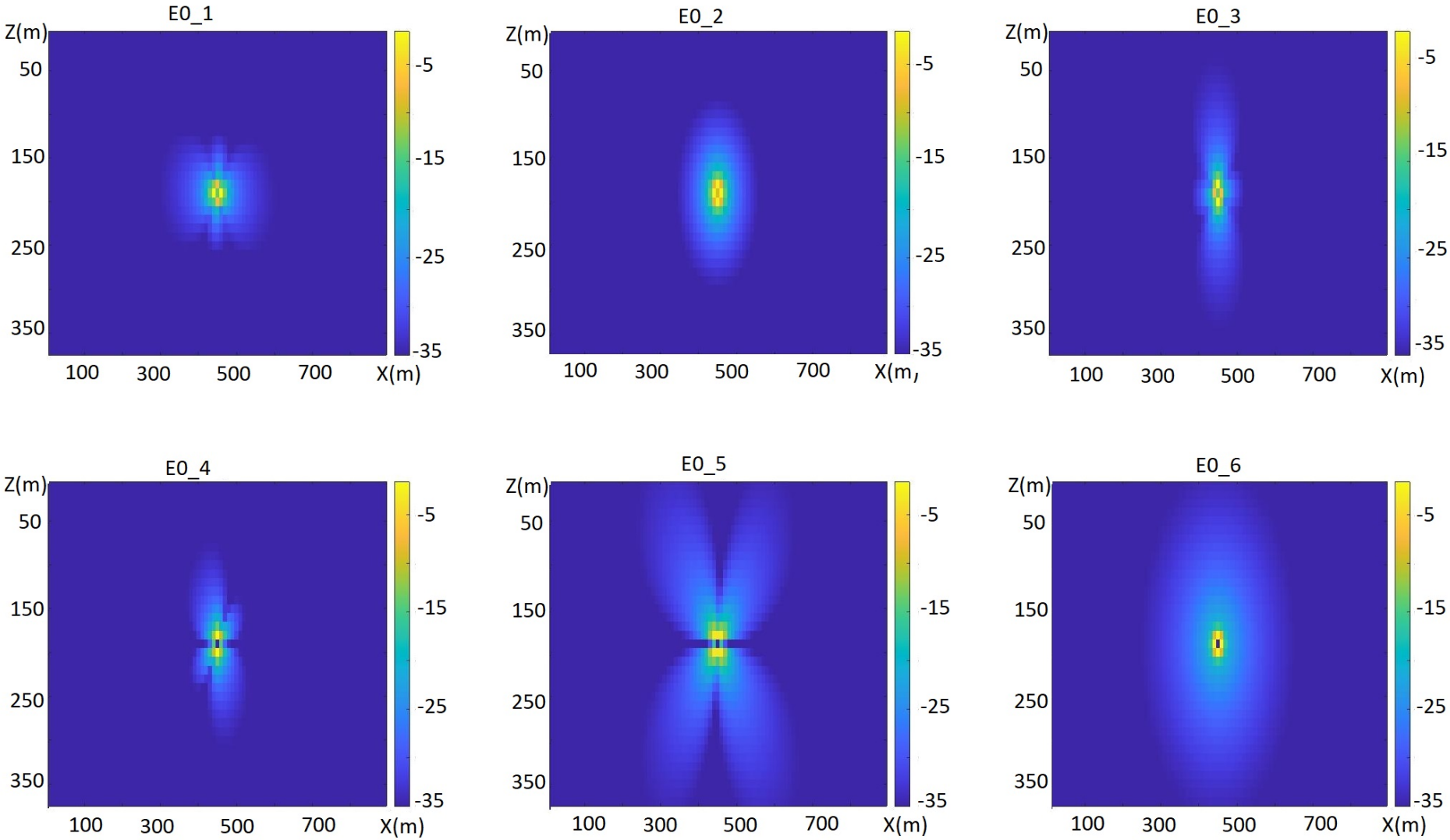
- The relative residual error (e) at each iteration step can be given as:

$$e = \frac{\|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}\|}{\|\boldsymbol{b}\|}$$





Strain field in homogeneous isotropic (background) medium at 10Hz



Alternative method for strain calculation

- We know that strain is the symmetric gradient of the displacement.

$$\epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$$

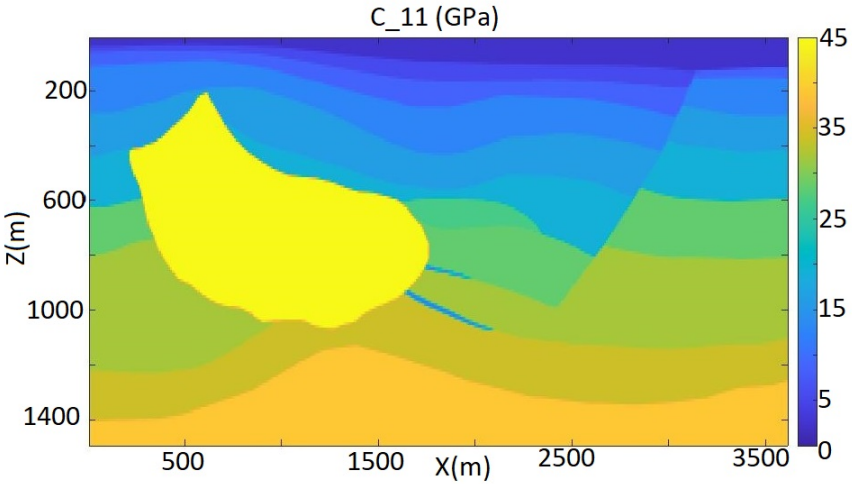
- The central finite difference scheme applied on the above equation is,

