I. INTRODUCTION

Perturbation theory is a useful tool for approximate study of wave properties in weakly anisotropic media. Especially its first-order approximation yields simple and transparent formulas only slightly more complicated than those for isotropic media. Thus it allows approximate but very effective solution of forward and inverse problems like ray tracing, travel-time computations, travel-time tomography, local determination of parameters of media, etc. Perturbation theory also plays an important role in the coupling ray theory and quasi-isotropic approximation in particular [see Červený (2001), for example].

Farra (2001) extended standard techniques for the calculation of the first-order approximations of the phase velocity and polarization vectors (see, e.g., Jech and Pšencík, 1989), and proposed a procedure how to calculate even higher-order approximations. In this contribution, we extend and apply results of Farra (2001). We extend it by a detailed description of the construction of perturbation series for \( qS \) waves, by introduction of new, useful, auxiliary vectorial bases, in which the perturbation formulas can be specified, and by the sensitivity study of the wave attributes. We then apply the formulas to study, both analytically and numerically, behavior of first-order perturbation formulas in the directions of acoustical axes.

In perturbation formulas of any order, an important role is played by a matrix, \( B_{mn} \), whose elements control various attributes of elastic waves. The matrix has several interesting and important properties. It is independent of the choice of a reference medium. It depends linearly on weak anisotropy (WA) parameters, which are, in turn, linearly related to elastic parameters. Explicit dependence of the matrix \( B_{mn} \) on various parameters of the medium or their combinations makes it possible to study sensitivity of various attributes of elastic waves propagating in an anisotropic medium to its parameters. This is important information for inversion of observed data into elastic moduli (Chu et al., 1994; Song et al., 2001; Zheng and Pšencík, 2002). Attention is also paid to alternative ways of constructing approximate formulas for the polarization vectors of \( qS \) waves.

A special attention is devoted to the study of effects of the first-order perturbation formulas on the position of acoustical axes in weakly anisotropic media. Acoustical axes are a characteristic feature of anisotropic media. We can distinguish two kinds of acoustical axes (Schoenberg and Helbig, 1997). The first are the well-known singular directions in which the two \( qS \) waves propagating in anisotropic media propagate with the same phase velocity. These directions correspond to the places on the slowness surfaces at which the two \( qS \)-wave sheets are in a contact. Characteristic features of singular directions and their vicinity are distortions of slowness surfaces and strong variations of \( qS \)-wave polarization vectors (Schoenberg and Helbig, 1997; Shuvalov and Every, 1997; Vavryčuk, 2003). The second kind of acoustical axes are longitudinal directions. A characteristic feature of

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Veronique Farra\(^a\) and Ivan Pšencík\(^b\)

\(^a\)Département de Sismologie, Institut de Physique du Globe de Paris, 4 Place Jussieu, 75252 Paris Cedex 05, France and Geophysical Institute, Academy of Sciences of the Czech Republic, Boční II, Praha 4, Czech Republic\(^b\)

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Use of the perturbation theory in the study of attributes of elastic waves propagating in weakly anisotropic media leads to approximate but transparent and simple formulas, which have many applications in forward and inverse wave modeling. We present and study such formulas. We show that all studied attributes depend on elements of a matrix linearly dependent on parameters of a medium. We study this dependence with the goal to understand which parameters of the medium, and in which combinations, affect individual wave attributes. Alternative auxiliar vector bases, in which the matrix can be specified, are proposed and studied. The vector bases offer alternative specifications of polarization vectors of \( qS \) waves. One of the important observations is that the higher-order (\( n \geq 2 \)) perturbation formulas for \( qS \) waves are obtained separately for \( qS1 \) and \( qS2 \) waves. We also study effects of the use of the perturbation theory on the accuracy of the determination of the acoustical axes in weakly anisotropic media. We show that longitudinal directions in the first-order approximation are identical with actual ones. In singular directions, however, the first-order formulas provide directions, which may deviate from the exact ones, or they may even indicate false singular directions. Again, the above-mentioned matrix depending linearly on the parameters of the medium plays a central role in this study. © 2003 Acoustical Society of America. [DOI: 10.1121/1.1591772]

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longitudinal directions is propagation of purely longitudinal and purely transverse waves along them, i.e., propagation of $qP$ waves with polarization parallel to the slowness vector (specifying the longitudinal direction) and propagation of $qS$ waves with polarization perpendicular to the slowness vector. Helbig (1993) and Schoenberg and Helbig (1997) propose to use the acoustical axes for inversion of observed data into elastic parameters specifying anisotropy of the medium. It thus seems that importance of the acoustical axes might increase in the near future. In this contribution, we study effects of frequently used approximations on the accuracy with which the acoustical axes can be determined.

In Sec. II, basic equations are given. Properties of the matrix $B_{mn}$ and its sensitivity to the parameters of a medium are studied. Alternative auxiliary vector bases, in which the matrix $B_{mn}$ can be specified, are proposed and discussed. Then basic results of Farra (2001) are briefly reviewed and extended. It is shown that the higher-order ($n \geq 2$) perturbation formulas for $qS$ waves are obtained separately for $qS1$ and $qS2$ waves. An alternative determination of two unit vectors specifying the $qS$-wave polarization, which leads to simplified polarization formulas, is proposed. In Sec. III, the conditions for the determination of acoustical axes in the first-order approximation of the perturbation theory are specified and discussed. The accuracy of the determination of acoustical axes determined by the perturbation formulas is studied. In Sec. IV, accuracy of perturbation formulas is illustrated on several numerical examples of media used in literature. The Appendix contains explicit expressions for the elements for one of many possible specifications of the matrix $B_{mn}$.

In the following, component notation is used. All the Roman lowercase indices range over the values 1, 2, and 3 and the uppercase indices over the values 1 and 2. Einstein summation convention is used for the repeated subscripts. Where necessary, superscripts are used. The superscripts in parentheses indicate an order of the quantity. In the matrices $B_{mn}^{(j)}$ and in the vectors $e_i^{(jm)}$, the upper indices $j$ in parentheses indicate that $B_{mn}^{(j)}$ and $e_i^{(jm)}$ are used in expressions for the quantities of the $j$th or $(j+1)$-st order (see the text). The superscripts in the curly brackets (for example, $g_i^{[m]}$) indicate the type of the wave, $m = 1, 2$ for the $qS$ waves and $m = 3$ for the $qP$ wave. Voigt notation $A_{a\beta}$ for density-normalized elastic parameters, with $a, \beta$ running from 1 to 6, is used in parallel with the tensor notation $a_{ijkl}$.

II. HIGHER-ORDER PERTURBATION FORMULAS

We first study properties of the matrix $B_{mn}$ playing the basic role in perturbation formulas for many wave attributes. Then we briefly review the formulas for the calculation of higher-order terms for the phase velocity and polarization of $qP$ and $qS$ waves derived by Farra (2001), and present their alternative form for $qS$ waves.

A. Basic equations

Let us start from the Christoffel matrix

$$\Gamma_{jk} = a_{ijkl} n_i n_l. \tag{1}$$

The symbol $\delta_{ijkl}$ denotes a tensor of density-normalized elastic parameters and $n_i$ is a unit wave vector. The Christoffel matrix $\Gamma_{jk}$ is symmetric and positive definite, see, e.g., Červený (2001). Thus for any direction of the wave vector $n_i$, $\Gamma_{jk}$ has three positive and real eigenvalues $G_m = G_m(n_i)$ and three corresponding eigenvectors $g_i^{[m]}(n_i)$. They satisfy the following system of equations:

$$\left(\Gamma_{jk} - G_m \delta_{jk}\right) g_i^{[m]} = 0. \tag{2}$$

Two eigenvalues $G_1$ and $G_2$ with corresponding eigenvectors $g_i^{[1]}$ and $g_i^{[2]}$ correspond to the two $qS$ waves, and the remaining eigenvalue $G_3$ with the eigenvector $g_i^{[3]}$ belongs to the $qP$ wave. The eigenvalues are related to the squares of corresponding phase velocities $c_m(n_i)$.

$$G_m(n_i) = c_m^2(n_i). \tag{3}$$

The eigenvectors $g_i^{[m]}$ specify polarization vectors of the corresponding waves. The eigenvalues $G_1$ and $G_2$ are for most geological materials smaller than $G_3$ (Schoenberg and Helbig, 1997). Since the relation between $G_1$ and $G_2$ can be chosen arbitrarily, we choose it as follows: $G_2 \leq G_1 < G_3$.

