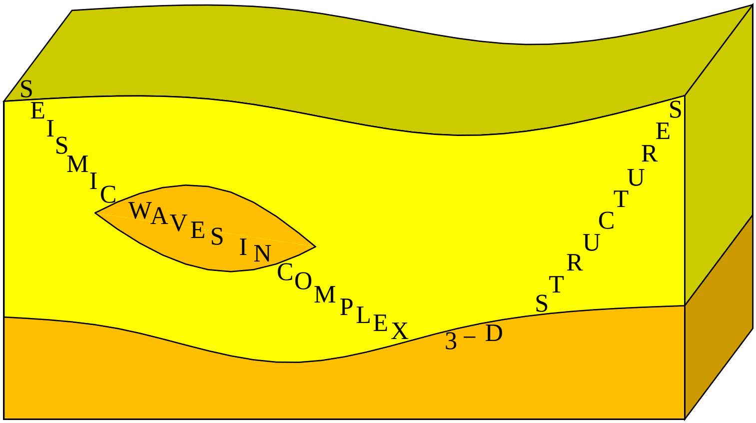
Ray series for electromagnetic waves in static heterogeneous bianisotropic dielectric media

Luděk Klimeš

Department of Geophysics
Faculty of Mathematics and Physics
Charles University in Prague



http://sw3d.cz

Notation

Einstein summation. Indices $i, j, ... = 1, 2, 3$; $\alpha, \beta, ... = 1, 2, 3, 4$.

Maxwell equations

System of 8 first-order partial differential equations for 12 quantities:
E_i ... electric field
B^i ... magnetic induction
D^i ... electric displacement
H_i ... magnetic field

System of 6 constitutive equations

Tellegen representation:

$$D^i = D^i(E_m, H_n)$$
$$B^i = B^i(E_m, H_n)$$

Boys-Post representation:

$$D^i = D^i(E_m, B^n)$$
$$H_i = H_i(E_m, B^n)$$

Special cases of the constitutive equations in the Boys-Post representation

Linear isotropic medium:

$$D^i = \varepsilon\, E_i$$ ε ... permitivity

$$H_i = \mu^{-1} B^i$$ μ^{-1} ... inverse permeability

Linear isotropic chiral medium:

$$D^i = \varepsilon\, E_i + \alpha\, B^i$$ α ... chirality parameter

$$H_i = \alpha\, E_i + \mu^{-1} B^i$$

Linear biisotropic medium:

$$D^i = \varepsilon\, E_i + \alpha\, B^i$$ α, β ... magnetoelectric parameters

$$H_i = \beta\, E_i + \mu^{-1} B^i$$

Linear anisotropic medium:

$$D^i = \varepsilon^{ij}\, E_j$$

$$H_i = \mu^{-1} B^i$$

Counterpart of elastic anisotropy:

$$D^i = \varepsilon\, E_i$$

$$H_i = \mu_{ij}^{-1} B^j$$

Linear bianisotropic medium

$$D^i = \varepsilon^{ij}\, E_j + \alpha^i{}_j\, B^j$$ $\alpha^i{}_j$, $\beta_i{}^j$ ... magnetoelectric matrices

$$H_i = \beta_i{}^j\, E_j + \mu_{ij}^{-1} B^j$$

Vector potential

We may express 6 components E_i and B^i in terms of 6 skew combinations $A_{\alpha,\beta} - A_{\beta,\alpha}$ of the derivatives of the components of covariant 4-vector potential A_α.

Then 4 Maxwell equations for E_i and B^i are identically satisfied.

Remaining 4 Maxwell equations:
for $D^i = D^i(A_{\alpha,\beta} - A_{\beta,\alpha})$ and $H_i = H^i(A_{\alpha,\beta} - A_{\beta,\alpha})$.

In this case, the Boys-Post representation is superior to the Tellegen representation.

Aharonov-Bohm experiment

Electrons propagate around a solenoid through a region where $E_i = 0$ and $B^i = 0$, but $A_\alpha \neq 0$.

Interference of electrons depends on A_α.