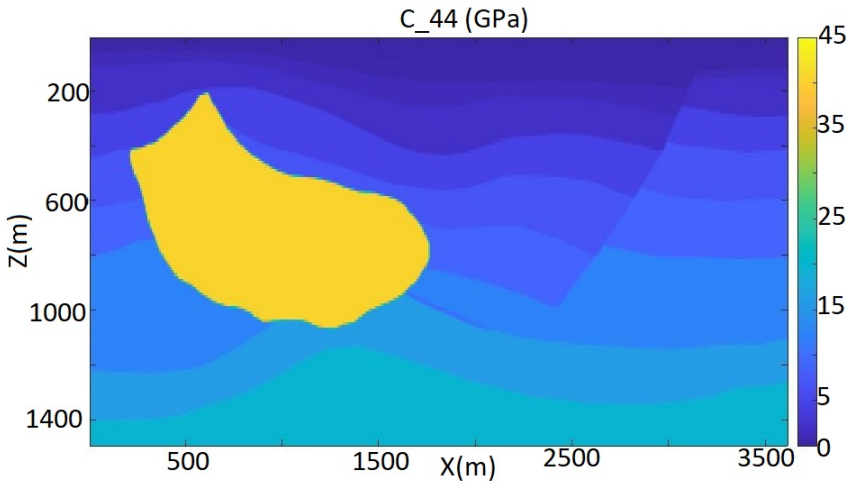
$$\epsilon_{ij} = \frac{1}{2}\left[\frac{u_i(\boldsymbol{x}+\delta\boldsymbol{x}) - u_i(\boldsymbol{x}-\delta\boldsymbol{x})}{2\delta x_j} + \frac{u_j(\boldsymbol{x}+\delta\boldsymbol{x}) - u_j(\boldsymbol{x}-\delta\boldsymbol{x})}{2\delta x_i}\right]$$

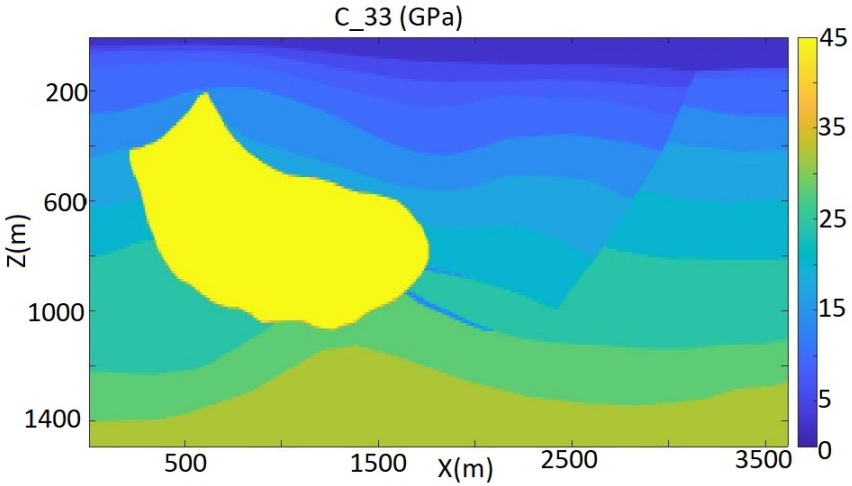
- Now, density contrast can be included easily.

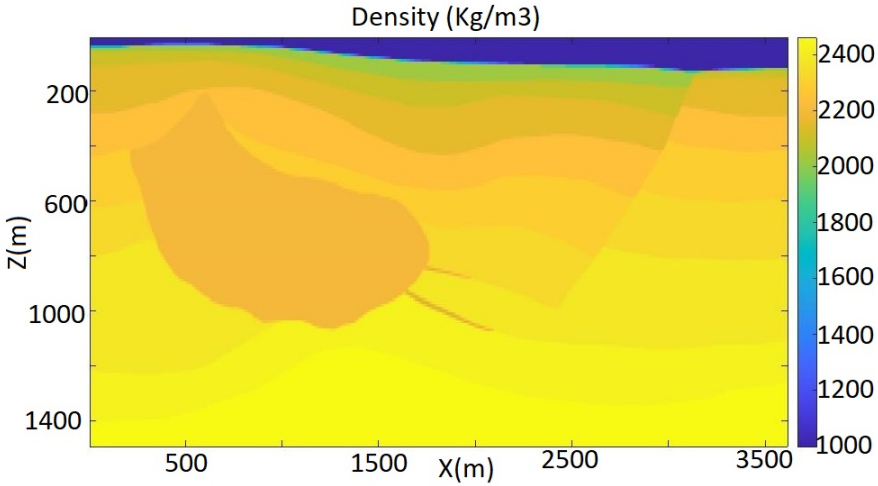
$$u_i - [H_{iJ}^{(0)}\Delta C_{JK}\epsilon_K + \omega^2 G_{ij}^{(0)}\Delta\rho u_j] = u_i^{(0)}$$

Hess model (SEG)

