Representation theorem for viscoelastic waves with a non-symmetric stiffness matrix

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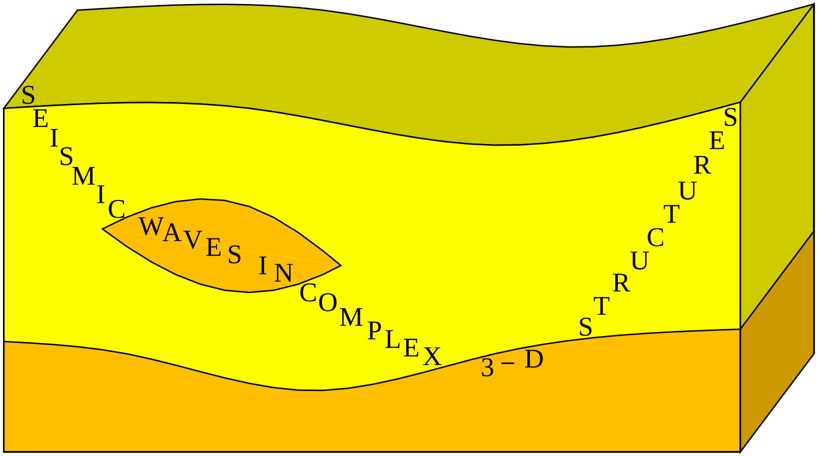

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Notation

Indices $i, j, ... = 1, 2, 3$. The Einstein summation over repetitive indices.

Frequency-domain viscoelastic stiffness tensor

Symmetry of the frequency-domain elastic or viscoelastic stiffness tensor: $c^{ijkl} = c^{ijkl}(x^m, \omega)$:

$$c^{ijkl} = c^{jikl} \quad , \qquad c^{ijkl} = c^{ijlk} \quad .$$

Additional symmetry of the frequency-domain stiffness tensor proved in an elastic medium but not in a viscoelastic medium:

$$c^{ijkl} = c^{klij} \quad .$$

Frequency-domain viscoelastic stiffness tensor:

$$c^{ijkl} \neq c^{klij} \quad .$$

Unfortunately, we currently do not know a real viscoelastic material with a non-symmetric stiffness matrix.

However, we derive the representation theorem for viscoelastic waves with a stiffness tensor which is non-symmetric with respect to the exchange of the first pair of indices and the second pair of indices.

Viscoelastodynamic equation in the frequency domain

Anisotropic viscoelastodynamic equation for displacement $u_i(\mathbf{x},\omega)$ in the frequency domain:

$$\left[c^{ijkl}(\mathbf{x},\omega)\,u_{k,l}(\mathbf{x},\omega)\right]_{,j} + \omega^2 \varrho(\mathbf{x})\,u_i(\mathbf{x},\omega) + f^i(\mathbf{x},\omega) = 0 \quad ,$$

$\varrho = \varrho(x^m)$... density,
ω... circular frequency,
$f^i(\mathbf{x},\omega)$... force density.

Subscript $_{,k}$ following a comma denotes the partial derivative with respect to corresponding spatial coordinate x^k.

If the definition volume for the viscoelastodynamic equation is not infinite, we assume homogeneous boundary conditions (Aki & Richards, 1980, box 2.4).

Green function

The frequency-domain Green function $G_{im}(\mathbf{x},\mathbf{x}',\omega)$ for complex-valued displacement in a viscoelastic medium is the solution of equation

$$\left[c^{ijkl}(\mathbf{x},\omega)\, G_{km,l}(\mathbf{x},\mathbf{x}',\omega)\right]_{,j} + \omega^2 \varrho(\mathbf{x})\, G_{im}(\mathbf{x},\mathbf{x}',\omega) + \delta^i_m\, \delta(\mathbf{x}-\mathbf{x}') = 0$$

analytical with respect to the inverse Fourier transform.
The partial derivatives are related to coordinates $\mathbf{x}$.

Taking scalar product of the equation for the frequency-domain Green function with $f^m(\mathbf{x}',\omega)$ and integrating over the subset V of the definition volume for the viscoelastodynamic equation containing the support of force density $f^m(\mathbf{x}',\omega)$, we see that

$$u_i(\mathbf{x},\omega) = \int_V \mathrm{d}^3\mathbf{x}'\, G_{im}(\mathbf{x},\mathbf{x}',\omega)\, f^m(\mathbf{x}',\omega)$$

is the solution of the frequency-domain viscoelastodynamic equation.

Complementary medium

Analogously to Kamenetskii (2001, eq. 12), we define complementary medium $\tilde{c}^{ijkl}(\mathbf{x},\omega)$ as

$$\tilde{c}^{ijkl}(\mathbf{x},\omega) = c^{klij}(\mathbf{x},\omega) \quad .$$

Complementary Green function

Frequency-domain complementary Green function $\tilde{G}_{km}(\mathbf{x},\mathbf{x}',\omega)$ is the frequency-domain Green function in the complementary medium:

$$\left[c^{klij}(\mathbf{x},\omega)\,\tilde{G}_{km,l}(\mathbf{x},\mathbf{x}',\omega)\right]_{,j} + \omega^2\varrho(\mathbf{x})\,\tilde{G}_{im}(\mathbf{x},\mathbf{x}',\omega) + \delta^i_m\,\delta(\mathbf{x}-\mathbf{x}') = 0 \ .$$

The partial derivatives are related to coordinates $\mathbf{x}$.