The electromagnetic field cannot be completely described by E_i and B^i.
The electromagnetic field is better described by A_α.

Electromagnetic wave equation

Maxwell equations for a linear bianisotropic medium in the Boys-Post representation:

$$[\chi^{\alpha\beta\gamma\delta}(x^\mu)\, A_{\delta,\gamma}(x^\nu)]_{,\beta} = J^\alpha(x^\mu) \quad . \tag{23}$$

Electromagnetic vector potential A_α is a covariant 4-vector.
Constitutive tensor $\chi^{\alpha\beta\gamma\delta}$ is a contravariant tensor density of weight -1.
Current density J^α is a contravariant 4-vector density of weight -1.

The constitutive tensor is skew with respect to its first and second indices, and with respect to its third and fourth indices:

$$\chi^{\alpha\beta\gamma\delta} = -\chi^{\beta\alpha\gamma\delta} \quad , \tag{20}$$

$$\chi^{\alpha\beta\gamma\delta} = -\chi^{\alpha\beta\delta\gamma} \quad . \tag{21}$$

The equation numbers correspond to Klimeš (2016).

Components of the constitutive tensor

Constitutive tensor $\chi^{\alpha\beta\gamma\delta}$ has 36 independent components represented by 36 constitutive parameters ε^{ij}, $\alpha^i{}_j$, $\beta_i{}^j$, μ^{-1}_{ij}:

$$\begin{pmatrix} \chi^{1414} & \chi^{1424} & \chi^{1434} \\ \chi^{2414} & \chi^{2424} & \chi^{2434} \\ \chi^{3414} & \chi^{3424} & \chi^{3434} \end{pmatrix} = -\begin{pmatrix} \varepsilon^{11} & \varepsilon^{12} & \varepsilon^{13} \\ \varepsilon^{21} & \varepsilon^{22} & \varepsilon^{23} \\ \varepsilon^{31} & \varepsilon^{32} & \varepsilon^{33} \end{pmatrix}$$

$$\begin{pmatrix} \chi^{1423} & \chi^{1431} & \chi^{1412} \\ \chi^{2423} & \chi^{2431} & \chi^{2412} \\ \chi^{3423} & \chi^{3431} & \chi^{3412} \end{pmatrix} = -\begin{pmatrix} \alpha^1{}_1 & \alpha^1{}_2 & \alpha^1{}_3 \\ \alpha^2{}_1 & \alpha^2{}_2 & \alpha^2{}_3 \\ \alpha^3{}_1 & \alpha^3{}_2 & \alpha^3{}_3 \end{pmatrix}$$

$$\begin{pmatrix} \chi^{2314} & \chi^{2324} & \chi^{2334} \\ \chi^{3114} & \chi^{3124} & \chi^{3134} \\ \chi^{1214} & \chi^{1224} & \chi^{1234} \end{pmatrix} = \begin{pmatrix} \beta_1{}^1 & \beta_1{}^2 & \beta_1{}^3 \\ \beta_2{}^1 & \beta_2{}^2 & \beta_2{}^3 \\ \beta_3{}^1 & \beta_3{}^2 & \beta_3{}^3 \end{pmatrix}$$

$$\begin{pmatrix} \chi^{2323} & \chi^{2331} & \chi^{2312} \\ \chi^{3123} & \chi^{3131} & \chi^{3112} \\ \chi^{1223} & \chi^{1231} & \chi^{1212} \end{pmatrix} = \begin{pmatrix} \mu^{-1}_{11} & \mu^{-1}_{12} & \mu^{-1}_{13} \\ \mu^{-1}_{21} & \mu^{-1}_{22} & \mu^{-1}_{23} \\ \mu^{-1}_{31} & \mu^{-1}_{32} & \mu^{-1}_{33} \end{pmatrix}$$

Differentiating the above Maxwell equations, we obtain the continuity equation

$$J^{\alpha}_{,\alpha} = 0 \quad . \tag{24}$$

We can thus replace the fourth Maxwell equation by its initial conditions and by the continuity equation for the source terms. For the electromagnetic wave propagation, we then need just the first three of four Maxwell equations

$$(\chi^{i\beta\gamma\delta} A_{\delta,\gamma})_{,\beta} = J^{i} \quad . \tag{25}$$