The tensor $a_{ijkl}$ can be expressed as follows:

$$a_{ijkl} = a_{ijkl}^{(0)} + \Delta a_{ijkl}. \tag{4}$$

Here $a_{ijkl}^{(0)}$ is a tensor of density-normalized elastic parameters in a reference isotropic medium:

$$a_{ijkl}^{(0)} = (\alpha^2 - 2\beta^2) \delta_{ij} \delta_{kl} + \beta^2(\delta_{ik} \delta_{jl} + \delta_{jl} \delta_{ik}) \tag{5}$$

and $\Delta a_{ijkl}$ is its perturbation. In (5), $\alpha$ and $\beta$ denote $P$- and $S$-wave velocities of the reference isotropic medium.

B. Matrix $B_{mn}$

We introduce three mutually perpendicular unit vectors $e_i^1, e_i^2$, and $e_i^3$ so that $e_i^3 = n_i$. The vectors $e_i^1$ and $e_i^2$, situated in the plane perpendicular to $e_i^3$, can be chosen arbitrarily.

We use the vectors $e_i^k$ to define a matrix $B_{mn}$,

$$B_{mn} = \Gamma_{jk} e_j^m e_k^n. \tag{6}$$

The matrix $B_{mn}$ is independent of the choice of the reference isotropic medium. It is related to the weak anisotropy (WA) matrix $B_{mn} = \Delta a_{ijkl} \gamma_{mn} e_i^m e_j^n$, which, in contrast to $B_{mn}$, depends on the choice of the reference medium. Equation (13) of Pšenčík and Gajewski (1998) yields the relation between the two matrices:

$$B_{mn} = B_{mn} + c_0^2 \delta_{mn}. \tag{7}$$

The symbol $c_0$ in Eq. (7) stands for the phase velocity of the reference isotropic medium. For $m = n = 1$ or 2, $c_0 = \beta$; for $m = n = 3$, $c_0 = \alpha$. From (7), we can see that the off-diagonal terms of the matrix $B_{mn}$ are of the first order.

In the following we frequently use two specifications of the vectors $e_i^k$ and, thus, of the matrix $B_{mn}$. In one we use a matrix $\hat{B}_{mn}^{(0)}$ defined by Eq. (6) with vectors $e_i^1$ and $e_i^2$ substituted by vectors $\hat{e}_i^{(0)}$ and $\hat{e}_i^{(1)}$ chosen so that

$$\hat{B}_{12}^{(0)} = 0, \hat{B}_{11}^{(0)} > 0, \hat{B}_{22}^{(0)} < 0. \tag{8}$$

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The second condition in Eq. (8) excludes singular directions from our considerations. The singular directions are studied in Sec. III. The diagonal elements of the matrix \( \bar{B}_{mn} \) specify the first-order approximations \( G^{(1)}_m \) of the eigenvalues of the Christoffel matrix, see Farra (2001).

\[
G^{(1)}_1 = \bar{B}_{11}, \quad G^{(1)}_2 = \bar{B}_{22}, \quad G^{(1)}_3 = \bar{B}_{33}. \tag{9}
\]

Since the elements of the matrix \( B_{mn} \) are independent of the velocities \( \alpha \) and \( \beta \) of the reference medium, \( G^{(1)}_m \) in (9) are independent of them, too.

In the other specification, we substitute the vectors \( e^K_i \) by vectors \( c^K_i \) given by Eqs. (A1) and (A2) of the Appendix. We denote the corresponding matrix \( B_{mn} \) by a bar: \( \bar{B}_{mn} \).

Explicit expressions for the elements of the matrix \( \bar{B}_{mn} \) can be found in Eqs. (A4) in the Appendix. Elements of a matrix \( B_{mn} \) specified for arbitrarily chosen vectors \( e_i \) are related to \( \bar{B}_{mn} \), by simple linear relations. If we denote by \( \phi \) the acute angle between the vectors \( e^K_i \) and \( e^K_i \), the elements of the matrix \( \bar{B}_{mn} \) transform into \( B_{mn} \), in the following simple way, see, e.g., Pšenčík and Vravčuk (2002):

\[
\begin{align*}
B_{11} &= \bar{B}_{11} \cos^2 \phi + 2 \bar{B}_{12} \cos \phi \sin \phi + \bar{B}_{22} \sin^2 \phi, \\
B_{22} &= \bar{B}_{22} \sin^2 \phi - 2 \bar{B}_{12} \cos \phi \sin \phi + \bar{B}_{11} \cos^2 \phi, \\
B_{12} &= (\bar{B}_{22} - \bar{B}_{11}) \cos \phi \sin \phi + \bar{B}_{12} (\cos^2 \phi - \sin^2 \phi), \\
B_{13} &= \bar{B}_{13} \cos \phi + \bar{B}_{23} \sin \phi, \\
B_{23} &= -\bar{B}_{13} \sin \phi + \bar{B}_{23} \cos \phi, \\
B_{33} &= \bar{B}_{33}.
\end{align*}
\] (10)

We can see that the elements \( \bar{B}_{11} \), \( \bar{B}_{12} \), and \( \bar{B}_{22} \) transform into the elements of the matrix \( B_{mn} \) with the same subscripts. The same holds for the elements \( \bar{B}_{13} \) and \( \bar{B}_{23} \). The element \( \bar{B}_{33} \) is unaffected by the rotation.

In what follows it is shown that in the first-order approximation the elements of the above matrices control phase velocities, polarizations, and orientations of acoustical axes of elastic waves propagating in weakly anisotropic media. The elements \( B_{11} \), \( B_{22} \), and \( B_{22} \) are related to the \( qS \)-wave phase velocities and control orientation of singular directions. The element \( B_{33} \) is related to the \( qP \)-wave phase velocity. The elements \( B_{13} \) and \( B_{23} \) are related to the polarization vectors of \( qP \) and \( qS \) waves and control orientation of the longitudinal directions. Since, for given \( c^K_i \), the elements of the matrix \( B_{mn} \) are linear functions of the WA or elastic parameters, it is desirable to deduce which parameters can affect individual elements of the matrix \( B_{mn} \) and which parameters and in which form (individually or in a combination with others) can be retrieved from individual elements of the matrix \( \bar{B}_{mn} \).

By a simple inspection of Eqs. (A4) in the Appendix, we can find that the elements \( \bar{B}_{11} \), \( \bar{B}_{12} \), and \( \bar{B}_{22} \) are controlled by 15 mutually independent coefficients, which depend linearly on all 21 WA parameters. The coefficients are \( e_{13} \), \( e_{24} \), \( e_{34} \), \( x_3 - e_{35} \), \( y_3 - e_{35} \), \( e_{45} - \delta_{x} + e_{x} \), \( e_{y} - \delta_{y} + e_{y} \), \( e_{z} - \delta_{z} + e_{z} \), \( x_5 - e_{35} \), \( x_6 - e_{35} \), \( \gamma_{13} \), \( \gamma_{14} \), \( \gamma_{23} \), \( \gamma_{24} \), \( e_{46} \), and \( e_{56} \). We can see that from the above elements of the matrix \( \bar{B}_{mn} \) we can retrieve individually only six WA parameters. All remaining 15 WA parameters can be found only in combinations with others.

