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Weak-contrast reflection–transmission coefficients in a generally anisotropic background

Luděk Klimeš*

ABSTRACT

Explicit equations for approximate linearized reflection–transmission coefficients at a generally oriented weak-contrast interface separating two generally and independently anisotropic media are presented. The equations are derived also for all singular directions and are thus valid in degenerate cases (e.g., in an isotropic background). The equations are expressed in general Cartesian coordinates, with arbitrary orientation of the interface. The explicit equations for linearized reflection–transmission coefficients have a very simple form—much simpler than the equations published previously. The equations for all reflection–transmission coefficients, with the exception of the unconverted transmitted wave, have a common form. The form of the equations is very suitable for inversion and for analyzing the sensitivity of seismic data to discontinuities in individual elastic moduli. The factors of proportionality of the contrasts of elastic moduli and density are expressed in terms of the slowness and polarization vectors of the corresponding generated wave and incident wave.

INTRODUCTION

Approximate linearized reflection–transmission coefficients for weak-contrast interfaces are very useful in the stratigraphic analysis and structural inversion of seismic data, especially in amplitude versus offset (AVO) and amplitude versus azimuth (AVA) analyses. The linearized weak-contrast reflection–transmission coefficients are especially relevant for analyzing the sensitivity of seismic data to discontinuities in individual elastic moduli.

The linearized reflection–transmission coefficients for a weak-contrast interface between two isotropic media have been known for a long time (Bortfeld, 1961; Richards and Frasier, 1976; Aki and Richards, 1980).

Banik (1987), Thomsen (1993), Rueger (1996, 1997), and Haugen and Ursin (1996) assume a weak-contrast interface between two weakly anisotropic media of transversely isotropic (TI) symmetry. They obtain the formulas for the linearized reflection–transmission coefficients for several particular orientations of the crystal axes with respect to the interface and plane of incidence.

Vavryčuk and Pšenčík (1998), Zillmer et al. (1998), and Pšenčík and Vavryčuk (1998) consider general weakly anisotropic media, i.e., a generally anisotropic perturbation to an isotropic background. They assume a horizontal weak-contrast interface, represented by a weak-contrast interface coinciding with a coordinate plane in Cartesian coordinates. Vavryčuk and Pšenčík (1998) derive the linearized P–P reflection coefficient; Zillmer et al. (1998) develop the equations for the linearized P–P, SV–SV, and SH–SH reflection coefficients; and Pšenčík and Vavryčuk (1998) obtain the expressions for both the linearized P–P reflection and P–P transmission coefficients.

Vavryčuk (1999) also assumes general weakly anisotropic media but considers a weak-contrast interface generally oriented with respect to the coordinate system. He finds the relations for the linearized reflection and transmission coefficients corresponding to an incident P-wave, including the P–S converted waves.

Ursin and Haugen (1996) abandon the assumption of weak anisotropy and allow for a transversely isotropic background with a vertical axis of symmetry. They assume a horizontal weak-contrast interface (coinciding with a coordinate plane) separating two transversely isotropic media with a vertical (perpendicular to the interface) axis of symmetry and find the explicit expressions for the corresponding linearized reflection–transmission coefficients.

Unlike these authors, we allow for a generally anisotropic background and consider a weak-contrast interface generally oriented with respect to the Cartesian coordinates. We derive explicit equations for the linearized reflection–transmission coefficients, including all converted waves, at a generally oriented weak-contrast interface separating two generally and

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*Charles University, Department of Geophysics, Ke Karlovu 3, 121 16 Prague 2, Czech Republic. E-mail: klimes@seis.karlov.mff.cuni.cz.

independently anisotropic media. In other words, the weak contrasts in 21 elastic moduli may be of general anisotropy, fully independent of the generally anisotropic background. The resulting equations for the weak-contrast linearized reflection–transmission coefficients are also derived for all singular directions. The equations are thus also valid in degenerate cases (e.g., in an isotropic background). The equations are expressed in general Cartesian coordinates, with arbitrary orientation of the interface. The resulting explicit equations for linearized reflection–transmission coefficients have a very simple form—much simpler than the equations published by the above-mentioned authors. The resulting equations for all reflection–transmission coefficients, with the exception of the unconverted transmitted wave, have a common form. The form of the equations is very suitable for inversion and for analyzing the sensitivity of seismic data to discontinuities in the individual elastic moduli. The factors of proportionality of the contrasts of elastic moduli and density are expressed in terms of the slowness and polarization vectors of the corresponding generated wave and incident wave. The expressions for the linearized weak-contrast reflection–transmission coefficients, derived in this paper, are independent of the background and obey the same reciprocity relations as the exact reflection–transmission coefficients.

Let us emphasize that the linearized weak-contrast reflection–transmission coefficients, like other linearizations and perturbations, are designed primarily for inversion, during which the exact reflection–transmission coefficients cannot be applied. The linearized weak-contrast reflection–transmission coefficients should not replace the exact reflection–transmission coefficients in forward modeling in any case.

The numerical example at the end of our paper, comparing the weak-contrast reflection–transmission coefficients with the exact, is not included to demonstrate the accuracy or inaccuracy of the expressions derived for the linearized weak-contrast reflection–transmission coefficients and applied to a finite-contrast interface. The accuracy cannot be checked by a few numerical examples. This is the task of a thorough analytical investigation. Rather, the numerical example serves as an illustration and a simple test that can reveal gross mistypes in the final equations for the linearized weak-contrast reflection–transmission coefficients.

A reader interested in application of the linearized weak-contrast reflection–transmission coefficients rather than in their derivation may proceed directly to the final equation (71).

SPECIFICATION OF GIVEN QUANTITIES AND REFLECTION–TRANSMISSION COEFFICIENTS

Denote the elastic moduli (components of the stiffness tensor) and density at the incident side of a plane interface by

$$c_{ijkl}, \quad \varrho \tag{1}$$

and the elastic moduli and density at the other side of the interface by

$$c_{ijkl} + \Delta c_{ijkl}, \quad \varrho + \Delta\varrho, \tag{2}$$

where Δc_{ijkl} and $\Delta\varrho$ are assumed to be small quantities (strictly speaking, infinitesimally small quantities). Denote the slowness vector and unit polarization vector of the incident plane wave and the normal to the interface by

$$P_i, \quad E_i, \quad n_i. \tag{3}$$

Denote the slowness vectors and the unit polarization vectors of the reflected waves and the corresponding displacement reflection coefficients by

$$p_i^\alpha, \quad e_i^\alpha, \quad R^\alpha, \quad \alpha = 1, 2, 3. \tag{4}$$

Denote the slowness vectors and the unit polarization vectors of the transmitted waves and the corresponding displacement transmission coefficients by

$$p_i^\alpha, \quad e_i^\alpha, \quad R^\alpha, \quad \alpha = 4, 5, 6, \tag{5}$$

where index $\alpha = 6$ corresponds to the unconverted wave, also called the nontransformed wave (Červený and Ravindra, 1971). The slowness vectors p_i^α of the six generated waves are determined by the slowness vector P_i of the incident wave through Snell's law. Unit polarization vectors E_i and e_i^α are determined by slowness vectors P_i and p_i^α through the corresponding Christoffel equations.