In our coordinate system, we choose the Weyl gauge condition

$$A_4 = 0 \quad . \tag{26}$$

Maxwell equations (25) then simplify to

$$(\chi^{i\beta\gamma l} A_{l,\gamma})_{,\beta} = J^{i} \quad . \tag{27}$$

We assume that the structure is time-independent (static) in our coordinate system,

$$\chi^{\alpha\beta\gamma\delta} = \chi^{\alpha\beta\gamma\delta}(x^m) \quad . \tag{30}$$

We can thus apply the ray-theory approximation in the frequency domain.

In frequency domain, the Maxwell equations for $A_i = A_i(x^m, \omega)$ with linear constitutive relations in the Boys-Post representation read

$$(\chi^{ijkl} A_{l,k})_{,j} - \mathrm{i}\omega(\chi^{ij4l} A_l)_{,j} - \mathrm{i}\omega\chi^{i4kl} A_{l,k} - \omega^2\chi^{i44l} A_l = J^i \quad , \tag{32}$$

where ω is the circular frequency. Electric current density J^i represents the source term and vanish outside the source.

Electric field strength:

$$E_k = \mathrm{i}\omega\, A_k \quad . \tag{34}$$

Magnetic induction:

$$B^k = \epsilon^{klm} A_{m,l} \quad . \tag{4}$$

Standard ray series

We express frequency-domain magnetic vector potential $A_j = A_j(x^m, \omega)$ in terms of its vectorial amplitude $a_i = a_i(x^m, \omega)$ and travel time $\tau = \tau(x^m)$ as

$$A_i = a_i \exp(\mathrm{i}\omega\tau) \quad . \tag{35}$$

We express the vectorial amplitude in the form of asymptotic series

$$a_i = \sum_{n=0}^{\infty} (\mathrm{i}\omega)^{-n} \, a_i^{[n]} \quad , \tag{36}$$

where $a_i^{[n]} = a_i^{[n]}(x^m, \omega)$ is the n-th order vectorial amplitude.

We consider standard anisotropic ray theory assuming strictly decoupled waves, and proceed according to Červený (2001) using differential operators

$$N^i(a_m, \tau_{,n}) = \Gamma^{il}(x^m, \tau_{,n}, -1)\, a_l \quad , \tag{42}$$

$$M^i(a_m, \tau_{,n}) = \chi^{ijkl}\tau_{,j}a_{l,k} + (\chi^{ijkl}\tau_{,k}a_l)_{,j} - \chi^{i4jl}a_{l,j} - (\chi^{ij4l}a_l)_{,j} \quad , \tag{43}$$

$$L^i(a_m) = (\chi^{ijkl}a_{l,k})_{,j} \quad . \tag{44}$$

Kelvin-Christoffel matrix

$$\Gamma^{il}(x^m, p_n, p_4) = \chi^{i\beta\gamma l}(x^m)p_\beta p_\gamma \tag{41}$$

is a function of six phase-space coordinates x^m, p_n formed by three spatial coordinates x^m and three slowness-vector components p_n. We shall insert $p_4 = -1$.

The Kelvin-Christoffel matrix is not symmetric. Its right-hand eigenvectors differ from its left-hand eigenvectors.

Right-hand eigenvector $g_i = g_i(x^m, \tau_{,n})$, corresponding to selected eigenvalue $G = G(x^m, \tau_{,n})$ of the Kelvin-Christoffel matrix:

$$\Gamma^{il}\, g_l = G\, g_i \quad .$$

Corresponding left-hand eigenvector $\vec{g}_i = \vec{g}_i(x^m, \tau_{,n})$:

$$\vec{g}_i\, \Gamma^{il} = \vec{g}_l\, G \quad .$$

We denote by $G^{\perp}$ the other two eigenvalues of the Kelvin-Christoffel matrix, by $g_i^{\perp}$ the corresponding right-hand eigenvectors, and by $\vec{g}_i^{\perp}$ the corresponding left-hand eigenvectors. Superscript $^{\perp}$ takes two values. The three right-hand eigenvectors of the Kelvin-Christoffel matrix and the three left-hand eigenvectors of the Kelvin-Christoffel matrix are mutually biorthogonal, and we choose them mutually biorthonormal.