Provisional form of the representation theorem

We consider volume V which is the subset of the definition volume for the viscoelastodynamic equation and need not contain the support of force density $f^i(\mathbf{x},\omega)$.

We apply the procedure analogous to the symmetric stiffness matrix, but for the complementary Green function rather than the Green function.

We multiply the equation for the frequency-domain complementary Green function by $u_i(\mathbf{x},\omega)$, subtract the product of the frequency-domain viscoelastodynamic equation with $\tilde{G}_{im}(\mathbf{x},\mathbf{x}',\omega)$, integrate over volume V, and obtain the representation theorem in its provisional form:

$$\begin{aligned} u_m(\mathbf{x}',\omega) = &\int_V \mathrm{d}^3\mathbf{x}\, \tilde{G}_{im}(\mathbf{x},\mathbf{x}',\omega)\, f^i(\mathbf{x},\omega) \\ &+ \oint_{\partial V} \mathrm{d}^2\mathbf{x}\, \big[\tilde{G}_{im}(\mathbf{x},\mathbf{x}',\omega)\, n_j(\mathbf{x})\, c^{ijkl}(\mathbf{x},\omega)\, u_{k,l}(\mathbf{x},\omega) \\ &\qquad - \tilde{G}_{im,j}(\mathbf{x},\mathbf{x}',\omega)\, c^{ijkl}(\mathbf{x},\omega)\, u_k(\mathbf{x},\omega)\, n_l(\mathbf{x})\big] \quad , \end{aligned}$$

where $n_i(\mathbf{x})$ is the unit normal to the surface ∂V of volume V pointing outside volume V.

Reciprocity relation

The provisional form of the representation theorem yields the reciprocity relation:

$$G_{mn}(\mathbf{x}',\mathbf{x}'',\omega) = \tilde{G}_{nm}(\mathbf{x}'',\mathbf{x}',\omega) \quad .$$

Wave-field differences between symmetric and non-symmetric stiffness matrices

For the differences between viscoelastic waves with symmetric and non-symmetric stiffness matrices in the ray-theory approximation, and for the corresponding differences between the Green function and the complementary Green function refer to Klimeš (2018).

Representation theorem

We insert the reciprocity relation into the provisional form of the representation theorem and obtain the final version of the representation theorem:

$$\begin{aligned} u_m(\mathbf{x}', \omega) = &\int_V \mathrm{d}^3\mathbf{x}\, G_{mi}(\mathbf{x}', \mathbf{x}, \omega)\, f^i(\mathbf{x}, \omega) \\ &+ \oint_{\partial V} \mathrm{d}^2\mathbf{x} \left[G_{mi}(\mathbf{x}', \mathbf{x}, \omega)\, n_j(\mathbf{x})\, c^{ijkl}(\mathbf{x}, \omega)\, u_{k,l}(\mathbf{x}, \omega) \right. \\ &\left. - G_{mi,j}(\mathbf{x}', \mathbf{x}, \omega)\, c^{ijkl}(\mathbf{x}, \omega)\, u_k(\mathbf{x}, \omega)\, n_l(\mathbf{x}) \right] \quad . \end{aligned}$$

The integral over volume V represents the wave field corresponding to the sources situated inside volume V.

The integral over the surface ∂V of volume V represents the wave field corresponding to the sources situated outside volume V, and is zero if all sources are situated inside volume V.

Conclusions

The representation theorem for viscoelastic waves with a stiffness tensor which is non-symmetric with respect to the exchange of the first pair of indices and the second pair of indices has the same form as the representation theorem for viscoelastic waves with a symmetric stiffness tensor.

For the detailed derivation refer to Klimeš (2017).

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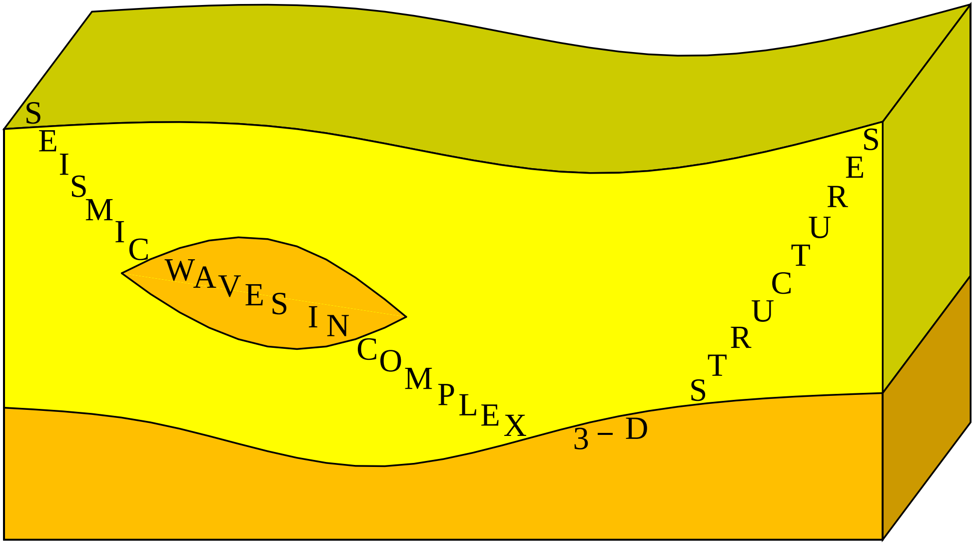

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