By a similar inspection of the element \( \bar{B}_{33} \) we can find that the element is controlled by 15 different mutually independent coefficients, which depend on 15 \( qP \)-wave WA parameters. The coefficients are \( e_{13} \), \( e_{24} \), \( e_{34} \), \( e_{16} - \delta_{x} + e_{x} \), \( e_{y} - \delta_{y} + e_{y} \), \( e_{z} - \delta_{z} + e_{z} \), \( x_3 - e_{35} \), \( x_4 - e_{35} \), \( x_5 - e_{35} \), \( x_6 - e_{35} \), \( \gamma_{13} \), \( \gamma_{14} \), \( \gamma_{23} \), \( \gamma_{24} \), \( e_{46} \), and \( e_{56} \). We can see that in this case all 15 involved WA parameters can be determined from \( \bar{B}_{33} \) individually.

The elements \( \bar{B}_{13} \) and \( \bar{B}_{23} \) are controlled by 14 mutually independent coefficients, which depend on 15 \( qP \)-wave WA parameters. The coefficients have the same form as for \( B_{33} \) except \( e_{13} \), \( e_{24} \), \( e_{34} \). Instead of these three coefficients we have their two combinations: \( e_{13} - e_{16} \) and \( e_{24} - e_{26} \). This means that from the elements \( \bar{B}_{13} \) and \( \bar{B}_{23} \) we can retrieve only nine individually WA parameters. The remaining six can be determined only in combinations. This has been observed by Zheng and Pšenčík (2002) during their inversion of synthetic polarization data in a VTI model.

A similar analysis in terms of the elastic parameters \( \alpha_{q\beta} \) yields the following conclusions, which can be simply deduced from Eqs. (A3). The elements \( B_{11} \), \( B_{12} \), and \( B_{22} \) depend on all 21 elastic parameters. Only six of them, \( A_{44} \), \( A_{55} \), \( A_{66} \), \( A_{45} \), \( A_{56} \), and \( A_{46} \), can be found individually. The remaining 15 elastic parameters can be found only in combinations with other parameters.

The element \( B_{33} \) depends also on all 21 elastic parameters. But only nine of them, \( A_{11} \), \( A_{16} \), \( A_{24} \), \( A_{26} \), \( A_{34} \) and \( A_{35} \), can be determined individually. The remaining 12 elastic parameters can be found only in combinations with others.

The elements \( \bar{B}_{13} \) and \( \bar{B}_{23} \) depend again on all 21 elastic parameters. But only six of them, \( A_{15} \), \( A_{16} \), \( A_{24} \), \( A_{25} \), \( A_{34} \) and \( A_{35} \), can be determined individually. The remaining 15 elastic parameters can be obtained only in combinations.

### C. \( qP \) wave

We follow Farra (2001) and express the polarization vector \( g_1^{(3)} \), related to the \( qP \) wave, in the vector basis \( (\hat{e}^{(0)}_i, \hat{e}^{(0)}_j, \hat{e}^{(0)}_3) \). Inserting \( g_1^{(3)} \) specified in this way into (2) and multiplying the resulting equation successively by \( \hat{e}^{(0)}_i \), \( \hat{e}^{(0)}_j \), and \( \hat{e}^{(0)}_3 \), we get a set of three equations from which we can determine a vector parallel to the exact polarization vector \( g_1^{(3)} \):

\[
\frac{\hat{B}_{13}^{(0)}}{G_3 - G_1^{(1)}} \hat{e}^{(0)}_i + \frac{\hat{B}_{23}^{(0)}}{G_3 - G_2^{(1)}} \hat{e}^{(0)}_j + \hat{e}^{(0)}_3. \tag{11}
\]

Note that the vector (11) is generally not unit. Equation (11) is derived under the assumption that the expressions in the denominators are nonzero. As mentioned above, in most geological materials this is guaranteed.

The vector (11) specifies the exact direction of polarization vector of the \( qP \) wave. Since the determination of the vectors \( \hat{e}^{(0)}_i \) and \( \hat{e}^{(0)}_j \) of the elements of the matrix \( \bar{B}_{mn}^{(0)} \) and of the first-order approximations of the eigenvalues is straightforward, only knowledge of the exact eigenvalue \( G_3 \) is nec-
ecessary for the determination of the direction of the polarization vector \( g_i^{(S)} \) from Eq. (11). Equation (11) can be also used for the determination of the directions of the polarization vectors of different order of approximation, see Farra (2001). In the zeroth-order approximation, the polarization vector is \( e_i^3 \). In the first-order approximation we get

\[
 g^{(1)}_i = \frac{\hat{B}_{13}^{(0)}}{G_3 - G_1^{(0)} e_i^1} + \frac{\hat{B}_{23}^{(0)}}{G_3 - G_2^{(0)}} e_i^2 + e_i^3. 
\]  

(12)

It is easy to show that in contrast to the general Eq. (11), Eq. (12) holds for arbitrarily chosen vectors \( e_i^k \), not only for \( e_i^{(0)k} \). It can thus be rewritten in the form

\[
 g^{(1)}_i = \frac{B_{13}}{\alpha^2 - \beta^2} e_i^1 + \frac{B_{23}}{\alpha^2 - \beta^2} e_i^2 + e_i^3. 
\]  

(13)

We took into account that \( G_3^{(0)} = \alpha^2 \) and \( G_1^{(0)} = G_2^{(0)} = \beta^2 \). Equation (13) corresponds to the expression for the first-order perturbation of the \( qP \)-wave polarization vector of Jech and Pšencík (1989). Since \( B_{13} \) and \( B_{23} \) are the first-order quantities, the vector \( g_i^{(1)[3]} \) is a unit vector in the first-order approximation.

If we substitute \( g_k^{(m)} \) in the Christoffel equation (2) by the vector (11), and multiply the resulting equation by \( e_i^3 \), we get

\[
 G_3 = G_3^{(1)} + \frac{(\hat{B}_{13}^{(0)})^2}{G_3 - G_1^{(1)}} + \frac{(\hat{B}_{23}^{(0)})^2}{G_3 - G_2^{(1)}}. 
\]  

(14)

This is a basic equation for the iterative determination of the higher-order approximations of the eigenvalue \( G_3 \), see Eq. (30) of Farra (2001). We can proceed in the same way as Farra (2001) to obtain approximations of \( G_3 \) of different orders. Specifically, in the first-order approximation we obtain

\[
 G_3^{(1)[3]} = \hat{B}_{33}^{(0)} = \Gamma_{jk} e_i^3 e_i^3, 
\]  

(15)

see Eq. (9). It is interesting to note that an odd-order approximation of the eigenvalue \( G_3 \) can be determined from a lower odd-order approximation. The same holds for even-order approximations, see Farra (2001). The odd-order approximations of the eigenvalue \( G_3 \) are independent of the choice of the reference medium while the even-order approximations depend on its choice.

Since the elements \( B_{13} \), \( B_{23} \), and \( B_{13} \) appearing in Eqs. (13) and (15) are related only to the elements \( \hat{B}_{13} \), \( \hat{B}_{23} \), and \( \hat{B}_{33} \), see Eq. (10), from the discussion in Sec. II B it follows that the first-order approximations of the polarization vector (13) and of the eigenvalue (15) (representing square of the phase velocity of the \( qP \) wave in the first-order approximation) depend on 15 WA parameters. From the phase velocity all 15 WA parameters can be determined. Nine individual WA parameters and the remaining six in five combinations can be determined from the polarization vector. The polarization vector depends, in addition, on the difference of squared phase velocities in the reference isotropic medium: \( \alpha^2 - \beta^2 \).