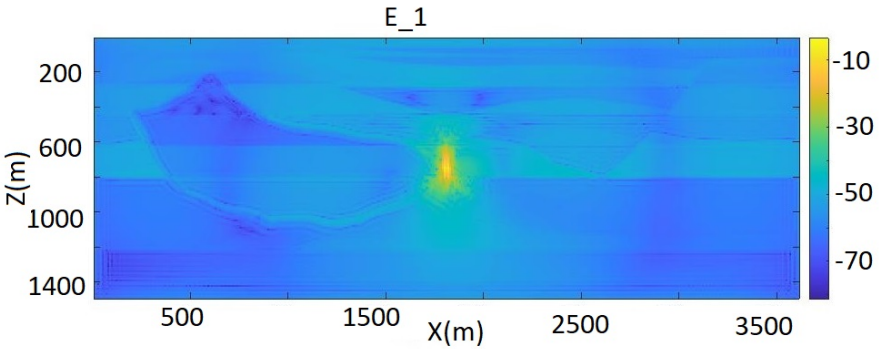


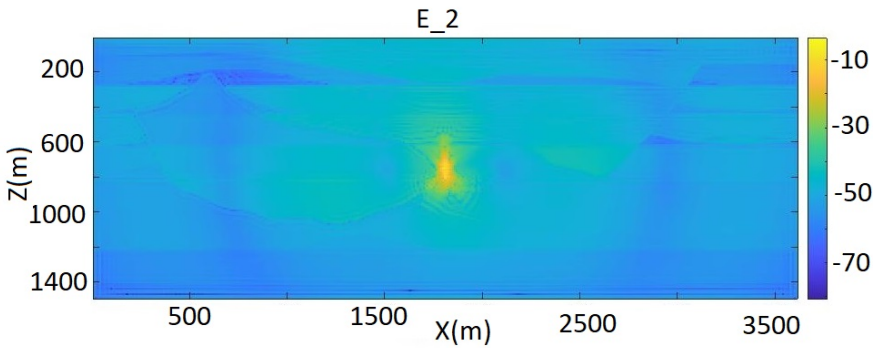


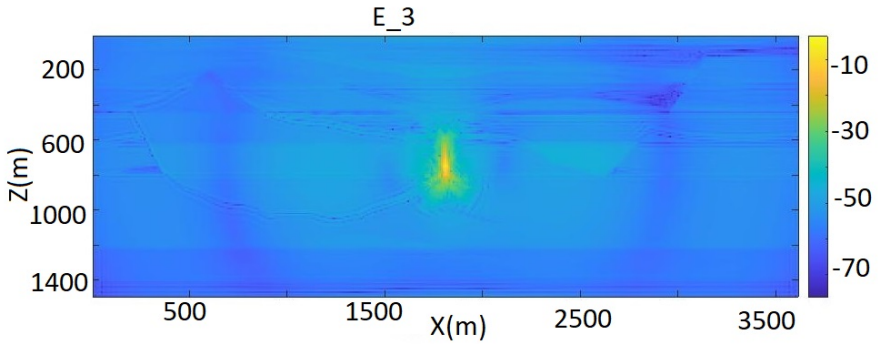


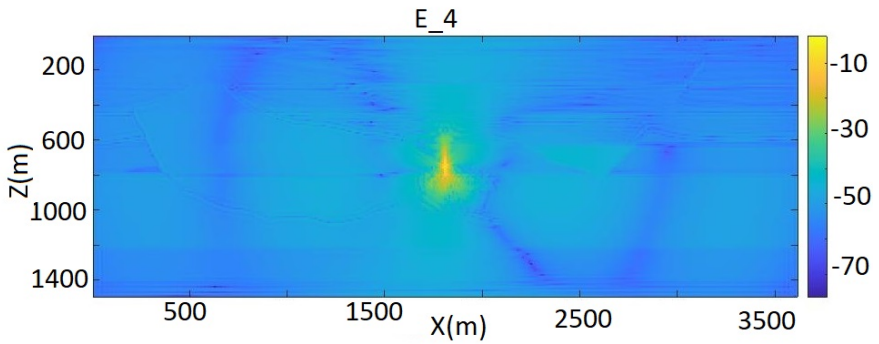


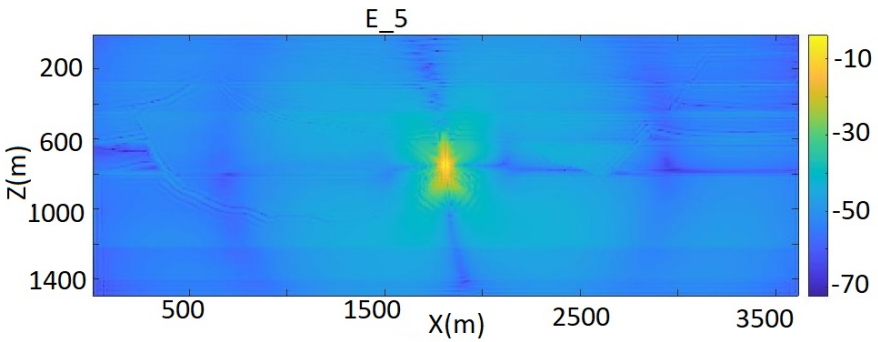
Strain field at 10Hz

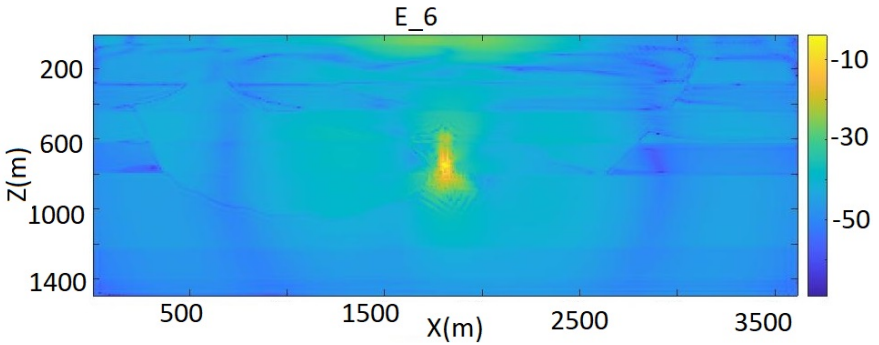




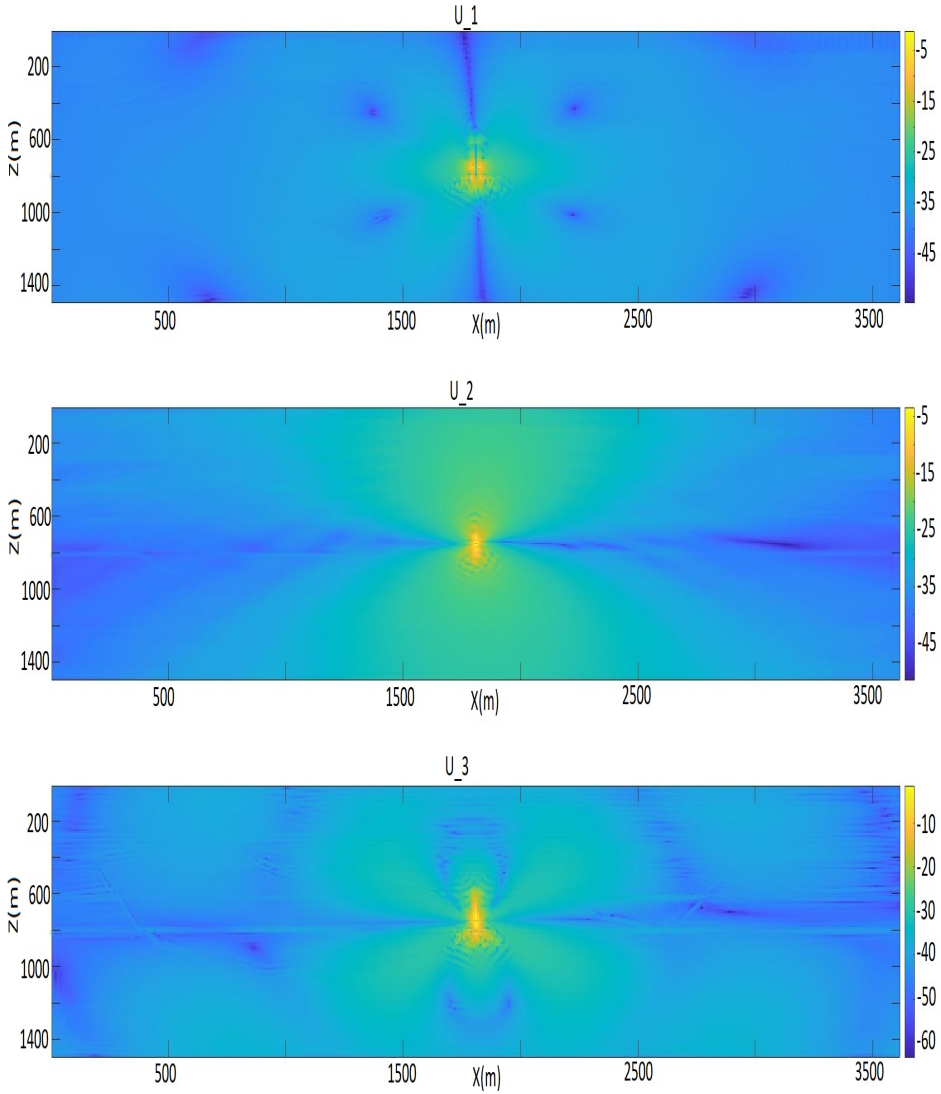




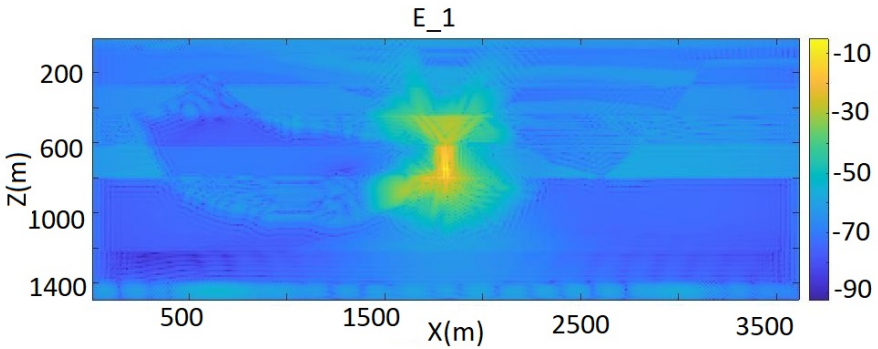


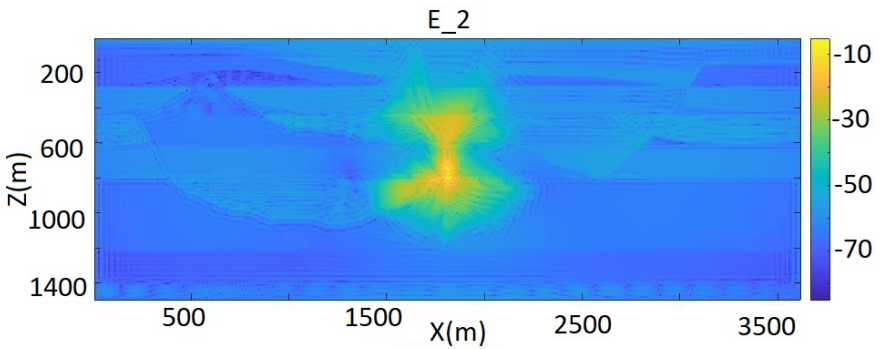