We speak about a degenerate incident wave if E_i is a linear combination of the eigenvectors corresponding to two or three identical or almost identical eigenvalues of the Christoffel matrix $\Gamma_{ik} = c_{ijkl} P_j P_l$ of the incident wave. Here we refer to the eigenvalues as almost identical if their difference is small, i.e., on the order of Δc_{ijkl} and $\Delta\varrho$. For a degenerate incident wave, we choose polarization vector e_i^6 in such a way that the difference between vectors E_i and e_i^6 is small (on the order of Δc_{ijkl} and $\Delta\varrho$).

We do not present the derivation in the degenerate case of three identical or almost identical eigenvalues of the Christoffel matrix of the incident wave. This case is completely improbable for seismic waves because of the elastic properties of rocks, while the derivation is analogous to the degenerate case of two almost identical eigenvalues.

We express the reflection–transmission coefficients in terms of new coefficients ΔR^α, $\alpha = 1, 2, 3, 4, 5, 6$, as

$$R^1 = \Delta R^1, \quad R^2 = \Delta R^2, \quad R^3 = \Delta R^3, \quad R^4 = -\Delta R^4, \quad R^5 = -\Delta R^5, \quad R^6 = 1 - \Delta R^6. \tag{6}$$

All coefficients ΔR^α, $\alpha = 1, 2, 3, 4, 5, 6$, are small quantities on the order of Δc_{ijkl} and $\Delta\varrho$:

$$\Delta R^\alpha = O(\Delta c_{ijkl}, \Delta\varrho). \tag{7}$$

If the slowness vectors are real valued, the projections of the ray-velocity vectors of the incident wave and transmitted waves onto the normal to the interface have the same sign, whereas the projections of the ray-velocity vectors of the reflected waves onto the surface normal have the opposite sign. Then, for $\alpha = 1, 2, 3, 4, 5$, $p_i^\alpha \neq P_i$, P_i and p_i^α real valued, we can also express definitions (6) in the unified form

$$\begin{aligned} R^\alpha &= -\Delta R^\alpha \operatorname{sgn}\left(V_i n_i v_j^\alpha n_j\right) \\ &= \Delta R^\alpha \operatorname{sgn}\left[V_i\left(p_i^\alpha - P_i\right) v_j^\alpha \left(P_j - p_j^\alpha\right)\right], \end{aligned} \tag{8}$$

where

$$V_j = \varrho^{-1} c_{ijkl} E_i E_k P_l, \quad v_j^\alpha = (\varrho^\alpha)^{-1} c_{ijkl}^\alpha e_i^\alpha e_k^\alpha p_l^\alpha, \tag{9}$$

with

$$c^1_{ijkl} = c^2_{ijkl} = c^3_{ijkl} = c_{ijkl}, \qquad \varrho^1 = \varrho^2 = \varrho^3 = \varrho,$$
$$c^4_{ijkl} = c^5_{ijkl} = c^6_{ijkl} = c_{ijkl} + \Delta c_{ijkl}, \quad \varrho^4 = \varrho^5 = \varrho^6 = \varrho + \Delta\varrho, \tag{10}$$

are the ray-velocity (group-velocity) vectors. The difference in the slowness vectors is oriented in the direction of the normal to the interface because the tangential component of the slowness vector is preserved according to Snell's law. That is why we may use the difference $p^\alpha_i - P_i$ in place of the normal n_i in equation (8).

Note that the Einstein summation convention is applied to repeated lowercase subscripts throughout this paper. On the other hand, no summation is implied by repeated Greek superscripts indexing the generated waves.

EQUATIONS FOR WEAK-CONTRAST DISPLACEMENT REFLECTION–TRANSMISSION COEFFICIENTS

Snell's law

Snell's law for the generated waves can be expressed as

$$p^\alpha_i = P_i + n_i a^\alpha, \quad \alpha = 1, 2, 3, 4, 5, 6. \tag{11}$$

The normal slowness vector discontinuities a^α can be calculated as the relevant solutions of the sixth-order equations obtained by inserting Snell's law [equation (11)] into characteristic equations (29) and (30) for the Christoffel matrix. Normal slowness vector discontinuities a^α for $\alpha = 1, 2, 3, 4, 5$ are finite quantities for a nondegenerate incident wave, whereas a^6 is a small quantity:

$$a^6 = O(\Delta c_{ijkl}, \Delta\varrho). \tag{12}$$

For a degenerate incident wave, we assume that $\alpha = 5$ corresponds to the other degenerate polarization, approximately perpendicular to the incident polarization:

$$E_i e^5_i = O(\Delta c_{ijkl}, \Delta\varrho). \tag{13}$$

In this case of a degenerate incident wave, a^5 is another small quantity:

$$a^5 = O(\Delta c_{ijkl}, \Delta\varrho). \tag{14}$$

Weakly anisotropic elastic moduli c_{ijkl}, differing from the isotropic elastic moduli only by small quantities $O(\Delta c_{ijkl}, \Delta\varrho)$, represent one important example of the degeneracy.

Continuity of displacement

The continuity of displacement across the interface is expressed by the equation

$$E_i + \sum_{\alpha=1}^{3} e^\alpha_i R^\alpha - \sum_{\alpha=4}^{6} e^\alpha_i R^\alpha = 0. \tag{15}$$

Inserting expressions (6) for the reflection–transmission coefficients into equation (15), the small discontinuity

$$\Delta e^6_i = e^6_i - E_i \tag{16}$$

in the unit polarization vector of the unconverted wave can be expressed in the form

$$\Delta e^6_i = \sum_{\alpha=1}^{6} e^\alpha_i \Delta R^\alpha. \tag{17}$$

In summary, we have introduced ten small quantities ΔR^α, $\alpha = 1, 2, 3, 4, 5, 6$, a^6 and Δe^6_i, $i = 1, 2, 3$, on the order of Δc_{ijkl} and $\Delta\varrho$ for a nondegenerate incident wave. For a degenerate incident wave, a^5 is the eleventh small quantity, while ΔR^5 becomes a small free parameter parametrizing the projection of Δe^6_i onto the second degenerate polarization vector e^5_i, which will become clear during the derivation. Since the polarization vectors have unit lengths,

$$E_i \Delta e^6_i = O^2(\Delta c_{ijkl}, \Delta\varrho). \tag{18}$$

The symbol $O^2(\Delta c_{ijkl}, \Delta\varrho)$ stands for the quantities of the order of $(\Delta c_{ijkl})^2$ and $(\Delta\varrho)^2$. Multiplying equation (17) by E_i and considering equation (18), we can express coefficient ΔR^6 in terms of five other coefficients:

$$\Delta R^6 = -\sum_{\alpha=1}^{5} E_i e^\alpha_i \Delta R^\alpha + O^2(\Delta c_{ijkl}, \Delta\varrho). \tag{19}$$