D. \( qS \) wave

We consider the two \( qS \) waves propagating in anisotropic media. We denote the faster wave \( qS1 \) and associate it with the index \( K = 1 \). The slower \( qS2 \) wave is associated with the index \( K = 2 \). For the \( qS \) waves we must use a procedure designed for the degenerate cases. In this section, we assume that the considered direction is not a singular direction. Singular directions are considered in Sec. III. We again follow Farra (2001) and express the polarization vector of the \( Kth \) \( qS \) wave, \( g_i^{(K)} \), in the basis \( (e_i^1, e_i^2, e_i^3) \). Note that arbitrarily chosen vectors \( e_i^k \) are considered. Inserting \( g_i^{(K)} \) specified in this way into (2) and multiplying the resulting equation successively by \( e_i^1 \), \( e_i^2 \), and \( e_i^3 \), we get, after some algebra, the following set of three equations for the projections of the vector \( g_i^{(K)} \) into \( e_i^1 \), \( e_i^2 \), and \( e_i^3 \):

\[
 (M_{11}^{(K)} - G_K) e_i^1 + M_{12}^{(K)} e_i^2 = 0, 
\]  

(16a)

\[
 M_{21}^{(K)} e_i^1 + (M_{22}^{(K)} - G_K) e_i^2 = 0, 
\]  

(16b)

\[
 B_{13}^{(K)} e_i^1 + B_{23}^{(K)} e_i^2 + (G_3^{(1)K} - G_K) e_i^3 = 0. 
\]  

(16c)

Here we use the notation of Farra (2001):

\[
 M_{ij}^{(K)} = B_{ij} + \frac{B_{13} B_{j3}}{G_K - G_3^{(1)K}}. 
\]  

(17)

Note that \( M_{ij}^{(K)} \) in Eq. (17) depends, through the index \( K \), on the type of the considered \( qS \) wave. From the condition of solvability of Eqs. (16a) and (16b), we get expressions for the eigenvalues \( G_K \) of the two \( qS \) waves:

\[
 G_1 = \frac{1}{3} [M_{11}^{(1)} + M_{22}^{(1)} + \sqrt{D_1}], 
\]  

(18a)

\[
 G_2 = \frac{1}{3} [M_{11}^{(2)} + M_{22}^{(2)} - \sqrt{D_2}], 
\]  

(18b)

where

\[
 D_K = (M_{11}^{(K)} - M_{22}^{(K)})^2 + 4(M_{12}^{(K)})^2. 
\]  

(19)

From Eqs. (16) and (17), we can determine vectors parallel to the exact polarization vectors \( g_i^{(1)} \) and \( g_i^{(2)} \). They can be written as follows:

\[
 P_K e_i^1 + Q_K e_i^2 + R_K e_i^3. 
\]  

(20)

The symbols \( P_K \) and \( Q_K \) in (20) have the following meaning:

\[
 P_1 = \sqrt{\frac{1}{2} \left( 1 + \frac{M_{11}^{(2)} - M_{22}^{(2)}}{\sqrt{D_1}} \right)}, 
\]  

(21a)

\[
 Q_1 = \text{sgn}(M_{12}^{(1)}) \sqrt{\frac{1}{2} \left( 1 - \frac{M_{11}^{(1)} - M_{22}^{(1)}}{\sqrt{D_1}} \right)}, 
\]  

and

\[
 P_2 = -\text{sgn}(M_{12}^{(2)}) \sqrt{\frac{1}{2} \left( 1 + \frac{M_{11}^{(2)} - M_{22}^{(2)}}{\sqrt{D_2}} \right)}, 
\]  

(21b)

\[
 Q_2 = \sqrt{\frac{1}{2} \left( 1 + \frac{M_{11}^{(2)} - M_{22}^{(2)}}{\sqrt{D_2}} \right)}. 
\]  

(21b)
The symbol $R_K$ in (20) reads
$$R_K = \frac{B_{13}^K P_K + B_{25}^K Q_K}{G_K - G_3^{(1)}}.$$ \hspace{1cm} (21c)

As in the case of $qS$ waves, the vectors (20) are generally not unit.

Equations (18) and (20), which hold for arbitrarily chosen vectors $\hat{e}_i^1$ and $\hat{e}_i^2$, are basic equations for the iterative determination of the higher-order approximations of the eigenvalues and eigenvectors of the two $qS$ waves propagating in anisotropic media, see Farra (2001). It is only necessary to find higher-order approximations $M^{(n)}_{ij}(K)$ of the matrix $M_{ij}^K$ and insert them into (18) and (20). From (17), the first-order approximation $M_{ij}^{(1)(K)}$ is obviously $M_{ij}^{(1)(K)} = B_{ij}$, and it is thus the same for both $qS$ waves. The $n$th-order approximation $M_{ij}^{(n)(K)}$ of the matrix $M_{ij}^K$, for $n \geq 2$, is defined by Eq. (17) with $G_K$ substituted by $G_K^{(n-2)}$, see Farra (2001). Let us consider, for example, the $qS1$ wave. Inserting $M_{ij}^{(1)(n)}$ into (18a) yields $n$th-order approximation $G_1^{(n)}$ of the eigenvalue $G_1$. Inserting $M_{ij}^{(1)(n)}$ into (20), specified for $K = 1$, yields $(n-1)$-st-order approximation $g_i^{(n-1)(1)}$ of the polarization vector $g_i^{(1)}$. For the $qS2$ wave, the procedure is analogous. As in the case of $qP$ wave, the odd-order approximations of the eigenvalues (18) are independent of the choice of the reference medium while the even-order approximations depend on it.

Let us again consider the $qS1$ wave, for example, and let us introduce vectors $\hat{e}_i^{(n-1)(1)}$, $\hat{e}_i^{(n-1)(2)}$ chosen in an analogous way to the vectors $\hat{e}_i^{(1)}$, $\hat{e}_i^{(2)}$ in Eq. (8). Specifically, let us choose them so that they define the $n$th-order approximation $M_{ij}^{(n)(1)}$ of the matrix $M_{ij}^{(1)}$, which satisfies
$$\hat{M}_{12}^{(n)(1)} = \hat{M}_{11}^{(n)(1)} = 0, \quad \hat{M}_{11}^{(n)(1)} > \hat{M}_{22}^{(n)(1)}.$$ \hspace{1cm} (22)

We can see that for $n = 1$, Eq. (22) reduces to Eq. (8) since $M_{11}^{(1)(K)} = B_{11}(0)$. This is a consequence of the equation $M_{11}^{(1)(K)} = B_{11}$, which holds for arbitrarily chosen vectors $\hat{e}_i^1$. Using vectors $\hat{e}_i^{(n-1)(1)}$ and $\hat{e}_i^{(n-1)(2)}$ chosen in the described way, we get from (18a) and (19) the expression for the $n$th approximation $G_1^{(n)}$ of the eigenvalue $G_1$: \hspace{1cm} (23)

In the first-order approximation ($n = 1$), Eq. (23) yields the first equation of (9):
$$G_1^{(1)} = \hat{M}_{11}^{(1)(1)} = B_{11}(0).$$ \hspace{1cm} (24)

The direction of the vector $g_i^{(n)(1)}$, which represents the $n$th-order approximation $(n \geq 1)$ of the polarization vector $g_i^{(1)}$ of the considered $qS1$ wave is specified by the vector
$$\hat{e}_i^{(n)(1)} = \frac{\hat{B}_{13}^{(n)}}{G_{1}^{(n-1)}} - G_3^{(1)} e_i^1.$$ \hspace{1cm} (25)

Note again that the vector (25) is not unit. Equation (25) follows from Eq. (20) specified for $K = 1$. For $n = 0$, Eq. (20) specified for $K = 1$ yields the vector $\hat{e}_i^{(0)(1)}$ as the zeroth-order approximation of the vector $g_i^{(1)}$. The symbol $\hat{B}_{13}^{(n)}$ denotes a matrix defined by Eq. (6), in which the vectors $e_i^1$ are sub-

stituted by the vectors $\hat{e}_i^{(n)(1)}$. For $n = 0$, the matrix $\hat{B}_{k_l}^{(n)}$ reduces to $\hat{B}_{13}^{(0)}$ satisfying (8). In the first-order approximation ($n = 1$), Eq. (25) yields the direction of the vector $g_i^{(1)(1)}$:
$$\hat{e}_i^{(1)(1)} = \frac{\hat{B}_{13}^{(1)}}{\beta^2 - \alpha^2} e_i^1.$$ \hspace{1cm} (26)

Note that when using the vectors $\hat{e}_i^{(n)(1)}$, the $n$th-order approximation of the polarization vector $g_i^{(1)}$ has zero projection into the vector $\hat{e}_i^{(1)}$. Since the vector $\hat{e}_i^{(1)}$ used in (26) differs from $\hat{e}_i^{(0)(1)}$, Eq. (26) differs from a similar equation for the first-order perturbation of the $qS1$-wave polarization vector given by Jech and Půmpa (1989).