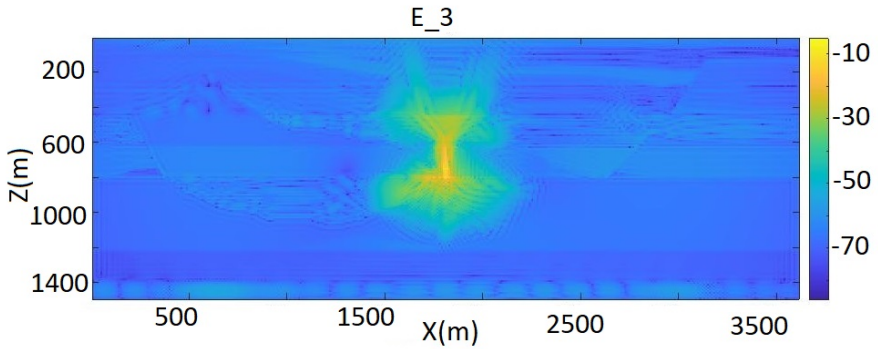
Displacement field at 10Hz

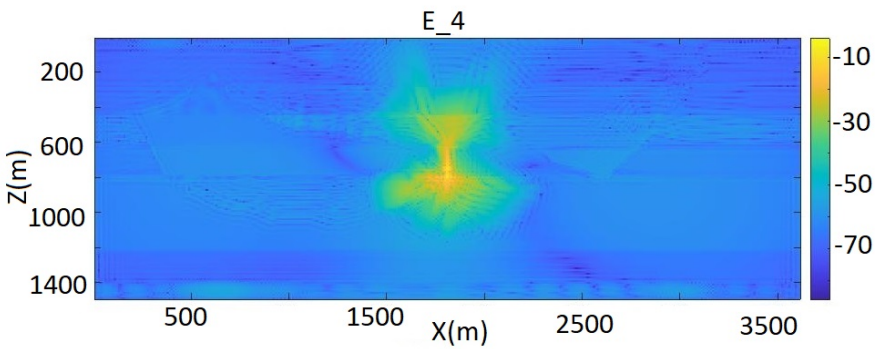


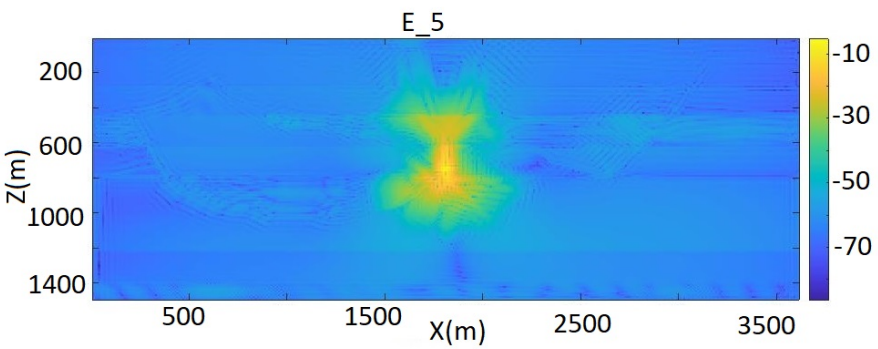
Strain field at 5Hz

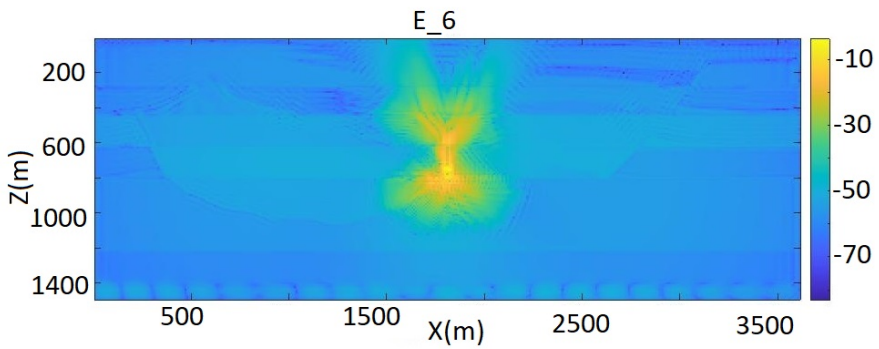




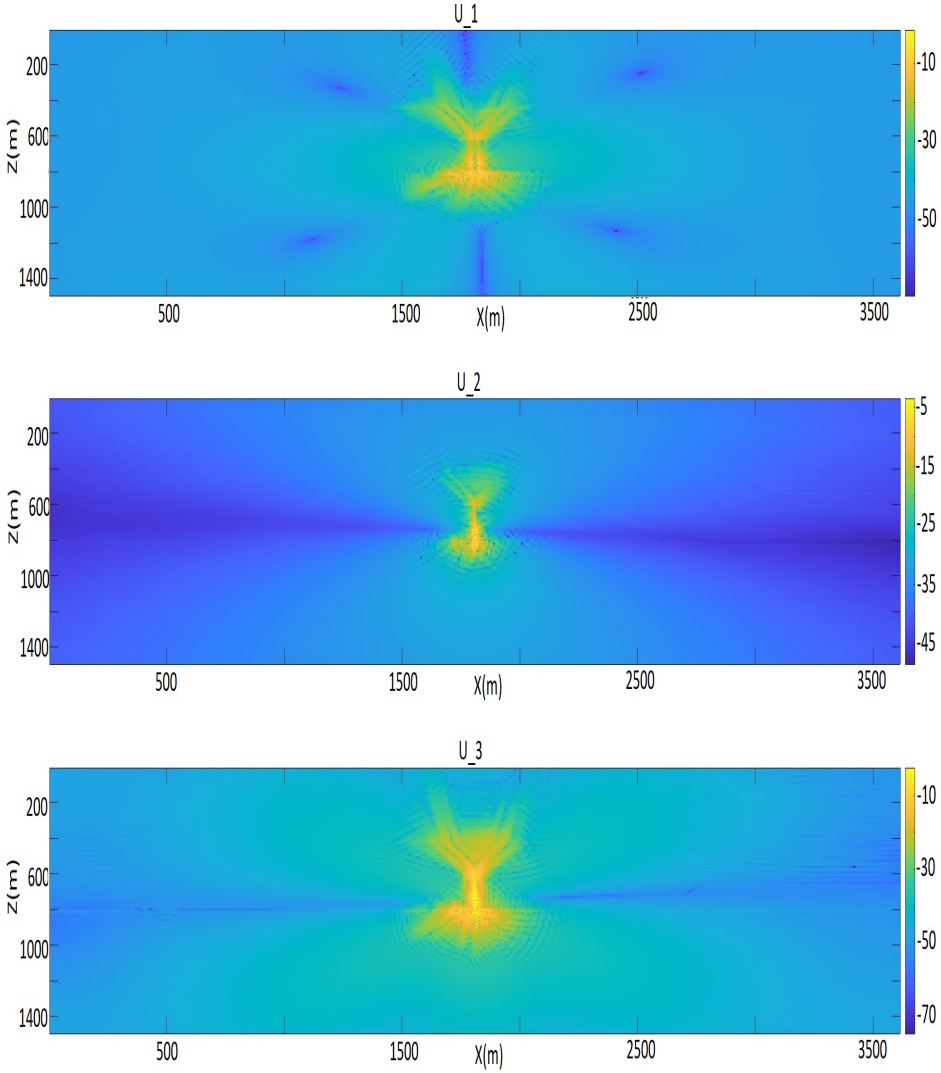








Displacement field at 5Hz



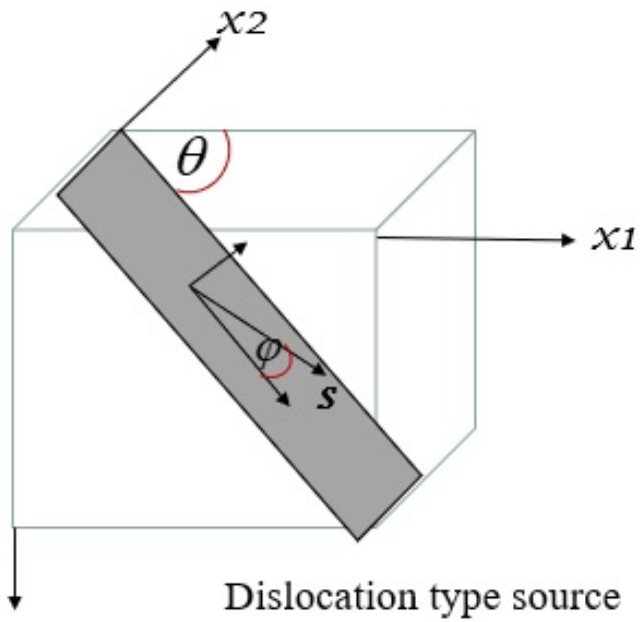
Derivation of momemt tensor components in VTI media