Continuity of traction

The continuity of traction across the interface is expressed by the equation

$$c_{ijkl} n_j E_k P_l + \sum_{\alpha=1}^{3} c_{ijkl} n_j e^\alpha_k p^\alpha_l R^\alpha - \sum_{\alpha=4}^{6} (c_{ijkl} + \Delta c_{ijkl}) n_j e^\alpha_k p^\alpha_l R^\alpha = 0. \tag{20}$$

Inserting expressions (6) for the transmission coefficients, equation (20) reads

$$\sum_{\alpha=1}^{3} c_{ijkl} n_j e^\alpha_k p^\alpha_l \Delta R^\alpha + \sum_{\alpha=4}^{6} (c_{ijkl} + \Delta c_{ijkl}) n_j e^\alpha_k p^\alpha_l \Delta R^\alpha + c_{ijkl} n_j E_k P_l - (c_{ijkl} + \Delta c_{ijkl}) n_j e^6_k p^6_l = 0. \tag{21}$$

Keeping just the first-order small quantities and using Snell's law for p^6_i results in

$$\sum_{\alpha=1}^{6} c_{ijkl} n_j e^\alpha_k p^\alpha_l \Delta R^\alpha - c_{ijkl} n_j \left(\Delta e^6_k P_l + E_k n_l a^6\right) - \Delta c_{ijkl} n_j E_k P_l = O^2(\Delta c_{ijkl}, \Delta\varrho). \tag{22}$$

Inserting equation (17) for Δe^6_k, we arrive at

$$\sum_{\alpha=1}^{6} c_{ijkl} n_j e^\alpha_k \left(p^\alpha_l - P_l\right) \Delta R^\alpha - c_{ijkl} n_j E_k n_l a^6 - \Delta c_{ijkl} n_j E_k P_l = O^2(\Delta c_{ijkl}, \Delta\varrho). \tag{23}$$

Inserting Snell's law [equation (11)] for p^α_l and remembering that a^6 is a small quantity, equation (23) becomes

$$\sum_{\alpha=1}^{5} A_{ik} e^\alpha_k a^\alpha \Delta R^\alpha - A_{ik} E_k a^6 - \Delta c_{ijkl} n_j E_k P_l = O^2(\Delta c_{ijkl}, \Delta\varrho), \tag{24}$$

where we introduce

$$A_{ik} = c_{ijkl} n_j n_l. \tag{25}$$

For a degenerate incident wave, both a^5 and ΔR^5 are small quantities, and the summation in equation (24) is performed only over α from 1 to $\underline{4}$ (the last summation index is underlined to highlight the difference between a nondegenerate and a degenerate incident wave).

Christoffel equations

The Christoffel equations for the incident wave and six generated waves are

$$c_{ijkl} P_j E_k P_l - \varrho E_i = 0, \tag{26}$$

$$c_{ijkl} p_j^\alpha e_k^\alpha p_l^\alpha - \varrho e_i^\alpha = 0, \quad \alpha = 1, 2, 3, \tag{27}$$

and

$$(c_{ijkl} + \Delta c_{ijkl}) p_j^\alpha e_k^\alpha p_l^\alpha - (\varrho + \Delta\varrho) e_i^\alpha = 0, \quad \alpha = 4, 5, 6. \tag{28}$$

The characteristic equations for the Christoffel matrix, following from the Christoffel equations (27) and (28), are

$$\det\left[c_{ijkl} p_j^\alpha p_l^\alpha - \varrho\delta_{ik}\right] = 0, \quad \alpha = 1, 2, 3, \tag{29}$$

and

$$\det\left[(c_{ijkl} + \Delta c_{ijkl}) p_j^\alpha p_l^\alpha - (\varrho + \Delta\varrho)\delta_{ik}\right] = 0, \quad \alpha = 4, 5, 6. \tag{30}$$

Equations (29) and (30) with Snell's law [equation (11)] represent the sixth-order equations determining the slowness vectors p_i^α of the six generated waves.

Subtracting the Christoffel equation for the incident wave from the Christoffel equation for the unconverted wave and leaving only the first-order small quantities, we obtain

$$\begin{aligned} c_{ijkl}\left(n_j a^6 E_k P_l + P_j \Delta e_k^6 P_l + P_j E_k n_l a^6\right) - \varrho \Delta e_i^6 \\ + \Delta c_{ijkl} P_j E_k P_l - \Delta\varrho E_i = O^2(\Delta c_{ijkl}, \Delta\varrho), \end{aligned} \tag{31}$$

which may be expressed as

$$\begin{aligned} (\Gamma_{ik} - \varrho\delta_{ik})\Delta e_k^6 + B_{ik} E_k a^6 + \Delta c_{ijkl} P_j E_k P_l - \Delta\varrho E_i \\ = O^2(\Delta c_{ijkl}, \Delta\varrho), \end{aligned} \tag{32}$$

where

$$\Gamma_{ik} = c_{ijkl} P_j P_l \tag{33}$$

is the Christoffel matrix of the incident wave and

$$B_{ik} = c_{ijkl}(P_j n_l + n_j P_l). \tag{34}$$

Inserting equation (17) for Δe_k^6, equation (32) becomes

$$\begin{aligned} \sum_{\alpha=1}^{6} (\Gamma_{ik} - \varrho\delta_{ik}) e_k^\alpha \Delta R^\alpha + B_{ik} E_k a^6 + \Delta c_{ijkl} P_j E_k P_l \\ - \Delta\varrho E_i = O^2(\Delta c_{ijkl}, \Delta\varrho). \end{aligned} \tag{35}$$

Since $(\Gamma_{ik} - \varrho\delta_{ik}) e_k^6 = (\Gamma_{ik} - \varrho\delta_{ik})\Delta e_k^6$ is small,

$$\begin{aligned} \sum_{\alpha=1}^{\underline{5}} (\Gamma_{ik} - \varrho\delta_{ik}) e_k^\alpha \Delta R^\alpha + B_{ik} E_k a^6 + \Delta c_{ijkl} P_j E_k P_l \\ - \Delta\varrho E_i = O^2(\Delta c_{ijkl}, \Delta\varrho). \end{aligned} \tag{36}$$

For a degenerate incident wave, $(\Gamma_{ik} - \varrho\delta_{ik}) e_k^5$ is also small, and the summation in equation (36) is performed only over α from 1 to $\underline{4}$.

For a nondegenerate incident wave, equations (24) and (36) form a linear system of equations for six small quantities ΔR^α, $\alpha = 1, 2, 3, 4, \underline{5}$, and a^6.

For a degenerate incident wave, ΔR^5 is a small free parameter and equations (24) and (36) form a linear system of six equations for five small quantities ΔR^α, $\alpha = 1, 2, 3, \underline{4}$, and a^6. The redundant equation corresponds to the secular equation for E_i.