In an analogous way, we can derive corresponding approximations of the eigenvalue $G_2$ and of the direction of the polarization vector $g_i^{(2)}$ of the $qS2$ wave. In Eq. (22), we must substitute $M_{ij}^{(n)(1)}$ by $M_{ij}^{(n)(2)}$. This defines a new set of vectors $\hat{e}_i^{(n)(1)}$ and $\hat{e}_i^{(n)(2)}$, which are generally different from the set of vectors $\hat{e}_i^{(n-1)(1)}$ and $\hat{e}_i^{(n-1)(2)}$ related to the $qS1$ wave. Then, Eq. (23) becomes $G_2^{(n)} = M_{11}^{(n)(2)}$, and in Eq. (25), $\hat{e}_i^{(n)(1)}$, $\hat{B}_{13}^{(2)}$, and $G_1^{(n-1)}$ must be substituted by $\hat{e}_i^{(n)(2)}$, $\hat{B}_{13}^{(n)(2)}$, and $G_2^{(n-1)}$, respectively.

The vectors $\hat{e}_i^{(n)(K)}$ corresponding to one of the considered $qS$ waves, let us say the $qS1$ wave, can be expressed in terms of arbitrarily chosen, mutually perpendicular unit vectors $e_i^1$ and $e_i^2$ in the following way:
\begin{equation}
\hat{e}_i^{(n)(1)} = e_i^1 \cos \phi(n) + e_i^2 \sin \phi(n),
\end{equation}
\begin{equation}
\hat{e}_i^{(n)(2)} = -e_i^1 \sin \phi(n) + e_i^2 \cos \phi(n).
\end{equation}

The angle $\phi(n)$ can be determined from the equation
\begin{equation}
tan 2 \phi(n) = \frac{2 M_{12}^{(n-1)(1)}}{M_{11}^{(n-1)(1)} - M_{22}^{(n-1)(1)}}.
\end{equation}

Equation (28) follows from the first condition in Eq. (22) if we take into account that the matrix $M_{ij}^{(n-1)(1)}$ satisfies the same relations (10) as the matrix $B_{ij}$. The second condition in (22) guarantees unique determination of the angle $\phi(n)$ from Eq. (28). For $n = 0$, Eq. (28) yields
\begin{equation}
tan 2 \phi(0) = \frac{2 B_{12}}{B_{11} - B_{22}}.
\end{equation}

For the $qS2$ wave, we can proceed in the same way. We obtain, of course, generally different vectors $\hat{e}_i^{(n)(1)}$ and $\hat{e}_i^{(n)(2)}$ and different values of the angles $\phi(n)$. Only in the zeroth-order [see Eq. (29)] and the first-order approximation, $\phi(0)$ and $\phi(1)$ have universal meaning for both $qS$ waves since $M_{ij}^{(1)(1)} = M_{ij}^{(1)(2)}$ and $M_{ij}^{(2)(1)} = M_{ij}^{(2)(2)}$.

Let us mention that the procedure of selection of vectors $\hat{e}_i^{(n)(K)}$ satisfying Eq. (22) can be extended to the limiting case. For the $K$th $qS$ wave we can specify vectors $\hat{e}_i^k$ so that the elements of the matrix $\hat{M}_{ij}^{(K)}$ [the matrix $\hat{M}_{ij}^{(K)}$ is the exact matrix $M_{ij}^{(K)}$, see (17), specified for the vectors $\hat{e}_i^1$] satisfy similar relations to those in Eq. (22):
$$\hat{M}_{12}^{(K)} = 0, \quad \hat{M}_{11}^{(K)} > \hat{M}_{22}^{(K)}.$$ \hspace{1cm} (30)

For such a specification, we get for the $qS1$ wave from (18a) and (19) specified for $K = 1$, 1370 J. Acoust. Soc. Am., Vol. 114, No. 3, September 2003

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\[ G_1 = \hat{M}^{(1)}_{11}, \quad (31) \]
and for the direction of the corresponding exact polarization vector,
\[ \hat{e}_i^j + \frac{\hat{B}_{13}}{G_1 - G_{3}^{(1)}} e_i^3, \quad (32) \]

The matrix \( \hat{B}_{mn} \) is specified by the vectors \( \hat{e}_i^j \). Again, analogous expressions for the eigenvalue \( G_2 \) and for the direction of the polarization vector \( \hat{g}_i^{(2)} \) can be written for the \( qS2 \) wave. Let us again emphasize that the corresponding vectors \( e_i^1 \) and \( e_i^2 \) are generally different from the vectors \( \hat{e}_i^j \) related to the \( qS1 \) wave.

The first-order approximation of the \( qS \)-wave polarization vectors [see (26) for \( qS1 \) wave, for example] depends on the elements \( \hat{B}_{13} \) and \( \hat{B}_{23} \) of the matrix \( \hat{B}_{mn} \) [the elements \( \hat{B}_{13}^{(1)} \) and \( \hat{B}_{23}^{(1)} \) are functions of \( \hat{B}_{13} \) and \( \hat{B}_{23} \) only, see Eq. (10)] and, through the angle \( \phi_i^{(1)} \), which controls specification of the vectors \( e_i^1(\phi_i) \) [see (28) specified for \( n = 1 \)], also on elements \( \hat{B}_{11}, \hat{B}_{12}, \) and \( \hat{B}_{22} \). The first-order approximation of the eigenvalues (24) depends on elements \( B_{11}, B_{12}, \) and \( B_{22} \) [since \( \hat{B}_{11}^{(0)} \) and \( \hat{B}_{22}^{(0)} \) depend on \( B_{11}, B_{12}, \) and \( B_{22} \) only, see Eq. (10)]. From the discussion in Sec. II B, it follows that the first-order approximations of the eigenvalues (representing squares of the phase velocities in the first-order approximations) and of the polarization vectors of \( qS \) waves depend on the complete set of 21 WA parameters. Six individual WA parameters and remaining 15 in nine combinations can be determined from the phase velocity. Fifteen individual WA parameters and remaining six in five combinations can be determined from the polarization vectors. The polarization vectors are, in addition, also controlled by the difference of squared velocities of the reference isotropic medium: \( \alpha^2 - \beta^2 \).

### III. ACOUSTICAL AXES

Characteristic feature of any anisotropic medium are so-called acoustical axes. According to Schoenberg and Helbig (1997) the acoustical axes specify two kinds of directions: longitudinal and singular.