- Grechka (2020) derived the expression for moment tensor in VTI media,

$$\mathbf{M} = \begin{bmatrix} \frac{c_{11}-c_{13}}{2} sin2\theta cos\phi & c_{66} sin\theta sin\phi & c_{44} cos2\theta cos\phi \\ & \frac{c_{12}-c_{13}}{2} sin2\theta cos\phi & c_{44} cos\theta sin\phi \\ & & \frac{c_{13}-c_{33}}{2} sin2\theta cos\phi \end{bmatrix} \mathbf{s\hat{n}}$$

- Density normalized stiffness matrix for the VTI medium (Schoenberg and Helbig, 1997)

$$\mathbf{c_{ij}^{VTI}} = \begin{bmatrix} 10 & 4 & 2.5 & 0 & 0 & 0 \\ & 10 & 2.5 & 0 & 0 & 0 \\ & & 6 & 0 & 0 & 0 \\ & & & 2 & 0 & 0 \\ & & & & 2 & 0 \\ & & & & & 3 \end{bmatrix}$$



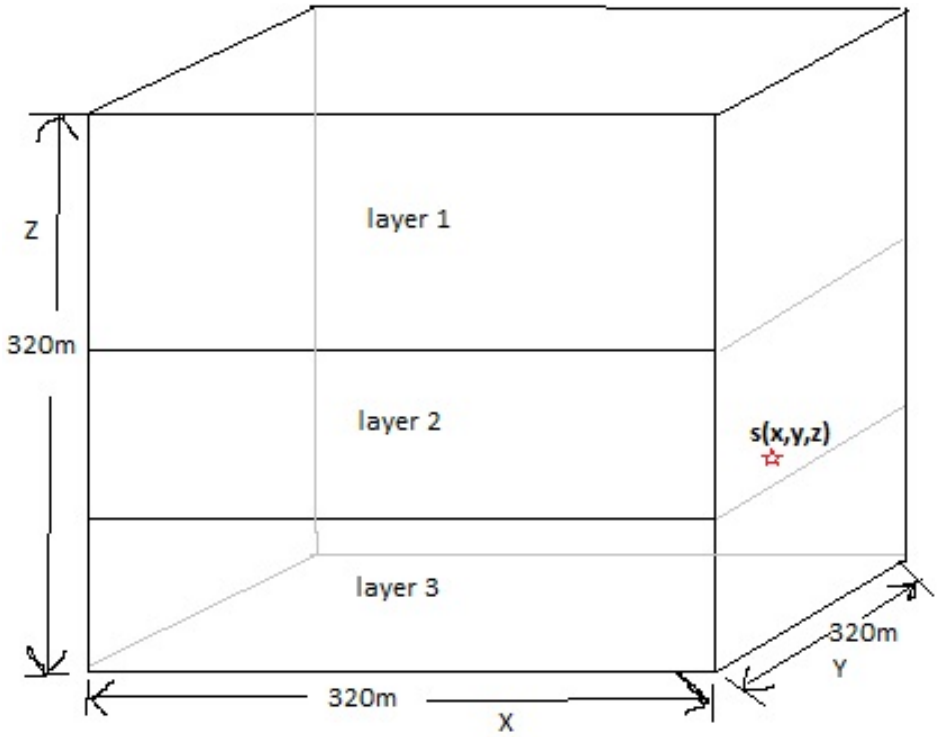
Stiffness Model

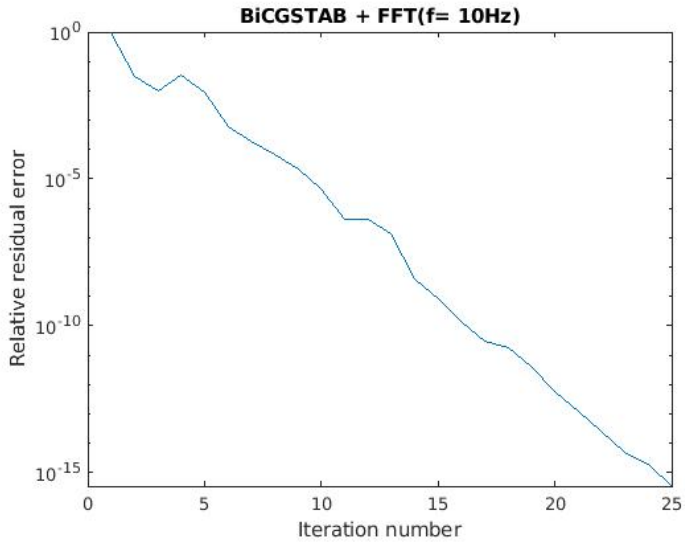
- Each layer possesses VTI symmetry.

$$C_{ij}(layer1) = \mathbf{c_{ij}^{VTI}}$$

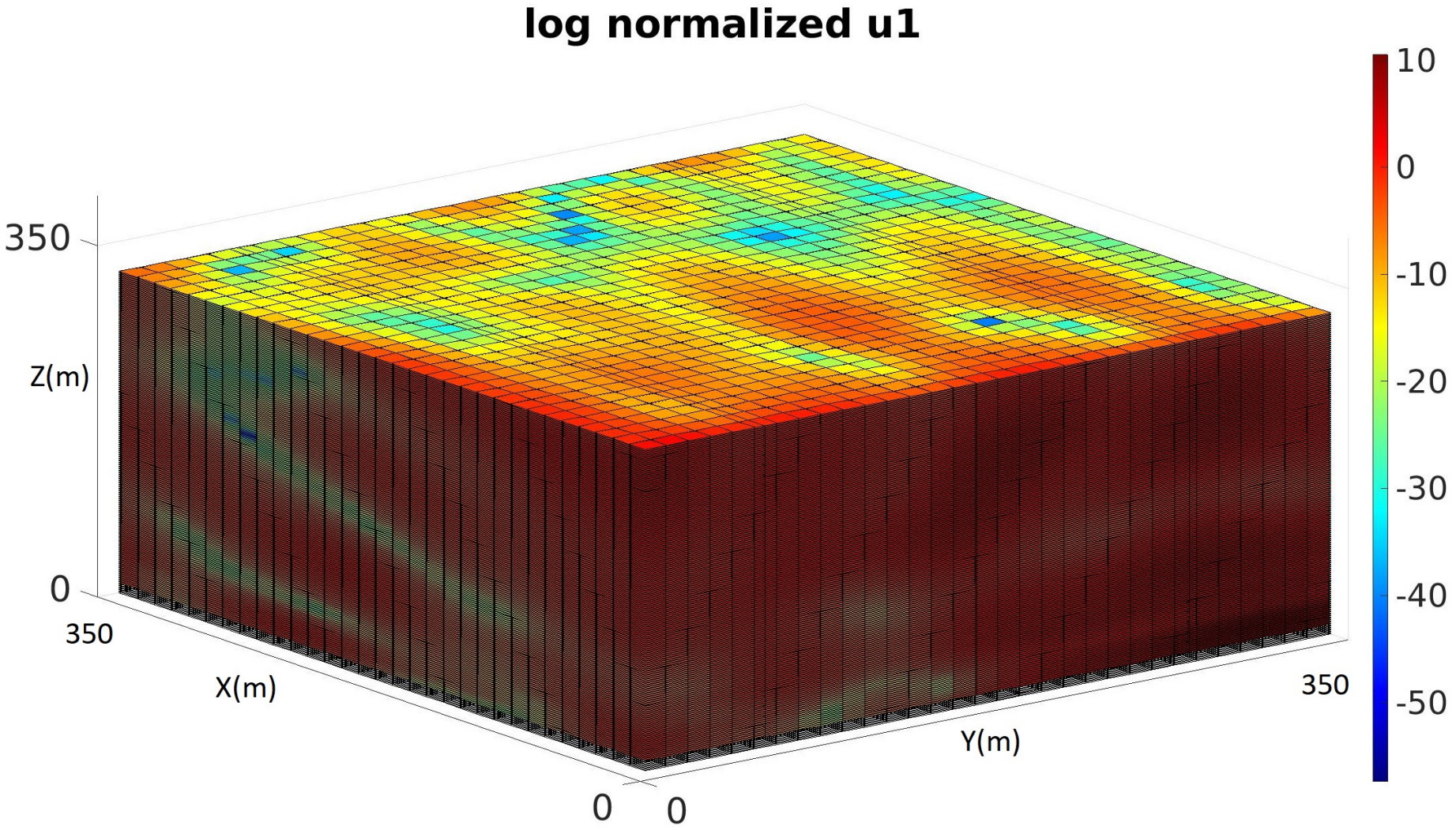
$$C_{ij}(layer2) = 1.1\mathbf{c_{ij}^{VTI}}$$