Approximate Christoffel equations

Leaving only finite quantities, Christoffel equations (27) and (28) for the generated waves may be approximated by

$$c_{ijkl} p_j^\alpha e_k^\alpha p_l^\alpha - \varrho e_k^\alpha = O(\Delta c_{ijkl}, \Delta\varrho), \quad \alpha = 1, 2, 3, 4, 5, 6. \tag{37}$$

Expressing the approximate Christoffel matrices

$$\Gamma_{ik}^\alpha = c_{ijkl} p_j^\alpha p_l^\alpha = c_{ijkl}\left(P_j + n_j a^\alpha\right)\left(P_l + n_l a^\alpha\right) \tag{38}$$

of the generated waves in terms of matrices (25), (33), and (34),

$$\Gamma_{ik}^\alpha = \Gamma_{ik} + B_{ik} a^\alpha + A_{ik} a^\alpha a^\alpha, \tag{39}$$

the approximate Christoffel equations (37) read

$$\left(\Gamma_{ik} - \varrho\delta_{ik} + B_{ik} a^\alpha + A_{ik} a^\alpha a^\alpha\right) e_k^\alpha = O(\Delta c_{ijkl}, \Delta\varrho). \tag{40}$$

SOLUTION OF THE LINEAR SYSTEM OF EQUATIONS

Multiplying equation (24) by $e_i^\alpha a^\alpha$ and subtracting equation (36) multiplied by e_i^α, we arrive at the system

$$\begin{aligned} \sum_{\beta=1}^{\underline{5}} e_i^\alpha \left(A_{ik} a^\alpha a^\beta - \Gamma_{ik} + \varrho\delta_{ik}\right) e_k^\beta \Delta R^\beta \\ - e_i^\alpha \left(A_{ik} a^\alpha + B_{ik}\right) E_k a^6 - e_i^\alpha \Delta c_{ijkl}\left(P_j + n_j a^\alpha\right) E_k P_l \\ + e_i^\alpha \Delta\varrho E_i = O^2(\Delta c_{ijkl}, \Delta\varrho) \end{aligned} \tag{41}$$

of six equations, with $\alpha = 1, 2, 3, 4, 5, 6$.

For a nondegenerate incident wave and $\alpha = 6$, or for a degenerate incident wave and $\alpha = 5, 6$, equation (41) reads

$$\begin{aligned} -e_i^\alpha B_{ik} E_k a^6 - e_i^\alpha \Delta c_{ijkl} P_j E_k P_l + e_i^\alpha \Delta\varrho E_i \\ = O^2(\Delta c_{ijkl}, \Delta\varrho). \end{aligned} \tag{42}$$

For $\alpha = 6$, equation (42) can be used to calculate a^6:

$$E_i B_{ik} E_k a^6 = \Delta\varrho - \Delta c_{ijkl} E_i P_j E_k P_l + O^2(\Delta c_{ijkl}, \Delta\varrho). \tag{43}$$

For a degenerate incident wave, equation (42) for $\alpha = 5$ represents the secular equation for E_i:

$$e_i^5\left(B_{ik} a^6 + \Delta c_{ijkl}\, p_j^5 P_l\right) E_k = O^2(\Delta c_{ijkl}, \Delta\varrho), \tag{44}$$

where e_i^5 is approximately perpendicular to E_i [see relation (13)]. The approximate Christoffel equations (40) yield

$$e_i^{\alpha}\big(A_{ik}a^{\alpha}+B_{ik}\big)E_k a^{\alpha} = e_i^{\alpha}(\varrho\delta_{ik}-\Gamma_{ik})E_k+O(\Delta c_{ijkl},\,\Delta\varrho), \tag{45}$$

where $(\varrho\delta_{ik}-\Gamma_{ik})E_k$ is identical to zero as a result of Christoffel equation (26):

$$e_i^{\alpha}\big(A_{ik}a^{\alpha}+B_{ik}\big)E_k a^{\alpha} = O(\Delta c_{ijkl},\,\Delta\varrho). \tag{46}$$

Note that the right-hand side of equation (46) is identical to zero for reflected waves, i.e., for $\alpha=1,2,3$ [Fryer and Frazer, 1984, their equation(3.20)]. For a nondegenerate incident wave and $\alpha=1,2,3,4,\underline{5}$, or for a degenerate incident wave and $\alpha=1,2,3,\underline{4}$, equation (46) yields

$$e_i^{\alpha}(A_{ik}a^{\alpha}+B_{ik})E_k = O(\Delta c_{ijkl},\,\Delta\varrho). \tag{47}$$

Considering equations (11) and (47), the system of equations (41) for coefficients ΔR^{α} reads

$$\sum_{\beta=1}^{\underline{5}} e_i^{\alpha}\big(A_{ik}a^{\alpha}a^{\beta}-\Gamma_{ik}+\varrho\delta_{ik}\big)e_k^{\beta}\Delta R^{\beta} = \Delta c_{ijkl}\, e_i^{\alpha}\, p_j^{\alpha} E_k P_l - \Delta\varrho e_i^{\alpha} E_i + O^2(\Delta c_{ijkl},\,\Delta\varrho), \tag{48}$$

where $\alpha=1,2,3,4,\underline{5}$ for a nondegenerate incident wave. For a degenerate incident wave (small a^5), $\alpha=1,2,3,\underline{4}$ and the summation in equation (48) is performed only over β from 1 to $\underline{4}$.

We now demonstrate that the system of equations (48) for coefficients ΔR^{α} is approximately decoupled, i.e.,

$$e_i^{\alpha}\big(A_{ik}a^{\alpha}a^{\beta}-\Gamma_{ik}+\varrho\delta_{ik}\big)e_k^{\beta} = O(\Delta c_{ijkl},\,\Delta\varrho) \tag{49}$$

for $\alpha\neq\beta$.

Decoupling for sufficiently different slowness vectors of generated waves

Subtracting equation (40) for e_i^{β} multiplied by e_i^{α} and equation (40) for e_i^{α} multiplied by e_i^{β}, we obtain the approximate identity

$$e_i^{\alpha}\big[B_{ik}(a^{\beta}-a^{\alpha})+A_{ik}(a^{\beta}a^{\beta}-a^{\alpha}a^{\alpha})\big]e_k^{\beta} = O(\Delta c_{ijkl},\,\Delta\varrho). \tag{50}$$

For $a^{\beta}-a^{\alpha}$ being a finite (i.e., not small) quantity, equation (50) may be divided by $a^{\beta}-a^{\alpha}$. We then obtain

$$e_i^{\alpha}\big[B_{ik}+A_{ik}(a^{\beta}+a^{\alpha})\big]e_k^{\beta} = O(\Delta c_{ijkl},\,\Delta\varrho). \tag{51}$$

We now multiply equation (51) by a^{β} and obtain

$$e_i^{\alpha}\big[B_{ik}a^{\beta}+A_{ik}(a^{\beta}+a^{\alpha})a^{\beta}\big]e_k^{\beta} = O(\Delta c_{ijkl},\,\Delta\varrho). \tag{52}$$

Subtracting equation (40) for e_i^{β} multiplied by e_i^{α} from equation (52), we arrive at equation (49). Equation (49) is thus derived for finite $a^{\beta}-a^{\alpha}$.