Eikonal equation

$$G(x^m, \tau_{,n}) = 0$$

can be solved by standard methods developed for solving the Hamilton-Jacobi equation (Hamilton, 1837; Červený, 1972; Klimeš, 2002; 2010).

Hamilton's equations of rays:

$$\frac{\mathrm{d}x^i}{\mathrm{d}\gamma} = \frac{\partial H}{\partial p_i}(x^m, p_n) \quad , \tag{77}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}\gamma} = -\frac{\partial H}{\partial x^i}(x^m, p_n) \quad . \tag{78}$$

Phase-space derivatives of the Hamiltonian function:

$$\frac{\partial H}{\partial x^i} = -\frac{1}{2\,\varrho}\;\vec{g}_a\,\chi^{a\beta\gamma d}_{,i}\,p_\beta\,p_\gamma\,g_d \quad , \tag{82}$$

$$\frac{\partial H}{\partial p_i} = -\frac{1}{2\,\varrho}\;\vec{g}_a\big(\chi^{ai\gamma d}+\chi^{a\gamma id}\big)p_\gamma\,g_d \quad , \tag{83}$$

where $p_4 = -1$ and

$$\varrho = -\tfrac{1}{2}\;\vec{g}_a\big(\chi^{a4\gamma d}+\chi^{a\gamma 4d}\big)p_\gamma\,g_d \quad . \tag{84}$$

Decomposition of a vectorial amplitude into principal amplitude component $a^{[n]}$ and two additional amplitude components $a^{\perp\,[n]}$:

$$a_i^{[n]} = a^{[n]} g_i + \sum_{\perp} a^{\perp\,[n]}\, g_i^{\perp} \quad . \tag{85}$$

Additional amplitude components:

$$a^{\perp\,[n]} = -\left[\vec{g}_i^{\perp} M^i(a_k^{[n-1]}, \tau_{,n}) + \vec{g}_i^{\perp} L^i(a_k^{[n-2]})\right] \left(G^{\perp}\right)^{-1} \tag{87}$$

with both $a^{\perp\,[0]} = 0$.

Zero-order principal amplitude component:

$$a^{[0]} = a_0^{[0]} \left(\varrho_0\, J_0\right)^{\frac{1}{2}} \left(\varrho\, J\right)^{-\frac{1}{2}} \exp\left(\int_{\tau_0}^{\tau} \mathrm{d}\gamma\, S\right) \quad . \tag{95}$$

Squared geometrical spreading

$$J = \det\left(\frac{\partial x^i}{\partial \gamma^a}\right) \tag{96}$$

represents the Jacobian of transformation from ray coordinates $\gamma^1, \gamma^2, \gamma^3$ to spatial coordinates x^i. These ray coordinates are composed of ray parameters γ^1 and γ^2, and of travel time $\gamma^3 = \tau$ along rays.

Amplitude factor $\exp\left(\int_{\tau_0}^{\tau} \mathrm{d}\gamma\, S\right)$ accounts for the non-reciprocity of the tensor Green function caused by the difference between symmetric and non-symmetric constitutive tensors with respect to the exchange of the first pair of indices and the second pair of indices:

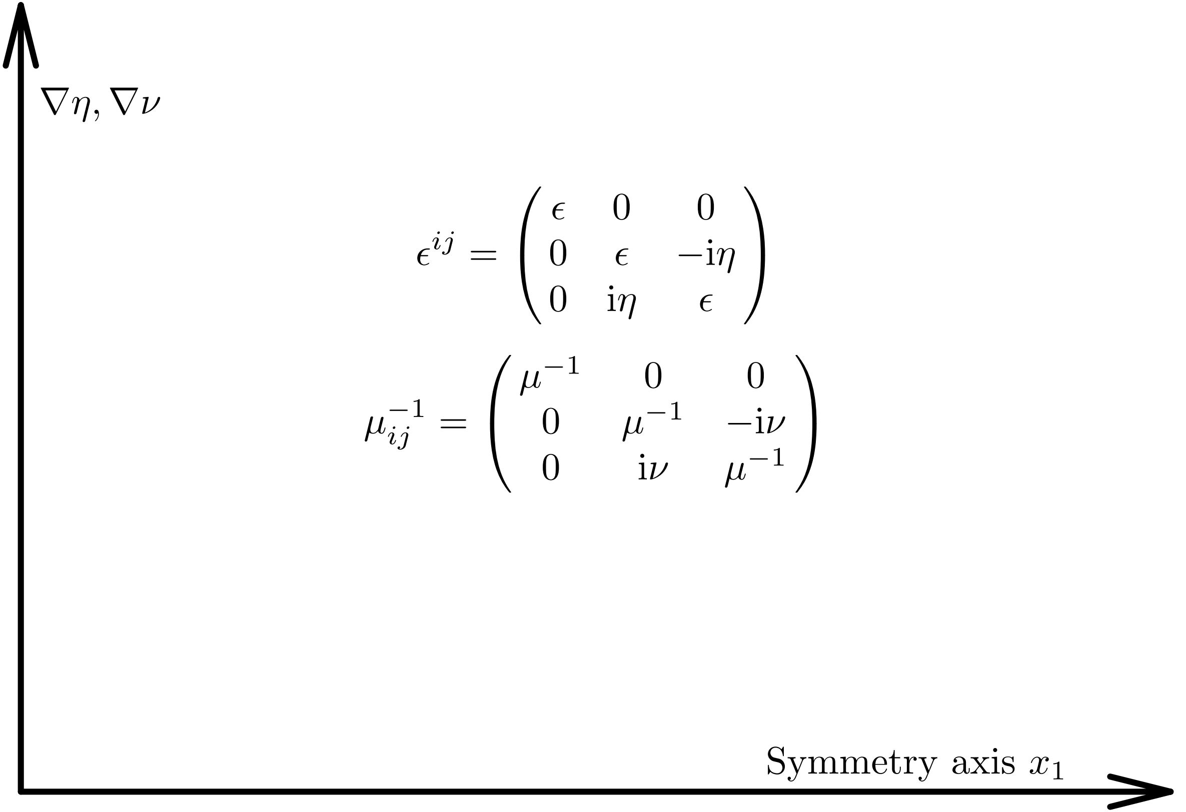
$$S = \frac{1}{4\,\varrho} \sum_{\perp} \left(\vec{g}_k \frac{\partial \Gamma^{kl}}{\partial x^j} g_l^{\perp} \vec{g}_r^{\perp} \frac{\partial \Gamma^{rs}}{\partial p_j} g_s - \vec{g}_k \frac{\partial \Gamma^{kl}}{\partial p_j} g_l^{\perp} \vec{g}_r^{\perp} \frac{\partial \Gamma^{rs}}{\partial x^j} g_s \right) \left(G^{\perp}\right)^{-1}$$
$$+ \frac{1}{4\,\varrho} \vec{g}_i \left(\chi^{ijkl} - \chi^{ikjl}\right)_{,j} \tau_{,k}\, g_l - \frac{1}{4\,\varrho} \vec{g}_i \left(\chi^{ij4l} - \chi^{i4jl}\right)_{,j} g_l - \vec{g}_i \frac{\mathrm{d}g_i}{\mathrm{d}\gamma} \ . \quad (115)$$

Term $\vec{g}_i \frac{\mathrm{d}g_i}{\mathrm{d}\gamma}$ represents just the correction of principal amplitude $a^{[n]}$ due to the undefined length of right-hand eigenvector g_i, and may be put to zero.

Quantity S may be singular at slowness-surface singularities, but is regular at spatial caustics.

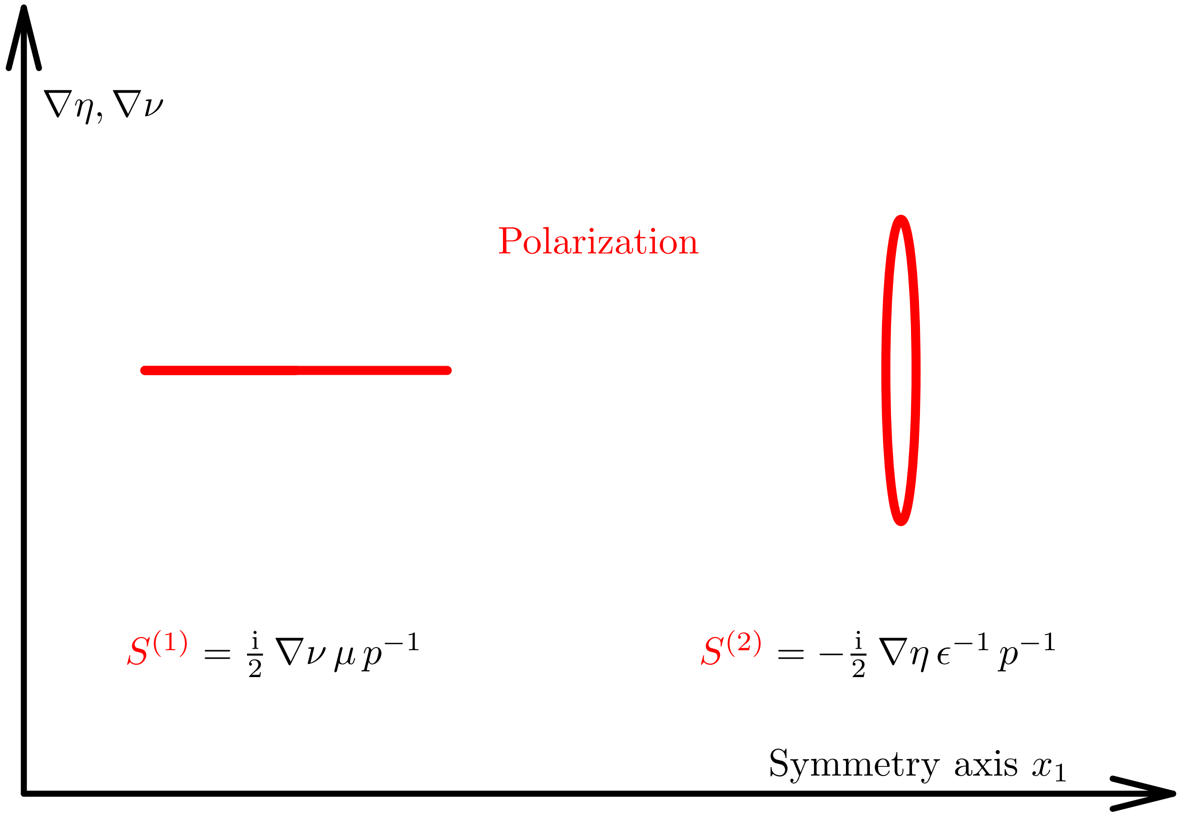
Quantity S vanishes for a constitutive tensor symmetric with respect to the exchange of the first pair of indices and the second pair of indices. For a non-symmetric constitutive tensor, quantity S vanishes in a homogeneous medium.

