Frequency-domain ray series for viscoelastic waves with a non-symmetric stiffness matrix

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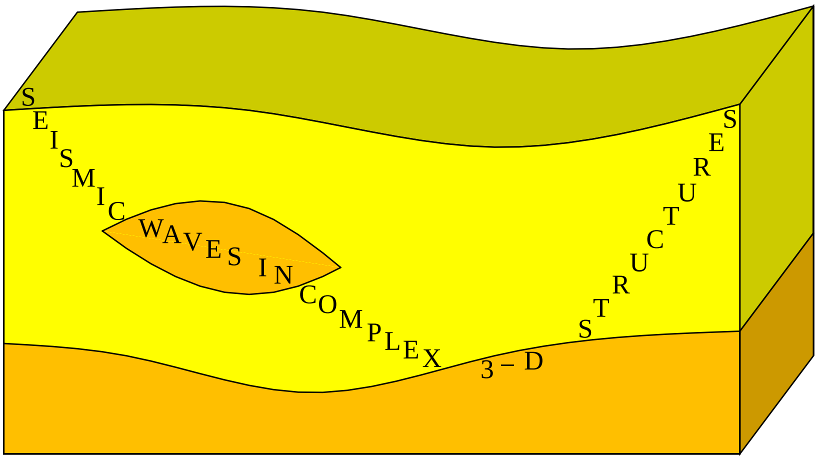

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Notation

Indices $i, j, ... = 1, 2, 3$. The Einstein summation over repetitive indices. The equation labels correspond to Klimeš (2018).

Frequency-domain viscoelastic stiffness tensor

Symmetry of the frequency-domain elastic or viscoelastic stiffness tensor $c^{ijkl} = c^{ijkl}(x^m, \omega)$:

$$c^{ijkl} = c^{jikl} \quad , \qquad c^{ijkl} = c^{ijlk} \quad . \tag{1,2}$$

Additional symmetry of the frequency-domain stiffness tensor proved in an elastic medium but not in a viscoelastic medium:

$$c^{ijkl} = c^{klij} \quad . \tag{3}$$

Frequency-domain viscoelastic stiffness tensor:

$$c^{ijkl} \neq c^{klij} \quad . \tag{4}$$

Unfortunately, we currently do not know a real viscoelastic material with a non-symmetric stiffness matrix.

However, we propose the frequency-domain ray series for viscoelastic waves with a stiffness tensor which is non-symmetric with respect to the exchange of the first pair of indices and the second pair of indices.

Viscoelastodynamic equation in the frequency domain

Anisotropic viscoelastodynamic equation for complex-valued displacement $u_i = u_i(x^m, \omega)$ in the frequency domain outside sources:

$$(c^{ijkl} u_{l,k})_{,j} - (\mathrm{i}\omega)^2 \varrho\, u_i = 0 \quad . \tag{5}$$

Lower-case Roman subscript $_{,k}$ following a comma denotes the partial derivative with respect to corresponding spatial coordinate x^k.
$\varrho = \varrho(x^m)$... density,
ω... circular frequency.

Ray series

Displacement in terms of its frequency-dependent complex-valued vectorial amplitude $U_i = U_i(x^m, \omega)$ and travel time $\tau = \tau(x^m)$:

$$u_i = U_i \exp(\mathrm{i}\omega\tau) \quad . \tag{6}$$

High-frequency asymptotic series

$$U_i = \sum_{n=0}^{\infty} (\mathrm{i}\omega)^{-n}\, U_i^{[n]} \quad . \tag{7}$$

We consider standard anisotropic ray theory assuming strictly decoupled S waves, and proceed according to Červený (2001) using differential operators

$$N^i(U_m, \tau_{,n}) = \varrho\,[\Gamma^{il}(x^m, \tau_{,n})\,U_l - U_i] \quad , \tag{9}$$

$$M^i(U_m, \tau_{,n}) = (c^{ijkl}\tau_{,k}U_l)_{,j} + c^{ijkl}\tau_{,j}U_{l,k} \quad , \tag{10}$$

$$L^i(U_m) = (c^{ijkl}U_{l,k})_{,j} \quad . \tag{11}$$

Christoffel matrix

$$\Gamma^{il}(x^m, p_n) = c^{ijkl}(x^m)\,p_j\,p_k\,\big[\varrho(x^n)\big]^{-1} \tag{12}$$

is a function of six phase-space coordinates x^m, p_n formed by three spatial coordinates x^m and three slowness-vector components p_n.

We insert high-frequency asymptotic series (7) into the viscoelastodynamic equation and sort the terms according to the order of iω, analogously to Červený (1972; 2001, sec. 5.7). We then obtain the system of equations

$$N^i\big(U_k^{[n]}, \tau_{,l}\big) + M^i\big(U_k^{[n-1]}, \tau_{,l}\big) + L^i\big(U_k^{[n-2]}\big) = 0 \tag{14}$$

for each order $n = 0, 1, 2, \ldots$. Here $U_k^{[-1]} = 0$ and $U_k^{[-2]} = 0$, i.e., operator M^i is missing in this equation for $n = 0$ and operator L^i is missing in this equation for $n = 0, 1$.

Eigenvectors and eigenvalues of the Christoffel matrix

The Christoffel matrix is not symmetric. Its right-hand eigenvectors differ from its left-hand eigenvectors.

Right-hand eigenvector $g_i = g_i(x^m, \tau_{,n})$, corresponding to selected eigenvalue $G = G(x^m, \tau_{,n})$ of the Christoffel matrix:

$$\Gamma^{il}\, g_l = G\, g_i \quad . \tag{16}$$

Corresponding left-hand eigenvector $\vec{g}_i = \vec{g}_i(x^m, \tau_{,n})$:

$$\vec{g}_i\, \Gamma^{il} = \vec{g}_l\, G \quad . \tag{17}$$

We denote by $G^{\perp}$ the other two eigenvalues of the Christoffel matrix, by $g_i^{\perp}$ the corresponding right-hand eigenvectors, and by $\vec{g}_i^{\perp}$ the corresponding left-hand eigenvectors. Superscript $^{\perp}$ takes two values. The three right-hand eigenvectors of the Christoffel matrix and the three left-hand eigenvectors of the Christoffel matrix are mutually biorthogonal, and we choose them mutually biorthonormal.

Eikonal equation

Eikonal equation

$$G(x^m, \tau_{,n}) = 1 \tag{20}$$

can be solved by the standard methods developed for solving the Hamilton-Jacobi equation (Hamilton, 1837; Červený, 1972; Klimeš, 2002; 2016).

Principal and additional amplitude components

Decomposition of a vectorial amplitude into principal amplitude component $U_i^{[n]}$ and two additional amplitude components $U^{\perp\,[n]}$:

$$U_i^{[n]} = U^{[n]} g_i + \sum_{\perp} U^{\perp\,[n]}\, g_i^{\perp} \quad . \tag{30}$$

Additional amplitude components:

$$U^{\perp\,[n]} = -\varrho^{-1}\left[\vec{g}_i^{\perp} M^i\big(U_k^{[n-1]}, \tau_{,n}\big) + \vec{g}_i^{\perp} L^i\big(U_k^{[n-2]}\big)\right]\big(G^{\perp} - 1\big)^{-1} \tag{32}$$

with both $U^{\perp\,[0]} = 0$.

Zero-order principal amplitude component

Zero-order principal amplitude component:

$$U^{[0]} = U_0^{[0]} \, (\varrho_0 \, J_0)^{\frac{1}{2}} \, (\varrho \, J)^{-\frac{1}{2}} \exp\Bigl(\int_{\tau_0}^{\tau} \mathrm{d}\gamma \, S\Bigr) \quad . \tag{40}$$

Subscript $_0$ denotes the initial conditions.

Squared geometrical spreading

$$J = \det\left(\frac{\partial x^i}{\partial \gamma^a}\right) \tag{41}$$

represents the Jacobian of transformation from ray coordinates $\gamma^1, \gamma^2, \gamma^3$ to spatial coordinates x^i. These ray coordinates are composed of ray parameters γ^1 and γ^2, and of travel time $\gamma^3 = \tau$ along rays.

“Non-reciprocity” due to a non-symmetric stiffness matrix

Difference between symmetric and non-symmetric stiffness matrices:

$$S = \frac{1}{4}\sum_{\perp}\left(\vec{g}_k \frac{\partial\Gamma^{kl}}{\partial x^j} g_l^{\perp} \vec{g}_r^{\perp} \frac{\partial\Gamma^{rs}}{\partial p_j} g_s - \vec{g}_k \frac{\partial\Gamma^{kl}}{\partial p_j} g_l^{\perp} \vec{g}_r^{\perp} \frac{\partial\Gamma^{rs}}{\partial x^j} g_s\right)\left(G - G^{\perp}\right)^{-1}$$
$$- \frac{1}{4\varrho}\vec{g}_i \left(c^{ijkl} - c^{ikjl}\right)_{,j}\tau_{,k} g_l - \vec{g}_i \frac{\mathrm{d}g_i}{\mathrm{d}\gamma} \quad . \tag{55}$$

Term $\vec{g}_i \frac{\mathrm{d}g_i}{\mathrm{d}\gamma}$ represents just the correction of principal amplitude $U^{[n]}$ due to the undefined length of right-hand eigenvector g_i, and may be put to zero.

Quantity S may be singular at slowness-surface singularities, but is regular at spatial caustics.

Quantity S vanishes for a symmetric stiffness matrix. For a non-symmetric stiffness matrix, quantity S vanishes in a homogeneous medium.

Quantity S is thus generated by a combination of a non-symmetric stiffness matrix and heterogeneities.

Higher-order principal amplitude components

Higher-order principal amplitude components:

$$U^{[n]} = U^{[0]} \left[\frac{U_0^{[n]}}{U_0^{[0]}} + \int_{\tau_0}^{\tau} \mathrm{d}\gamma \, \frac{Z^{[n-1]}}{U^{[0]} \, \sqrt{\varrho}} \right] \tag{42}$$

with

$$Z^{[n-1]} = -\frac{1}{2\sqrt{\varrho}} \left[\sum_{\perp} \vec{g}_i M^i \big(U^{\perp [n]} g_k^{\perp}, \tau_{,n} \big) + \vec{g}_i L^i \big(U_k^{[n-1]} \big) \right] \quad . \tag{39}$$

Conclusions

We have derived the anisotropic-ray-theory series for viscoelastic waves with a non-symmetric stiffness matrix. These ray series enable us to estimate which phenomena could be observed in the wave field if the stiffness matrix were non-symmetric.

Whereas the two S waves, which propagate with different velocities, are linearly polarized in elastic media, they may be elliptically or even circularly polarized in viscoelastic media. Whereas the two elliptically polarized S waves always display equal handedness for a symmetric stiffness matrix, they display opposite handedness for a sufficiently non-symmetric stiffness matrix, similarly as electromagnetic waves in optically active media.

The ray-theory amplitudes corresponding to a non-symmetric stiffness matrix are not reciprocal in the same way as the ray-theory amplitudes corresponding to a symmetric stiffness matrix. This "non-reciprocity" is expressed in terms of quantity S in the expression for the zero-order ray-theory amplitude. Refer to Klimeš (2017, eq. 18) for the sense in which the ray-theory Green function corresponding to a non-symmetric stiffness matrix is reciprocal.

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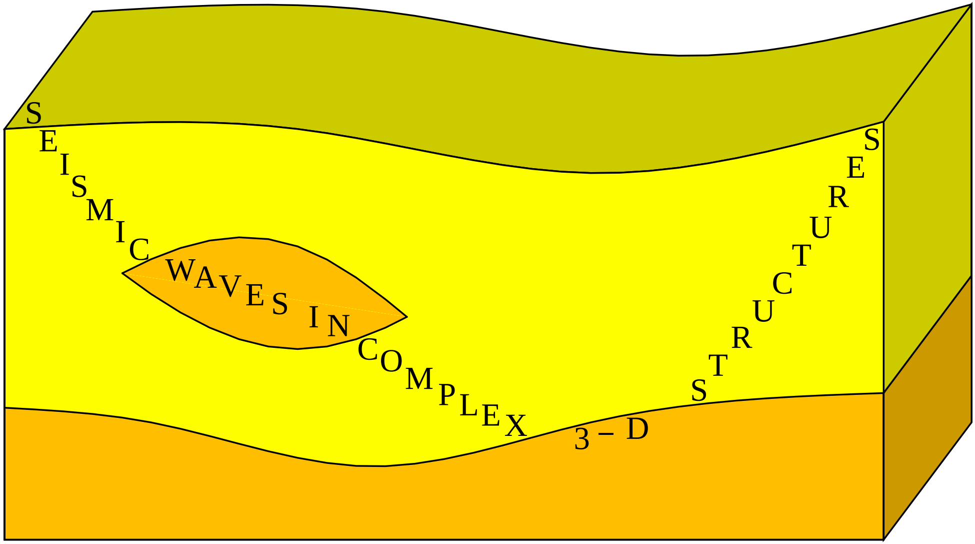
Acknowledgements

The research has been supported:

by the Grant Agency of the Czech Republic under contract 16-05237S,

and by the consortium "Seismic Waves in Complex 3-D Structures"



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