Decoupling for almost identical slowness vectors of generated waves

We now derive equation (49) for zero or small $a^{\beta}-a^{\alpha}$, i.e., for

$$a^{\beta}-a^{\alpha} = O(\Delta c_{ijkl},\,\Delta\varrho). \tag{53}$$

In this case, we can write

$$e_i^{\alpha}\big[A_{ik}a^{\alpha}a^{\beta}-\Gamma_{ik}+\varrho\delta_{ik}\big]e_k^{\beta} = e_i^{\alpha}\big[A_{ik}a^{\alpha}a^{\alpha}-\Gamma_{ik}+\varrho\delta_{ik}\big]e_k^{\beta} + O(\Delta c_{ijkl},\,\Delta\varrho), \tag{54}$$

where

$$e_i^{\alpha}\big[A_{ik}a^{\alpha}a^{\alpha}-\Gamma_{ik}+\varrho\delta_{ik}\big]e_k^{\beta} = e_i^{\alpha}\big\{c_{ijkl}\big[\big(p_j^{\alpha}-P_j\big)\big(p_l^{\alpha}-P_l\big)-P_jP_l\big]+\varrho\delta_{ik}\big\}e_k^{\beta}. \tag{55}$$

Using the approximate Christoffel equations (37), we obtain

$$e_i^{\alpha}\big\{c_{ijkl}\big[\big(p_j^{\alpha}-P_j\big)\big(p_l^{\alpha}-P_l\big)-P_jP_l\big]+\varrho\delta_{ik}\big\}e_k^{\beta} = c_{ijkl}\big(p_j^{\alpha}-P_j\big)p_l^{\alpha}\big(e_i^{\beta}e_k^{\alpha}+e_i^{\alpha}e_k^{\beta}\big) + O(\Delta c_{ijkl},\,\Delta\varrho). \tag{56}$$

For zero or small $a^{\beta}-a^{\alpha}$, equations (54)–(56) yield

$$e_i^{\alpha}\big[A_{ik}a^{\alpha}a^{\beta}-\Gamma_{ik}+\varrho\delta_{ik}\big]e_k^{\beta} = c_{ijkl}\big(p_j^{\alpha}-P_j\big)p_l^{\alpha} \times\big(e_i^{\beta}e_k^{\alpha}+e_i^{\alpha}e_k^{\beta}\big)+O(\Delta c_{ijkl},\,\Delta\varrho). \tag{57}$$

Considering the equation

$$p_i^{\beta}-p_i^{\alpha} = O(\Delta c_{ijkl},\,\Delta\varrho), \tag{58}$$

which follows from equation (53), the approximate Christoffel matrix (37) for e_i^{β} can be approximated by

$$c_{ijkl}\,p_j^{\alpha}e_k^{\beta}p_l^{\alpha}-\varrho e_k^{\beta} = O(\Delta c_{ijkl},\,\Delta\varrho). \tag{59}$$

For $\alpha\neq\beta$, e_i^{α} and e_i^{β} are thus approximately two mutually perpendicular eigenvectors of the same Christoffel equation (59):

$$e_i^{\alpha}e_i^{\beta} = O(\Delta c_{ijkl},\,\Delta\varrho). \tag{60}$$

Then

$$c_{ijkl}e_i^{\beta}p_j^{\alpha}e_k^{\alpha}p_l^{\alpha} = O(\Delta c_{ijkl},\,\Delta\varrho) \tag{61}$$

for $\alpha\neq\beta$.

At an intersection singularity (curvilinear intersection of slowness sheets, also called a line singularity), the polarization vectors e_k^{α} and e_k^{β} are continuous and differentiable with respect to p_j^{α} along the individual slowness sheets. The derivative of equation (61) with respect to p_j^{α} is

$$c_{ijkl}\big(e_i^{\beta}e_k^{\alpha}+e_i^{\alpha}e_k^{\beta}\big)p_l^{\alpha}+c_{imkl}\,p_m^{\alpha}p_l^{\alpha}\left(\frac{\partial e_i^{\beta}}{\partial p_j}e_k^{\alpha}+\frac{\partial e_i^{\alpha}}{\partial p_j}e_k^{\beta}\right) = O(\Delta c_{ijkl},\,\Delta\varrho) \tag{62}$$

for $\alpha\neq\beta$. The approximate Christoffel equations (37) and (59) yield

$$c_{imkl}\,p_m^{\alpha}p_l^{\alpha}\left(\frac{\partial e_i^{\beta}}{\partial p_j}e_k^{\alpha}+\frac{\partial e_i^{\alpha}}{\partial p_j}e_k^{\beta}\right) = \varrho\left(\frac{\partial e_i^{\beta}}{\partial p_j}e_i^{\alpha}+\frac{\partial e_i^{\alpha}}{\partial p_j}e_i^{\beta}\right) + O(\Delta c_{ijkl},\,\Delta\varrho). \tag{63}$$

Equations (60), (62), and (63) yield

$$c_{ijkl}\big(e_i^{\beta}e_k^{\alpha}+e_i^{\alpha}e_k^{\beta}\big)p_l^{\alpha} = O(\Delta c_{ijkl},\,\Delta\varrho) \tag{64}$$

for $\alpha\neq\beta$. Equations (57) and (64) yield equation (49) for the intersection singularity. Note that the equation

$$c_{ijkl}^{\alpha}\big(e_i^{\beta}e_k^{\alpha}+e_i^{\alpha}e_k^{\beta}\big)p_l^{\alpha} = 0, \tag{65}$$

derived similarly as the approximate equation (64) but from the exact Christoffel matrix, is suitable for defining the polarization vectors at an intersection singularity.

At a wedge singularity (a wedge intersection of slowness sheets at a single point), the polarization vectors may conveniently be defined by equation (65). Equation (49) for the wedge singularity then follows directly from equations (57) and (65).

At a conical singularity (a conical intersection of slowness sheets at a single point, also called a point singularity), we define the polarization vectors for finite $p_j^\alpha - P_j$ as the eigenvectors e_i^α and e_i^β of the Christoffel equation corresponding to p_j^α, satisfying the additional condition

$$c_{ijkl}^\alpha \big(e_i^\beta e_k^\alpha + e_i^\alpha e_k^\beta\big) P_j p_l^\alpha = 0. \tag{66}$$

Equations (57) and (66) yield equation (49) for a conical singularity.

At an isotropic singularity (identical slowness sheets) or a kiss singularity (touching slowness sheets), we may choose any couple of mutually perpendicular polarization vectors e_i^α and e_i^β. Equation (64) then follows directly from equation (60). Equations (57) and (64) then yield equation (49) for touching slowness sheets.

Thus, for an appropriate choice of polarization vectors e_i^α and e_i^β at the contact of two slowness sheets (discussed earlier in this section), equation (49) also holds for zero or small $a^\beta - a^\alpha$, $\alpha \neq \beta$.