Example of S in a bigyrotropic medium



$$\epsilon^{ij} = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & -\mathrm{i}\eta \\ 0 & \mathrm{i}\eta & \epsilon \end{pmatrix}$$

$$\mu^{-1}_{ij} = \begin{pmatrix} \mu^{-1} & 0 & 0 \\ 0 & \mu^{-1} & -\mathrm{i}\nu \\ 0 & \mathrm{i}\nu & \mu^{-1} \end{pmatrix}$$



$\nabla\eta, \nabla\nu$
Polarization
$S^{(1)} = \frac{\mathrm{i}}{2}\,\nabla\nu\,\mu\,p^{-1}$
$S^{(2)} = -\frac{\mathrm{i}}{2}\,\nabla\eta\,\epsilon^{-1}\,p^{-1}$
Symmetry axis x_1

Higher-order principal amplitude components

$$a^{[n]} = a^{[0]} \left[\frac{a_0^{[n]}}{a_0^{[0]}} + \int_{\tau_0}^{\tau} \mathrm{d}\gamma \, \frac{Z^{[n-1]}}{a^{[0]} \, \sqrt{\varrho}} \right] \quad , \tag{97}$$

where

$$Z^{[n-1]} = \frac{1}{2\sqrt{\varrho}} \left[\sum_{\perp} \vec{g}_i M^i \big(a^{\perp\,[n]} g_k^{\perp}, \tau_{,n} \big) + \vec{g}_i L^i \big(a_k^{[n-1]} \big) \right] \quad . \tag{93}$$

References:

Červený, V. (1972): Seismic rays and ray intensities in inhomogeneous anisotropic media. *Geophys. J. R. astr. Soc.*, **29**, 1–13.

Červený, V. (2001): *Seismic Ray Theory.* Cambridge Univ. Press, Cambridge.

Hamilton, W.R. (1837): Third supplement to an essay on the theory of systems of rays. *Trans. Roy. Irish Acad.*, **17**, 1–144, read January 23, 1832, and October 22, 1832.

Klimeš, L. (2002): Second-order and higher-order perturbations of travel time in isotropic and anisotropic media. *Stud. geophys. geod.*, **46**, 213–248.

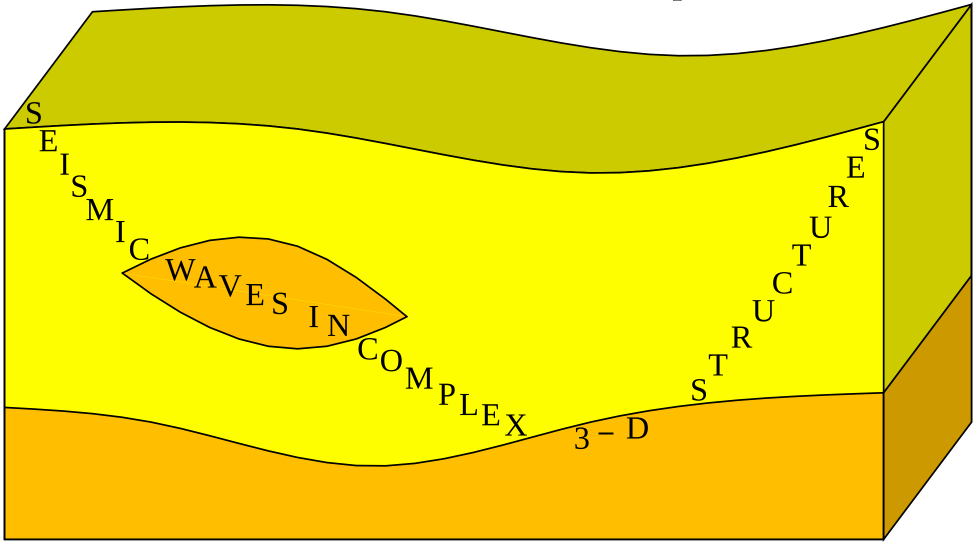
Klimeš, L. (2010): Transformation of spatial and perturbation derivatives of travel time at a general interface between two general media. *Seismic Waves in Complex 3-D Structures*, **20**, 103–114, online at "http://sw3d.cz".

Klimeš, L. (2016): Ray series for electromagnetic waves in static heterogeneous bianisotropic dielectric media. *Seismic Waves in Complex 3–D Structures*, **26**, 167–182, online at "http://sw3d.cz".

Acknowledgements

The research has been supported:
by the Grant Agency of the Czech Republic under contract 16-01312S,

and by the consortium “Seismic Waves in Complex 3-D Structures”



http://sw3d.cz