$$C_{ij}(layer3) = 1.2\mathbf{c_{ij}^{VTI}}$$





Elastic displacement field due to moment tensor source



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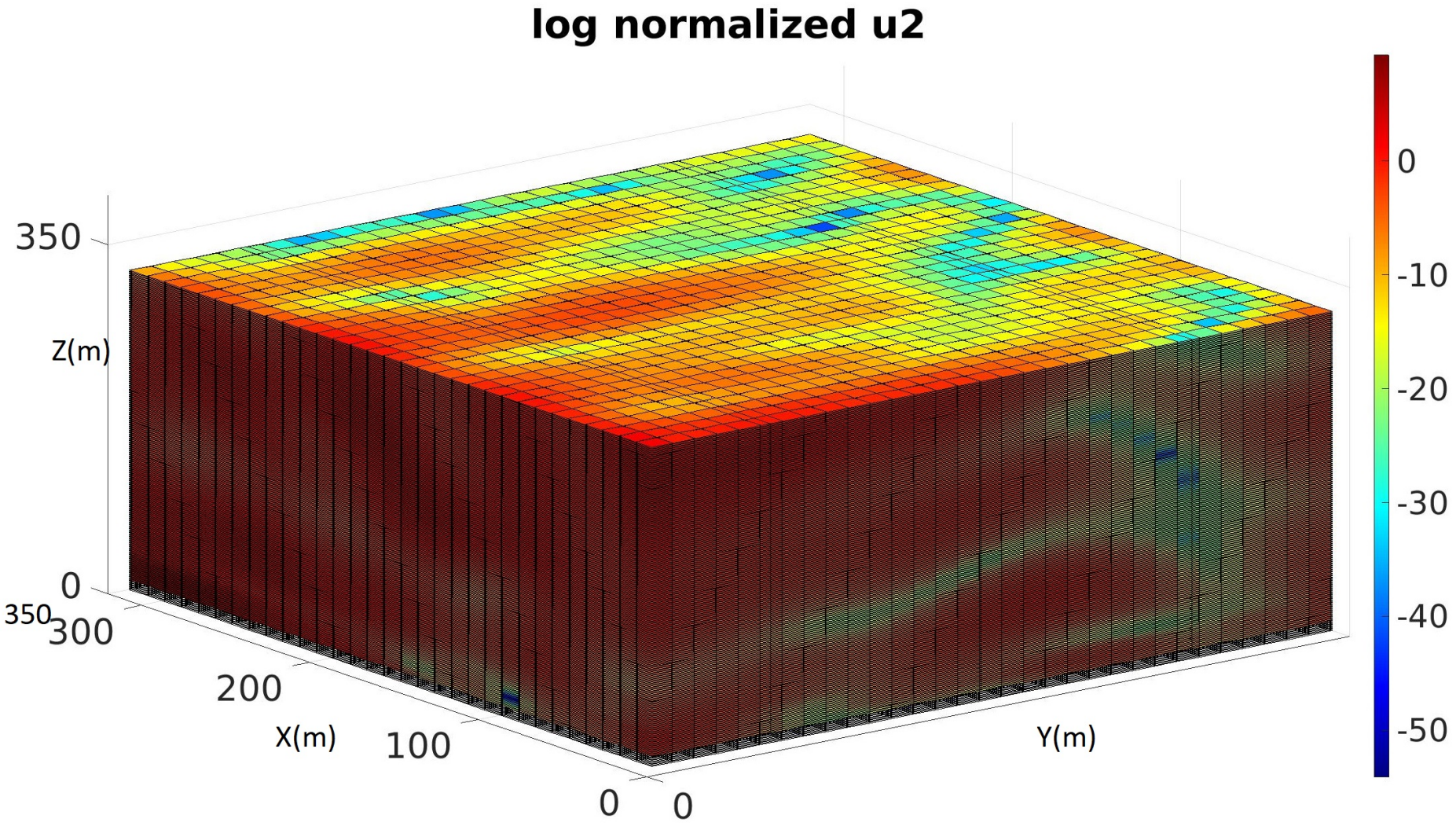

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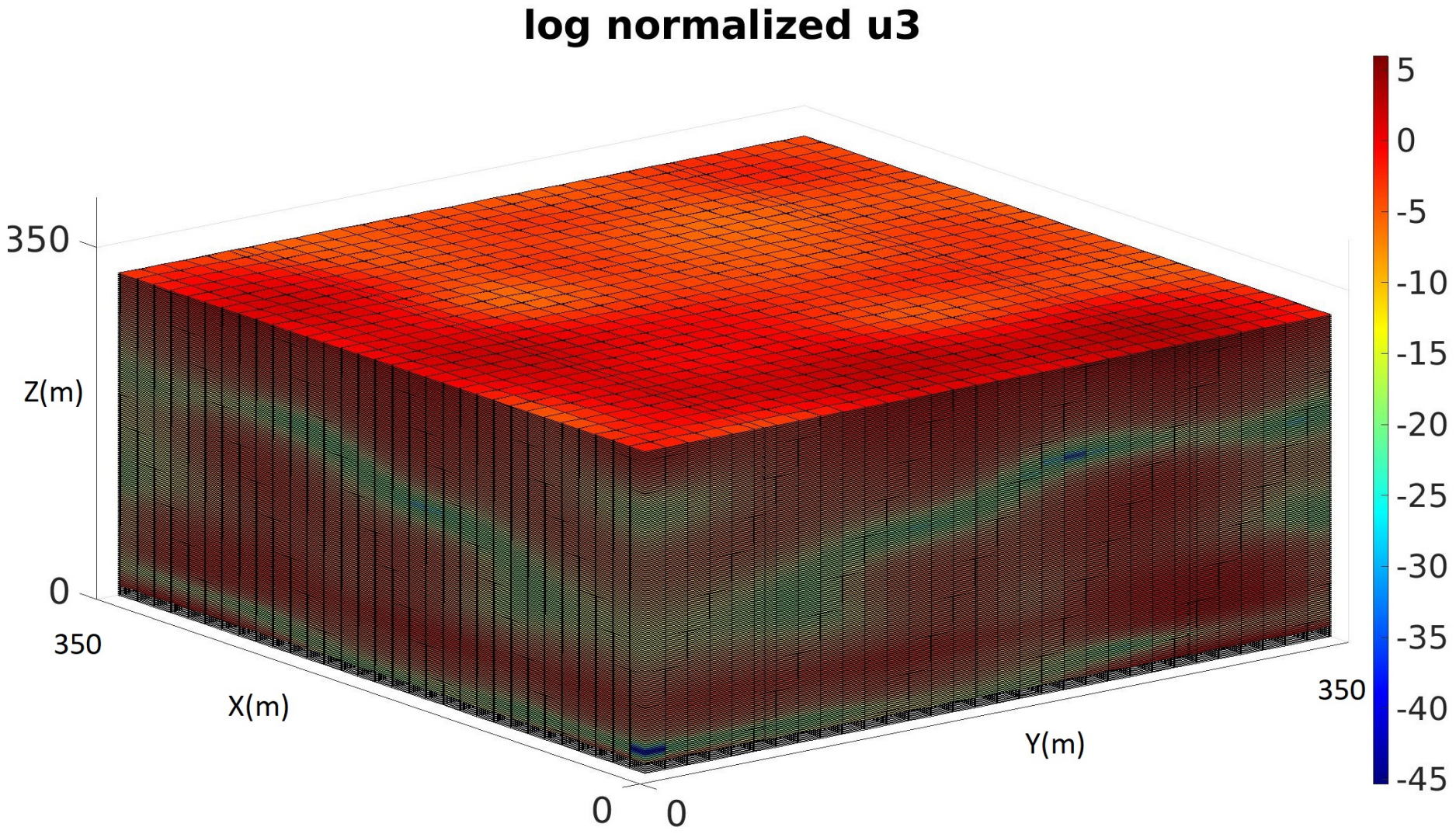

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