A contact of three slowness sheets is completely improbable for seismic waves because of the elastic properties of rocks. Thus, we only briefly outline the selection of polarization vectors and derivation of equation (49) at the corresponding singularities. Equation (65) applies to touching or intersecting slowness sheets. If the polarization vector is continuous along a slowness sheet intersecting a conical singularity, equation (65) relates the continuous polarization vector to the polarization vectors corresponding to the conical singularity, while the two perpendicular polarization vectors corresponding to the conical singularity are selected according to equation (66). If the polarization vectors are discontinuous along all three slowness sheets at a conical singularity, the three perpendicular polarization vectors are determined using equation (66).

Decoupled linear equations

We have demonstrated that equation (49) is valid for all $\alpha \neq \beta$. Equations (48) for coefficients ΔR^α are thus decoupled:

$$\begin{aligned} e_i^\alpha \big(A_{ik} a^\alpha a^\alpha - \Gamma_{ik} + \varrho\delta_{ik}\big) e_k^\alpha \Delta R^\alpha &= \Delta c_{ijkl} e_i^\alpha p_j^\alpha E_k P_l \\ &\quad - \Delta\varrho e_i^\alpha E_i + O^2(\Delta c_{ijkl}, \Delta\varrho), \end{aligned} \tag{67}$$

where $\alpha = 1, 2, 3, 4, \underline{5}$ for a nondegenerate incident wave and $\alpha = 1, 2, 3, \underline{4}$ for a degenerate incident wave.

The factor at ΔR^α in equation (67) may be converted into a simpler form. For $\alpha = \beta$, equation (57) reads

$$\begin{aligned} e_i^\alpha \big(A_{ik} a^\alpha a^\alpha - \Gamma_{ik} + \varrho\delta_{ik}\big) e_k^\alpha &= 2 c_{ijkl} e_i^\alpha \big(p_j^\alpha - P_j\big) e_k^\alpha p_l^\alpha \\ &\quad + O(\Delta c_{ijkl}, \Delta\varrho). \end{aligned} \tag{68}$$

Equation (68) can be expressed as

$$\begin{aligned} e_i^\alpha \big(A_{ik} a^\alpha a^\beta - \Gamma_{ik} + \varrho\delta_{ik}\big) e_k^\alpha &= 2\varrho^\alpha v_j^\alpha \big(p_j^\alpha - P_j\big) \\ &\quad + O(\Delta c_{ijkl}, \Delta\varrho), \end{aligned} \tag{69}$$

where the ray-velocity vector v_j^α of the generated wave is defined by equation (9). Factor (69) is finite where $\alpha = 1, 2, 3, 4, \underline{5}$ for a nondegenerate incident wave and $\alpha = 1, 2, 3, \underline{4}$ for a degenerate incident wave. Then

$$\Delta R^\alpha = \frac{\Delta\varrho e_i^\alpha E_i - \Delta c_{ijkl} e_i^\alpha p_j^\alpha E_k P_l}{2\varrho^\alpha v_i^\alpha \big(P_i - p_i^\alpha\big)} + O^2(\Delta c_{ijkl}, \Delta\varrho), \tag{70}$$

which can be converted using equations (6) into the weak-contrast reflection–transmission coefficients R^α. Since we have not yet made any assumption about the real-valued slowness vector, equation (70) applies even to evanescent generated waves.

If the generated wave is a homogeneous plane wave with a real-valued slowness vector, equation (70) can be converted into the weak-contrast reflection–transmission coefficient R^α using equation (8):

$$\begin{aligned} R^\alpha &= \frac{\Delta\varrho e_i^\alpha E_i - \Delta c_{ijkl} e_i^\alpha p_j^\alpha E_k P_l}{2\varrho^\alpha \big|v_i^\alpha \big(P_i - p_i^\alpha\big)\big|} \operatorname{sgn}\big[V_i\big(p_i^\alpha - P_i\big)\big] \\ &\quad + O^2(\Delta c_{ijkl}, \Delta\varrho), \end{aligned} \tag{71}$$

where $\alpha = 1, 2, 3, 4, \underline{5}$ for a nondegenerate incident wave and $\alpha = 1, 2, 3, \underline{4}$ for a degenerate incident wave. The quantities on the right-hand side of equation (71) are defined by equations (1)–(5), (9), and (10). The form of equation (71) is especially suitable for inversion, during which slowness vector P_i and polarization vector E_i of the incident wave and slowness vector p_i^α and polarization vector e_i^α of the generated wave are given rather than the normal to the interface.

Although equations (70) and (71) have been derived for incident polarization E_i satisfying secular equation (44) for a degenerate incident wave, equations (70) and (71) are linear in E_i and are thus valid for general incident polarization E_i. We thus ignore secular equation (44) because it might contradict definition (65) of the polarization vectors at an intersection singularity or definition (66) of the polarization vectors at a conical singularity of the slowness surface at p_i^α. Note that the sign factor of $\operatorname{sgn}[V_i(p_i^\alpha - P_i)]$ is present in equation (71) because the sign of $\Delta\varrho$ and Δc_{ijkl} is defined with respect to the direction V_i of incidence and not with respect to the discontinuity $p_i^\alpha - P_i$ in the slowness vector.

The linearized displacement weak-contrast reflection–transmission coefficients (71) are independent of the background and obey the same reciprocity relations as the exact displacement reflection–transmission coefficients.

The weak-contrast displacement transmission coefficient of the unconverted wave is given by

$$R^6 = 1 + \sum_{\alpha=1}^{\underline{5}} E_i e_i^\alpha \Delta R^\alpha + O^2(\Delta c_{ijkl}, \Delta\varrho), \tag{72}$$

which follows from equation (19). For a degenerate incident wave, both $E_i e_i^5$ and ΔR^5 are small quantities, and the summation in equation (72) is performed only over α from 1 to $\underline{4}$.

RECIPROCAL NORMALIZED WEAK-CONTRAST REFLECTION–TRANSMISSION COEFFICIENTS

The reciprocal normalized displacement reflection–transmission coefficients $\bar{R}^\alpha$ are defined by the relation

[Červený, 2001, his equation (5.4.13)]

$$\bar{R}^{\alpha} = R^{\alpha}\sqrt{\frac{|\varrho^{\alpha} v_i^{\alpha} n_i|}{|\varrho V_j n_j|}}, \tag{73}$$

where the ray-velocity vectors and ϱ^{α} are given by equations (9) and (10).

Equations (70) and (71), specified for normalized coefficients $\bar{R}^{\alpha}$, read

$$\Delta\bar{R}^{\alpha} = \frac{\Delta\varrho e_i^{\alpha} E_i - \Delta c_{ijkl} e_i^{\alpha} p_j^{\alpha} E_k P_l}{2\sqrt{\left|\varrho^{\alpha} v_i^{\alpha}\left(P_i - p_i^{\alpha}\right)\varrho V_j\left(p_j^{\alpha} - P_j\right)\right|}} \times \frac{1}{\operatorname{sgn}\left[v_i^{\alpha}\left(P_i - p_i^{\alpha}\right)\right]} + O^2(\Delta c_{ijkl}, \Delta\varrho) \tag{74}$$

and

$$\bar{R}^{\alpha} = \frac{\Delta\varrho e_i^{\alpha} E_i - \Delta c_{ijkl} e_i^{\alpha} p_j^{\alpha} E_k P_l}{2\sqrt{\left|\varrho^{\alpha} v_i^{\alpha}\left(P_i - p_i^{\alpha}\right)\varrho V_j\left(p_j^{\alpha} - P_j\right)\right|}} \times \operatorname{sgn}\left[V_i\left(p_i^{\alpha} - P_i\right)\right] + O^2(\Delta c_{ijkl}, \Delta\varrho), \tag{75}$$

where $\alpha = 1, 2, 3, 4, \underline{5}$ for a nondegenerate incident wave, and $\alpha = 1, 2, 3, \underline{4}$ for a degenerate incident wave. Equation (74), with $\operatorname{sgn}(x) = x/|x|$, also applies to evanescent generated waves, whereas equation (75) applies only to homogeneous generated plane waves with real-valued slowness vectors. Note that $\Delta\varrho \operatorname{sgn}[V_i(p_i^{\alpha} - P_i)]$ and $\Delta c_{ijkl} \operatorname{sgn}[V_i(p_i^{\alpha} - P_i)]$ are reciprocal, unlike $\Delta\varrho$ and Δc_{ijkl}, because the sign of $\Delta\varrho$ and Δc_{ijkl} is defined with respect to V_i.

The reciprocal normalized weak-contrast transmission coefficient $\bar{R}^6$ of the unconverted wave is approximately unity [Chapman, 1994, his equation (27); Vavryčuk, 1999, his equation (15)]:

$$\bar{R}^6 = 1 + O^2(\Delta c_{ijkl}, \Delta\varrho). \tag{76}$$

For a degenerate incident wave, the reciprocal normalized weak-contrast transmission coefficient $\bar{R}^5$, corresponding to the second degenerate polarization vector e_i^5, is a small free parameter related to the small parameter R^5 through equation (73).

The linearized normalized weak-contrast reflection–transmission coefficients (75) and (76) are independent of the background and obey the same reciprocity relations as the exact normalized reflection–transmission coefficients.

Definition (73) and equation (76) enable the weak-contrast displacement transmission coefficient of the unconverted wave to be expressed as

$$R^6 = \sqrt{\frac{\left|\varrho V_j\left(p_j^6 - P_j\right)\right|}{\left|\varrho^6 v_i^6\left(P_i - p_i^6\right)\right|}} + O^2(\Delta c_{ijkl}, \Delta\varrho). \tag{77}$$

Unlike the more complex expression (72), equation (77) does not explicitly express the approximately linear dependence of the weak-contrast displacement transmission coefficient of the unconverted wave on the contrasts Δc_{ijkl} and $\Delta\varrho$.

NUMERICAL EXAMPLE

We consider the model already used for analogous numerical illustrations by Vavryčuk and Pšenčík (1998), Pšenčík and Vavryčuk (1998), and Vavryčuk (1999), who call it model A/C. For a discussion of model A/C, refer to their papers. Model A/C is composed of homogeneous elastic half-spaces A and C, separated by an interface parallel with the x_1x_2 coordinate plane. We consider a P-wave incident from half-space A, as in the papers mentioned.

Half-space A is isotropic, with the squared velocities $v_P^2 = 16$ (km/s)^2, $v_S^2 = 16/3$ (km/s)^2, and density $\varrho = 2.65$ g/cm^3.

Half-space C is anisotropic, with the density-normalized elastic moduli $(c_{ijkl} + \Delta c_{ijkl})/(\varrho + \Delta\varrho)$ in (km/s)^2 given by the matrix

$$\begin{array}{c} \\ 11 \\ 22 \\ 33 \\ 23 \\ 13 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccccc} 11 & 22 & 33 & 23 & 13 & 12 \end{array} \\ \begin{pmatrix} 11.957 & 3.986 & 3.986 & 0.000 & 0.000 & 0.000 \\ & 15.551 & 4.884 & 0.000 & 0.000 & 0.000 \\ & & 15.551 & 0.000 & 0.000 & 0.000 \\ & & & 5.333 & 0.000 & 0.000 \\ & & & & 4.758 & 0.000 \\ & & & & & 4.758 \end{pmatrix} \end{array} \tag{78}$$

and with density $\varrho + \Delta\varrho = 2.60$ g/cm^3. Since both materials A and C are symmetric with respect to the x_1x_2 coordinate plane, we need not specify the orientation of the x_3 coordinate axis.

Since half-space A is isotropic, the slowness vector of the incident P-wave is situated in the plane of incidence, which is perpendicular to the interface. The azimuth of the plane of incidence is measured from the x_1x_3 coordinate plane. Because of the symmetry of the model with respect to the coordinate planes, the azimuth from 0° to 90° covers all reflection–transmission coefficients. We calculate the P–P displacement reflection coefficient for the reflected P-wave (Vavryčuk and Pšenčík, 1998; Pšenčík and Vavryčuk, 1998) and the P–SV and P–SH displacement reflection coefficients for the reflected SV- and SH-waves (Vavryčuk, 1999). The reflected SV-waves are polarized in the plane of incidence, whereas the SH-waves are polarized perpendicularly to the plane of incidence. The exact and linearized displacement reflection coefficients are calculated for the angles of incidence from 0° to 40°, as by Vavryčuk (1999).

The exact displacement reflection coefficients have been calculated simultaneously by RMATRIX software (Thomson, 1998a, b) and ANRAY software (Gajewski and Pšenčík, 1990; Pšenčík, 1998). The linearized weak-contrast displacement reflection coefficients calculated according to equation (71) are compared with the exact coefficients in Figure 1. Remember that no background medium appears in equation (71). The relative error is defined as

$$\varrho R = \frac{|R_{\text{linearized}} - R_{\text{exact}}|}{|R_{\text{exact}}|}. \tag{79}$$

Since exact reflection coefficients R_{exact} are roughly proportional to the contrasts in elastic moduli and density, while the absolute errors $|R_{\text{linearized}} - R_{\text{exact}}|$ are roughly homogeneous functions of the second degree with respect to the contrasts, the relative errors are roughly homogeneous functions of the first degree with respect to the contrasts.

Because of the degeneracy of the S-waves in isotropic half-space A, the polarization vectors of the reflected S-waves may be chosen arbitrarily in the plane defined by the SV and SH polarizations. If the polarization vectors are chosen so that one of the exact P–S reflection coefficients is very small, the corresponding relative error (79) becomes very large. That is why the individual relative errors of the P–SV and P–SH reflection coefficients have no physical meaning and why the relative error of the P–SH reflection coefficient is considerably greater than the relative error of the P–SV reflection coefficient. In this case, it is reasonable to assess the relative accuracy of both the

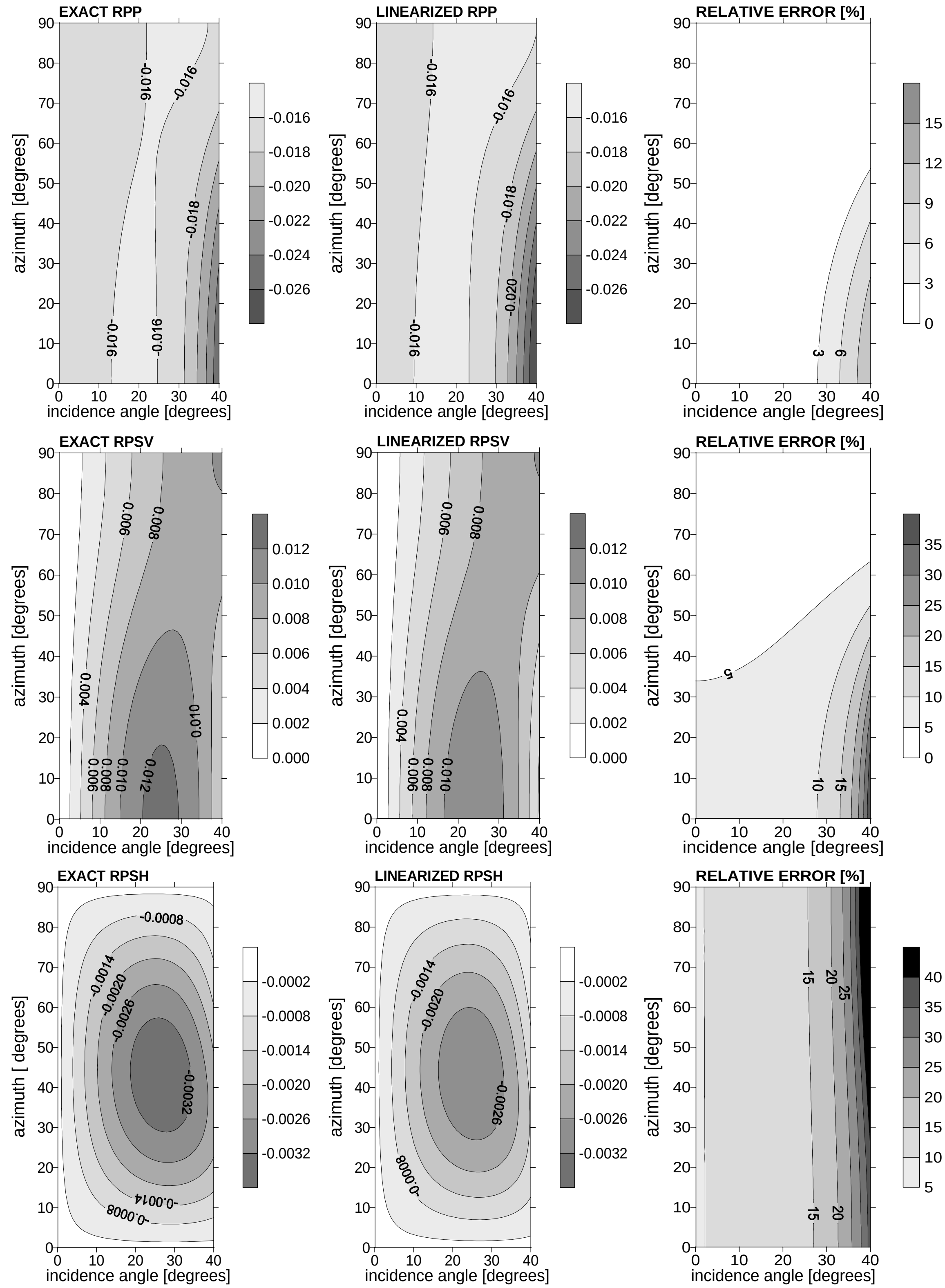

FIG. 1. Comparison of the exact and linearized displacement reflection coefficients corresponding to the incident P-wave in model A/C. The linearized weak-contrast reflection coefficients are calculated according to equation (71).

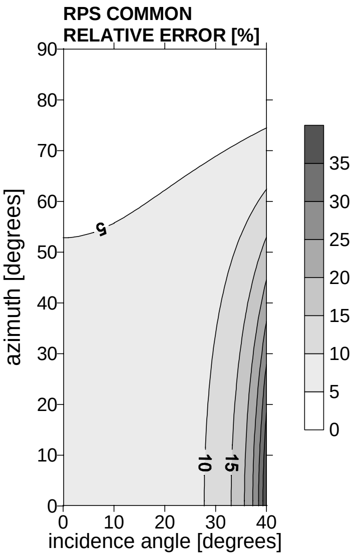

FIG. 2. Common relative error (80) of the P–SV and P–SH linearized displacement reflection coefficients in model A/C.

P–S reflection coefficients by means of their common relative error:

$$\varrho R_{\mathrm{S}} = \frac{\sqrt{|R_{\mathrm{SV\,linearized}} - R_{\mathrm{SV\,exact}}|^2 + |R_{\mathrm{SH\,linearized}} - R_{\mathrm{SH\,exact}}|^2}}{\sqrt{|R_{\mathrm{SV\,exact}}|^2 + |R_{\mathrm{SH\,exact}}|^2}}. \tag{80}$$

This common relative error is independent of the selection of the polarization vectors of the reflected S-waves. The common relative error (80) of the P–S reflection coefficients is displayed in Figure 2.

We have selected this model with a finite-contrast interface for numerical illustration because the results of calculations analogous to ours have already been published (Vavryčuk and Pšenčík, 1998; Pšenčík and Vavryčuk, 1998; Vavryčuk, 1999). We do not intend to demonstrate the applicability or inapplicability of the expressions derived for the linearized weak-contrast reflection–transmission coefficients to a finite–contrast interface. A thorough analytical investigation of the higher order perturbations of the reflection–transmission coefficients is necessary for estimating the regions of applicability of the linearized weak-contrast reflection–transmission coefficients.

CONCLUSIONS

The linearized weak-contrast displacement reflection–transmission coefficients are given by equations (71) and (72). The equations hold for general anisotropic media, including all singular directions or an isotropic background. The equations are expressed in general Cartesian coordinates for an arbitrary orientation of the interface. Equation (71) for linearized reflection–transmission coefficients has a very simple form—much simpler than the equations published previously by the authors mentioned in the Introduction, including the equations for weakly anisotropic or isotropic media. The form of equation (71) is especially suitable for inversion because, in inversion, the slowness vectors of the incident and generated waves rather than the normal to the interface are given. Analogously, the form of equation (71) is particularly relevant for the analysis of the sensitivity of seismic data to discontinuities in the individual elastic moduli. The resulting equations for linearized reflection–transmission coefficients, applied to a finite-contrast interface, are independent on the selection of the background. Note that the same equations for the linearized weak-contrast displacement reflection–transmission coefficients may also be derived by means of the Born approximation.

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