#### A. Longitudinal directions

A longitudinal direction is a direction in which the \( qP \) wave is purely longitudinal and the \( qS \) waves are purely transverse. The polarization vector \( \hat{g}_i^{(3)} \) of the \( qP \) wave is parallel to the corresponding wave normal \( n_1 \); the polarization vectors \( \hat{g}_i^{(K)} \) of the \( qS \) waves are perpendicular to \( n_1 \). By taking this into account in Eqs. (11) and in the expression for \( R_K \) in (21c), we can see that in the longitudinal direction
\[ \hat{B}_{13}^{(1)} = \hat{B}_{23}^{(0)} = B_{13} = B_{23} = 0. \quad (33) \]
We can see that Eq. (33) holds for arbitrary choice of vectors \( \hat{e}_i^j \). From Eqs. (12) or (13), we can see that if Eq. (33) holds, also the first-order approximation of the \( qP \)-wave polarization vector is parallel to the wave vector \( n_1 \). From Eq. (26), we can see that the first-order approximation of the \( qS1 \)-wave polarization vector is perpendicular to the wave vector \( n_1 \) if Eq. (33) holds. Thus the longitudinal directions determined from the first-order perturbation formulas are identical with exact longitudinal directions.

Let us mention that the condition (33) is automatically satisfied in isotropic media. This means that every direction in an isotropic medium is a longitudinal direction.

From Eq. (14) we can derive another interesting property of the first-order perturbation of the \( qP \)-wave eigenvalue. For \( G_3 > G_1^{(1)} \) and \( G_3 > G_2^{(1)} \), Eq. (14) indicates that \( G_3 = G_3^{(1)} \), i.e., the \( qP \)-wave first-order formula yields a value which is less than or at most equal to the exact eigenvalue. An alternative proof of this inequality can be found in the Appendix of Pšencík and Gajewski (1998). We can see from Eq. (14) that the equality \( G_3 = G_3^{(1)} \) occurs when (33) holds, i.e., in longitudinal directions.

A similar property as above holds also for the \( qS \) waves. The proof is only slightly more complicated. Let us consider vectors \( \hat{e}_i^{(0)J} \) chosen so that conditions (8) are satisfied and let us rewrite Eqs. (18) for this specification. We get
\[ G_1 = \frac{1}{2} \left( \hat{B}_{11}^{(0)} + \hat{B}_{22}^{(0)} + \frac{(\hat{B}_{13}^{(0)})^2 + (\hat{B}_{23}^{(0)})^2}{G_1 - G_3^{(1)}} + \sqrt{D_1} \right), \quad (34) \]
\[ G_2 = \frac{1}{2} \left( \hat{B}_{11}^{(0)} + \hat{B}_{22}^{(0)} + \frac{(\hat{B}_{13}^{(0)})^2 + (\hat{B}_{23}^{(0)})^2}{G_2 - G_3^{(1)}} - \sqrt{D_2} \right). \]
Let us also rewrite Eq. (19) for the same specification:
\[ D_K = \left( \hat{B}_{11}^{(0)} - \hat{B}_{22}^{(0)} \right) + \frac{(\hat{B}_{13}^{(0)})^2 + (\hat{B}_{23}^{(0)})^2}{G_1 - G_3^{(1)}} \leq D_K. \quad (36a) \]
\[ D_K = \left( \hat{B}_{11}^{(0)} - \hat{B}_{22}^{(0)} \right) - \frac{(\hat{B}_{13}^{(0)})^2 + (\hat{B}_{23}^{(0)})^2}{G_2 - G_3^{(1)}} \leq \sqrt{D_K}. \quad (36b) \]
The inequalities in (36) follow from
\[ -\left[ (\hat{B}_{13}^{(0)})^2 + (\hat{B}_{23}^{(0)})^2 \right] \leq (\hat{B}_{13}^{(0)})^2 - (\hat{B}_{23}^{(0)})^2 \]
\[ \leq (\hat{B}_{13}^{(0)})^2 + (\hat{B}_{23}^{(0)})^2 \]
and from \( G_3 \leq G_3^{(1)} \). Inserting (36b) to the expression for \( G_1 \) in (34) and (36a) to \( G_2 \), we get two important results:
\[ G_1 \leq \hat{B}_{11}^{(0)} = G_{1}^{(1)}, \quad G_2 \leq \hat{B}_{22}^{(0)} = G_{2}^{(1)}. \quad (38) \]
We can see that the first-order formulas yield values which are greater than or at least equal to exact eigenvalues of the \( qS \) waves. As in the case of \( qP \) wave, we can see from Eqs. (17)–(19) that the equality \( G_1 = G_1^{(1)} \) and \( G_2 = G_2^{(1)} \) occurs when (33) holds, i.e., in longitudinal directions.

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Let us mention that the equality of the first-order approximation of only one of the $qS$-wave eigenvalues to its exact counterpart does not imply a longitudinal direction. The $SH$ wave in a TI medium can be taken as an example.

Equations (33) can be thus used for the determination of longitudinal directions if the WA parameters are known or, vice versa, for the determination (or constraint) of the WA parameters if the longitudinal directions are known from observations. In the $(x,z)$ plane of an orthorhombic medium, the equation $\vec{B}_{13}=0$ yields two “bound” directions (Helbig, 1993) along coordinate axes $x$ and $z$ and an equation

$$2\varepsilon_{x,n_1^2} - 2\varepsilon_{z,n_1^2} + \delta_2(n_1^2 - n_z^2) = 0.$$

(39)

Equation $\vec{B}_{23}=0$ is satisfied automatically. For $\tan^2 \theta = n_z^2/n_x^2$, where $\theta$ is measured from the $z$-axis, and under one of the conditions

$$\frac{1}{2}\delta_2 < \min(\varepsilon_x, \varepsilon_z), \quad \frac{1}{2}\delta_2 > \max(\varepsilon_x, \varepsilon_z),$$

(40)

Eq. (39) yields

$$\tan^2 \theta = \frac{2\varepsilon_x - \delta_2}{2\varepsilon_z - \delta_2}.$$

(41)

Equation (41) and conditions (40) are equivalent to those given for the same situation by Schoenberg and Helbig (1997). In a similar way we get for the $(x,y)$ plane of an orthorhombic medium from Eqs. (33) two “bound” directions along coordinate axes $x$ and $y$ and the direction

$$\tan^2 \theta = \frac{2\varepsilon_x - \delta_2}{2\varepsilon_y - \delta_2}.$$

(42)

Here $\theta$ is measured from the $y$ axis. The longitudinal directions can exist if one of the conditions

$$\frac{1}{2}\delta_2 < \min(\varepsilon_x, \varepsilon_y), \quad \frac{1}{2}\delta_2 > \max(\varepsilon_x, \varepsilon_y),$$

(43)

is satisfied.

B. Singular directions

The second type of acoustical axes is connected with singular directions. Singular directions exist only for $qS$ waves and occur when the phase velocities (or the eigenvalues $G_{ij}$ of the two $qS$ waves are equal. This is the so-called degenerate situation in perturbation analysis. In such a situation, it is not possible to specify uniquely the polarization vectors of the $qS$ waves. It is only possible to find the plane in which the polarization vectors are situated, i.e., the plane perpendicular to the third polarization vector. This fact reflects the indeterminacy of corresponding formulas for the polarization vectors. The expression (17) is independent of the type of the considered wave in the singular direction because $G_1 = G_2$. Inspection of Eqs. (18) reveals that the equality $G_1 = G_2$ occurs if $D_1 = D_2 = 0$. Because $D_K$ is given, see (19), by a sum of two quadratic terms, each of these terms must be zero. This yields

$$M_{11}^{(K)} = M_{22}^{(K)}, \quad M_{12}^{(K)} = 0.$$

(44)

Equation (44) holds simultaneously for $K = 1$ and $K = 2$, and for an arbitrary choice of the vectors $\vec{e}_i^{(K)}$. In the first-order approximation, the singular direction is specified by

$$B_{11} = B_{22}, \quad B_{12} = 0.$$

(45)

In the $(x,z)$ plane of an orthorhombic medium, the equation $B_{12} = 0$ is automatically satisfied and equation $B_{11} = B_{22}$ has an explicit form

$$\alpha^2(\varepsilon_x - \delta_x + \varepsilon_z)\cos^4 \theta + \beta^2(\varepsilon_x - \gamma_x) = 0.$$

(46)

In the plane $(y,z)$, the equation $B_{11} = B_{22}$ has the form

$$\alpha^2(\varepsilon_y - \delta_y + \varepsilon_z)\cos^4 \theta + \beta^2(\varepsilon_y - \gamma_y) = 0.$$

(47)

If a singular direction differs from a longitudinal direction, the equality $G_1 = G_2$ does not imply generally $G_1^{(1)} = G_2^{(1)}$. This means that the singular direction in the first-order approximation, specified by (45), differs generally from the actual singular direction specified by (44). This also holds for approximations of higher orders. In the limit, the singular direction should converge to the actual one. If Eqs. (44) and (45) are satisfied for the same direction, this automatically implies that that direction is longitudinal. Let us mention that the conditions (44) and (45) are automatically satisfied in isotropic media. This means that every direction in an isotropic medium is a singular direction.

IV. NUMERICAL EXAMPLES

In applications, the most frequently used formulas are based on the first-order approximation. In the following, we shall compare results of the first- and second-order perturbation formulas for $qS$ waves with exact results for several types of anisotropic media. This will enable us to estimate accuracy of the first-order formulas in a vicinity of singular directions. Let us emphasize that the results of this section were obtained by using Eqs. (18) and (20). In order to calculate the $n$-th-order approximation of the square of the phase velocity from Eq. (18), the $n$-th-order approximation of the matrix $M_{ij}^{(K)}$ was used. The velocity was calculated as a square root from the corresponding eigenvalue. For the determination of the $n$-th-order approximation of polarization vectors from (20), the $(n+1)$-th-order approximation of $M_{ij}^{(K)}$ was used.

For illustration of effects of a singularity on the first-order approximation formulas, let us consider a simple transversely isotropic medium with vertical axis of symmetry. We use the model of thin water-filled cracks of Shearer and Chapman (1989). The model is specified by the density-normalized elastic parameters $A_{ij}$, in ($km/s$)$^2$, with values $A_{11} = 20.16, A_{33} = 19.63, A_{12} = 7.40, A_{13} = A_{23} = 7.26, A_{44} = A_{55} = 3.48$, and $A_{66} = 6.38$. Anisotropy of this model is rather strong, about 29% and 30% for $qS$ waves. Due to the axial symmetry of the medium, it is sufficient to investigate just a quadrant of a vertical plane containing the axis of symmetry. Figure 1 shows variations of phase velocities of the two $qS$ waves in such a section. The angle $\theta$ specifies the...
direction of the wave vector, $\theta=0^\circ$ corresponds to the direction along the axis of symmetry, and $\theta=90^\circ$ corresponds to the direction perpendicular to it. The two distinct curves correspond to the $SH$ and $qSV$ waves. Polarization of the $SH$ wave is perpendicular to the plane of propagation and thus to the wave vector (therefore we omit the letter $q$ in front of the $SH$); the $qSV$ wave is polarized in the plane of propagation, its polarization is generally not perpendicular to the wave vector. The first-order phase velocity formula for the $SH$ wave yields exact values of the phase velocity (because the zeroth-order approximation of the $SH$-wave polarization vector coincides with its exact counterpart). For the $qSV$ wave the values generally differ, the approximate values being always greater than or equal to the exact ones (see the analytical proof above). Results of the approximate formula coincide with exact for $\theta=0^\circ$ (kiss singularity), for approximately $45^\circ$ (longitudinal direction) and for $90^\circ$ (another longitudinal direction). For the goals of this paper, the most interesting is the intersection of the $SH$- and $qSV$-wave phase velocities at $58.59^\circ$, which corresponds to an intersection singularity. We can see that the singularity estimated by the first-order formula is shifted from the direction of the true singularity to the direction $\theta=59.7^\circ$ determined from (46). The first-order perturbation formula thus yields distorted result. It is of interest that there is no shift in case of the kiss singularity. The kiss singularity direction coincides with the longitudinal direction, which is determined exactly by the first-order formulas.

Let us now consider a more complicated situation, an orthorhombic model used by Farra (2001). The model is specified by the density-normalized elastic parameters $A_{ij}$, in $(\text{km/s})^2$, with values $A_{11}=10.8$, $A_{22}=11.3$, $A_{33}=8.5$, $A_{12}=2.2$, $A_{13}=1.9$, $A_{23}=1.7$, $A_{44}=3.6$, $A_{55}=3.9$, and $A_{66}=4.3$. Squares of the $P$- and $S$-wave velocities of the reference isotropic medium are chosen 10.04 and 4.01 $(\text{km/s})^2$, respectively. Figure 2 shows exact (top), first-order (middle), and second-order (bottom) maps of relative differences (in $\%$) of $qS1$ and $qS2$ phase velocities as functions of the polar angle $\theta$ and the azimuth $\varphi$ specifying the wave vector.

FIG. 1. $qSV$- and $SH$-wave phase velocity sections for the VTI medium specified in the text. Comparison of exact and first-order values.

FIG. 2. Exact, first-order, and second-order maps of relative differences of $qS1$- and $qS2$-wave phase velocities as functions of azimuth $\phi$ and polar angle $\theta$ for an orthorhombic medium of Farra (2001).
The relative differences are calculated as differences of the
$qS$-wave velocities normalized by their average. There is a
point singularity at $\varphi=0^\circ$ and $\theta=34.4^\circ$ and the region of
small differences extends to larger values of $\varphi$ in the map of
effect differences. Another region of small differences can be
observed on the right-hand side of the exact map for $\varphi=90^\circ$
and $\theta=35^\circ$. Let us now compare the exact map with its first-
order approximation in the middle frame. We can see several
interesting phenomena. The most significant is the shift of
the region of minimum differences to higher values of $\theta$
from $34.4^\circ$ to $38.91^\circ$ following from (46). Another important
feature is the appearence of two minima seemingly indicat-
ing existence of singularities for $\varphi=90^\circ$, $\theta=33^\circ$, and $\theta=45^\circ$.
The bottom of the valley crossing the plots for $\theta=30^\circ$ to $40^\circ$ is
deeper in the first-order map. For relative differences of $qS1$
and $qS2$ phase velocities greater than approximately $3\%$, the
differences between the top and middle frames are minimum.
From inspection of the frame of the second-order approxi-
mation, we can see that it gives effectively the same results
as the exact formula. Slight exception is the region of $\varphi=90^\circ$
and $\theta=35^\circ$.

In the following, we show effects of the first-order per-
trurbations on the model of orthorhombic medium proposed
and studied in detail by Schoenberg and Helbig (1997). The
model has been obtained by combining finely layered (VTI)
model with vertical fractures. The model is specified by the
density-normalized elastic parameters $A_{ij}$, in (km/s)$^2$, with
values $A_{11}=9.00$, $A_{22}=9.84$, $A_{33}=5.94$, $A_{12}=3.60$, $A_{13}$
$=2.25$, $A_{23}=2.40$, $A_{44}=2.00$, $A_{55}=1.60$, and $A_{66}=2.18$.
Squares of the $P$- and $S$-wave velocities of the reference
isotropic medium are chosen as 7.60 and 2.26 (km/s)$^2$, re-
spectively. Figure 3 shows again exact (top), first-order (middle), and second-order (bottom) maps of relative differ-
ences of $qS1$ and $qS2$ phase velocities. This model has four
singularities in the shown map: two singularities (conical
points) for $\varphi=0^\circ$ and $\theta=20.1^\circ$ and $\theta=59.8^\circ$, one conical
point for $\varphi=90^\circ$ and $\theta=72.5^\circ$, and another one for
$\varphi=44.89^\circ$ and $\theta=46.53^\circ$ (see Schoenberg and Helbig, 1997).
Comparison of the exact and first-order maps yields several
interesting phenomena. We can see that, as in Fig. 2, the
singularity $\varphi=0^\circ$, $\theta=59.8^\circ$ is shifted within the plane of
symmetry $\varphi=0^\circ$ to $\theta=66.23^\circ$ following from (46). The other
singularity in the same plane, $\varphi=0^\circ$, $\theta=20.1^\circ$, remains, how-
ever, at nearly the same position, specifically at $\theta=20.09^\circ$.
This is a result of the closeness of the longitudinal direction
specified by $\varphi=0^\circ$ and $\theta=20.3^\circ$. As mentioned above, the
first-order formulas yield exact results for the longitudinal
directions. Thus the differences between the first-order ap-
proximations of phase velocities of the $qS$ waves in a vicin-
ity of a longitudinal direction are also close to exact ones.
The singularity $\varphi=90^\circ$, $\theta=72.5^\circ$ is shifted in the plane
$\varphi=90^\circ$ to $\theta=76.74^\circ$ following from (47). Interesting is the
shift of the point singularity $\varphi=44.89^\circ$, $\theta=46.53^\circ$ situated
off the symmetry planes. The shift of the singularity in the
first-order approximation is in both directions. It is about $2^\circ$
in $\varphi$ and about $4^\circ$ in the polar angle $\theta$. As in Fig. 2, the
results of the second-order formula practically coincide with
exact results. The only exception is the singularity $\varphi=90^\circ$,
$\theta=72.5^\circ$, which remains shifted by about $2^\circ$. The use of the

![FIG. 3. Exact, first-order, and second-order maps of relative differences of $qS1$- and $qS2$-wave phase velocities as functions of azimuth $\phi$ and polar angle $\theta$ for an orthorhombic medium of Schoenberg and Helbig (1997).](image)
third-order formula would shift the singularity to the proper position.

The effect of singularities on polarization vectors is much more pronounced. In singular directions the formulas for polarization vectors are indetermined. In their vicinity, polarization vectors change fast. This can be seen in Fig. 4. Figure 4 shows deviations of the exact polarization vector of

FIG. 4. Deviations (in degrees) of the exact q51-wave polarization vector from its zeroth-, first- and second-order approximations as functions of azimuth $\phi$ and polar angle $\theta$ for an orthorhombic medium of Schoenberg and Helbig (1997).

FIG. 5. The same as in Fig. 4 but for the azimuths and polar angles in the interval $30^\circ$–$60^\circ$. 
V. CONCLUSIONS

A detailed analysis of higher-order formulas of Farra (2001) for the eigenvalues and directions of the polarization vectors of $qP$ and $qS$ waves in weakly anisotropic media was performed. Possible ways of constructing separate higher-order perturbation for $qS1$ and $qS2$ waves were shown. The formulas depend on various elements of the matrix $B_{mn}$, which, in turn, depend linearly on the WA or elastic parameters of a medium. Thus the sensitivity analysis of attributes of elastic waves to the parameters of the medium reduces to the sensitivity analysis of the elements of the matrix $B_{mn}$.

Analysis of the first-order perturbation formulas for phase velocities and polarization vectors yielded the following results. From inversion of $qP$-wave phase velocity, 15 independent WA parameters can be found. If information about $qP$-wave polarization and/or orientation of longitudinal axes is available, the number of determinable independent WA parameters increases to 15. In order to determine all 21 WA parameters, observations of both $qP$- and $qS$-wave phase velocities are necessary. From observations of $qP$- and $qS$-wave polarizations alone, only nine individual WA parameters can be found. Additional use of the acoustical axes, specifically of the singular directions, adds six further independent WA parameters. Using (A3), a similar analysis can be made for elastic parameters.

Another goal of this contribution was to show the effect of the first-order perturbation formulas on the determination of acoustical axes. For this purpose, explicit analytical conditions for the determination of the acoustical axis were given. We have found, both analytically and in numerical examples, that the approximations do not affect positions of the longitudinal directions. They may, however, affect quite significantly positions of singularities. The effects can be removed by using higher-order approximations.

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APPENDIX: EXPRESSIONS FOR ELEMENTS OF THE MATRIX $B_{mn}$

We introduce three mutually perpendicular unit vectors $\bar{e}_1^1$, $\bar{e}_2^2$, and $\bar{e}_3^3$ so that $\bar{e}_3^3 = n_i$, where $n_i$ is a unit wave vector. The vectors $\bar{e}_1^1$ and $\bar{e}_2^2$ are situated in the plane perpendicular to $\bar{e}_3^3$ and they are selected so that

$$\bar{e}_1^1 = D^{-1}(n_1 n_3, n_2 n_3, n_3^2 - 1),$$

$$\bar{e}_2^2 = D^{-1}(-n_2, n_1, 0), \quad \bar{e}_3^3 = n_i = (n_1, n_2, n_3),$$

where

$$D = (n_1^2 + n_2^2)^{1/2}, \quad n_1^2 + n_2^2 + n_3^2 = 1.$$  

In the following, a reference isotropic medium with the $P$-wave velocity $\alpha$ and $S$-wave velocity $\beta$ is considered. We specify the symmetric matrix $B_{mn}$ given in Eq. (6) for vectors (A1) and denote it $\bar{B}_{mn}$. The matrix $\bar{B}_{mn}$ can be expressed in terms of 21 WA parameters:
\[ \begin{align*}
\varepsilon_{11} &= \frac{A_{11} - \alpha^2}{2\alpha^2}, \quad \varepsilon_{12} = \frac{A_{12} - \alpha^2}{2\alpha^2}, \quad \varepsilon_{13} = \frac{A_{13} - \alpha^2}{2\alpha^2}, \\
\varepsilon_{22} &= \frac{A_{22} - \alpha^2}{2\alpha^2}, \quad \varepsilon_{23} = \frac{A_{23} - \alpha^2}{2\alpha^2}, \quad \varepsilon_{33} = \frac{A_{33} - \alpha^2}{2\alpha^2}, \\
\delta_{11} &= \frac{A_{11} + 2A_{55} - \alpha^2}{\alpha^2}, \quad \delta_{12} = \frac{A_{12} + 2A_{44} - \alpha^2}{\alpha^2}, \\
\delta_{13} &= \frac{A_{13} + 2A_{55} - \alpha^2}{\alpha^2}, \quad \delta_{22} = \frac{A_{22} + 2A_{44} - \alpha^2}{\alpha^2}, \\
\delta_{23} &= \frac{A_{23} + 2A_{55} - \alpha^2}{\alpha^2}, \quad \delta_{33} = \frac{A_{33} + 2A_{55} - \alpha^2}{\alpha^2}, \\
\chi_1 &= \frac{A_{14} + 2A_{56}}{\alpha^2}, \quad \chi_2 = \frac{A_{25} + 2A_{46}}{\alpha^2}, \quad \chi_3 = \frac{A_{36} + 2A_{45}}{\alpha^2}, \\
\chi_4 &= \frac{A_{15}}{\alpha^2}, \quad \chi_5 = \frac{A_{16}}{\alpha^2}, \quad \chi_6 = \frac{A_{24}}{\alpha^2}, \\
\chi_7 &= \frac{A_{26}}{\alpha^2}, \quad \chi_8 = \frac{A_{34}}{\alpha^2}, \quad \chi_9 = \frac{A_{35}}{\alpha^2},
\end{align*} \]

Although the WA parameters depend on velocities \( \alpha \) and \( \beta \) of the reference isotropic medium, the matrix \( \tilde{B}_{\alpha\alpha} \) is independent of these velocities. Its elements have the following form:


