GEOPHYSICS, VOL. 76, NO. 3 (MAY-JUNE 2011); P. WA51–WA62, 11 FIGS., 1 TABLE.
10.1190/1.3555174

Boundary attenuation angles for inhomogeneous plane waves in anisotropic dissipative media

Vlastislav Červený[1] and Ivan Pšenčík[2]

ABSTRACT

We study behavior of attenuation (inhomogeneity) angles γ, i.e., angles between real and imaginary parts of the slowness vectors of inhomogeneous plane waves propagating in isotropic or anisotropic, perfectly elastic or viscoelastic, unbounded media. The angle γ never exceeds the boundary attenuation angle γ^*. In isotropic viscoelastic media $\gamma^* = 90°$; in anisotropic viscoelastic media γ^* may be greater than, equal to, or less than 90°. Plane waves with $\gamma > \gamma^*$ do not exist. Because γ^* in anisotropic viscoelastic media is usually not known a priori, the commonly used specification of an inhomogeneous plane wave by the attenuation angle γ may lead to serious problems. If γ is chosen close to γ^* or even larger, indeterminate, unstable or even nonphysical results are obtained. We study properties of γ^* and show that the approach based on the mixed specification of the slowness vector fully avoids the problems mentioned above. The approach allows exact determination of γ^* and removes instabilities known from the use of the specification of the slowness vector by γ. For $\gamma = \gamma^*$, the approach yields zero phase velocity, i.e., the corresponding wave is a nonpropagating wave mode. The use of the mixed specification leads to the explanation of the deviation of γ^* from 90° as a consequence of different orientations of energy-flux and propagation vectors in anisotropic media. The approach is universal; it may be used for isotropic or anisotropic, perfectly elastic or viscoelastic media, and for homogeneous and inhomogeneous waves, including strongly inhomogeneous waves, like evanescent waves.

INTRODUCTION

In seismological literature, it has been common to express the complex-valued slowness vector $\mathbf{p}$ of the plane wave propagating in a dissipative medium in the following way:

$$\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A}. \quad (1)$$

Here the real-valued vectors $\mathbf{P}$ and $\mathbf{A}$ have been usually called the propagation vector and the attenuation vector, respectively. Vector $\mathbf{P}$ is perpendicular to the plane of constant phase, and oriented in the direction of propagation of the plane wave. The vector $\mathbf{A}$ is perpendicular to the plane of constant amplitude, and oriented in the direction of the maximum decay of amplitudes. The angle γ between the vectors $\mathbf{P}$ and $\mathbf{A}$ is called the attenuation (inhomogeneity) angle. The attenuation angle γ is defined in this paper as a nonoriented, nonnegative angle in the range $0° \le \gamma < 180°$. This definition has a clear physical meaning and satisfies the relation $\cos\gamma = \mathbf{P}\cdot\mathbf{A}/(|\mathbf{P}||\mathbf{A}|)$. For $\gamma = 0$, the vectors $\mathbf{P}$ and $\mathbf{A}$ are parallel, and the relevant plane wave is called homogeneous. For $\gamma \ne 0$, the relevant plane wave is called inhomogeneous.

In papers related to isotropic dissipative media, the *oriented* attenuation angle γ (including negative γ) often has been used. The oriented attenuation angle, however, cannot be uniquely defined using vectors $\mathbf{P}$ and $\mathbf{A}$ only; a strict orientation rule must be supplemented. Due to directional symmetry, this does not cause difficulties in isotropic dissipative media. In this paper, however, we consider anisotropic dissipative media, which require a 3D treatment. This is a basic reason why we use the nonoriented attenuation angle $\gamma \ge 0$ systematically. We emphasize, however, that this choice represents no loss of generality.

In dissipative media, isotropic and anisotropic, the nonoriented angle γ lies in the interval $0° \le \gamma \le \gamma^*$, where γ^* is the boundary attenuation angle. In isotropic dissipative media, $\gamma^* = 90°$ (Borcherdt, 2009). In anisotropic dissipative media, γ^* may be greater than, equal to, or less than 90°, but is always

Manuscript received by the Editor 30 July 2010; revised manuscript received 24 October 2010; published online 23 May 2011.
[1]Charles University, Faculty of Mathematics and Physics, Department of Geophysics, Prague, Czech Republic. E-mail: vcerveny@seis.karlov.mff.cuni.cz.
[2]Academy of Sciences of the Czech Republic, Institute of Geophysics, Prague, Czech Republic. E-mail: ip@ig.cas.cz.

positive and less than 180°. Inhomogeneous plane waves with nonoriented attenuation angles $\gamma > \gamma^*$ do not exist (Červený and Pšenčík, 2005a, b).

The boundary attenuation angle γ^* plays an important role in the study of inhomogeneous plane waves propagating in anisotropic dissipative media. It must be known in advance if the attenuation angle γ is used as a parameter specifying the inhomogeneous plane wave. If γ is chosen close to γ^* or greater, unstable or even nonphysical results are obtained. The well-known phenomena caused by the incorrect choice $\gamma > \gamma^*$ are the so-called forbidden directions, in which the square of the phase velocity becomes negative (Krebes and Le, 1994; Carcione and Cavallini, 1995; Carcione, 2007). If the angle γ^* is not known in advance, one can never be sure whether the chosen parameter γ specifying the inhomogeneous plane wave will lead to correct results (even for γ less than 90°). The above-mentioned problems can be fully avoided if an algorithm based on the mixed specification of the slowness vector (Červený and Pšenčík, 2005a) is used. This algorithm, in which γ appears as an evaluated quantity, never yields the attenuation angle γ greater than γ^*. Moreover, the algorithm offers an exact formula for the determination of γ^* valid universally for inhomogeneous plane waves in isotropic or anisotropic, perfectly elastic or dissipative media. The algorithm was successfully used to explain the phenomenon of forbidden directions (Červený and Pšenčík, 2005b). It also offers an alternative way how to estimate γ^* from the knowledge of the propagation and energy-flux vectors.

Notation analogous to equation 1 has been often used for the wave vector $\mathbf{k} = \omega\mathbf{p}$, not for the slowness vector $\mathbf{p}$ (Krebes and Le, 1994; Carcione, 2007; Behura and Tsvankin, 2009; Borcherdt, 2009; as examples). However, the difference between the use of $\mathbf{p}$ and $\mathbf{k}$ in our case is only formal, as we consider time-harmonic waves with a fixed positive circular frequency ω. If we wish to use equations derived in this paper in terms of the wave vector $\mathbf{k}$, we just use the relation $\mathbf{p} = \omega^{-1}\mathbf{k}$.

We begin with a brief review of the theory of time-harmonic, homogeneous and inhomogeneous plane waves propagating in viscoelastic anisotropic media and of the mixed specification of the slowness vector in the two following sections. The sections contain mostly results which we have already published (Červený and Pšenčík, 2005a,b; 2006), but which are basic for understanding and analysis of the subject of this paper. The two sections also contain some important, new, unpublished results. This applies to the symmetry relations involving parameters of the mixed specification, to the equation for the exact determination of γ^* in general anisotropic dissipative media, and to the observation that for $\gamma = \gamma^*$ we get the nonpropagating wave mode. The section which follows is devoted to plane waves propagating in isotropic viscoelastic media. There we illustrate advantages of the algorithm based on the mixed specification of the slowness vector on the basis of comparison of theoretical results obtained with the classical and mixed specifications. Next, we discuss the homogeneous and inhomogeneous SH plane waves, propagating in a symmetry plane of viscoelastic monoclinic, orthorhombic, or hexagonal media. We also offer a detailed explanation of the phenomenon of forbidden directions. In the next section we study energy attenuation and energy angles defined as nonoriented and nonnegative angles between the energy-flux vector and attenuation and propagation vectors, respectively. Relations of these angles to the boundary attenuation angle are explained and discussed. Finally we offer our conclusions.

Equations derived in this paper are mostly exact and generally valid for homogeneous and inhomogeneous plane waves, propagating in arbitrarily anisotropic and viscoelastic media, specified by 21 independent complex-valued viscoelastic moduli. Numerical examples are presented only for homogeneous and inhomogeneous P or S plane waves propagating in isotropic viscoelastic media, and for SH plane waves propagating in the plane of symmetry of a monoclinic viscoelastic medium. These simple models allow us to study and explain most of the important phenomena of plane-wave propagation in isotropic and anisotropic viscoelastic media in a simple and physically understandable manner. The models used also allow us to show important differences in plane-wave propagation in isotropic and anisotropic viscoelastic media.

We use Cartesian coordinates x_i, and denote time as t. Einstein summation convention over repeated indices is used throughout the paper. We consider time-harmonic plane waves with the time dependence chosen to be $\exp(-\mathrm{i}\omega t)$, where ω is a fixed, positive, circular frequency.

HOMOGENEOUS AND INHOMOGENEOUS PLANE WAVES IN ANISOTROPIC DISSIPATIVE MEDIA

We consider a time-harmonic plane wave propagating in an unbounded homogeneous anisotropic dissipative medium,

$$u_j(x_i, t) = aU_j \exp[-\mathrm{i}\omega(t - p_n x_n)]. \tag{2}$$

Here u_j, p_j, and U_j are complex-valued Cartesian components of displacement vector $\mathbf{u}$, slowness vector $\mathbf{p}$, and normalized polarization vector $\mathbf{U}$ ($\mathbf{U}\cdot\mathbf{U} = 1$), respectively. The symbol a denotes a scalar amplitude.

The anisotropic dissipative medium is described here as a linear viscoelastic medium, specified by independent complex-valued, frequency-dependent, density-normalized viscoelastic moduli:

$$a_{ijk\ell} = a^R_{ijk\ell} - \mathrm{i}a^I_{ijk\ell}. \tag{3}$$

The sign "−" in equation 3 is consistent with that used in the exponential factor in equation 2. As we consider ω fixed throughout, we do not show dependence of $a_{ijk\ell}$ on ω explicitly. We assume that $a_{ijk\ell}$ satisfy the symmetry relations (Carcione, 2007, section 2.1.1)

$$a_{ijk\ell} = a_{jik\ell\ell} = a_{ij} = a_{k\ell ij}, \tag{4}$$

reducing the number of independent moduli from 81 to 21.

Equation 2 represents a plane wave if, and only if, it satisfies the relevant equation of motion for anisotropic viscoelastic media. This requirement yields a system of three linear algebraic equations for the Cartesian components $U_i (i = 1, 2, 3)$ of the polarization vector $\mathbf{U}$,

$$a_{ijk\ell}p_j p_\ell U_k = U_i, \quad i = 1, 2, 3, \tag{5}$$

and the constraint relation for p_i:

$$\det\left[a_{ijk\ell}p_j p_\ell - \delta_{ik}\right] = 0. \tag{6}$$

Equation 6 is the condition of solvability of the system of equations 5. For the density-normalized viscoelastic moduli $a_{ijk\ell}$, equation 6 does not depend on the polarization vector. It will be used frequently in this paper.

SLOWNESS VECTOR OF A PLANE WAVE IN AN ANISOTROPIC DISSIPATIVE MEDIUM

We now describe properties of the slowness vector $\mathbf{p}$ and introduce certain definitions. These are valid for homogeneous and inhomogeneous plane waves in viscoelastic isotropic and anisotropic media, and for evanescent waves in perfectly elastic media. The formulae, which are important for the study of boundary attenuation angles, are taken — without derivation — from Červený and Pšenčík (2005a). More details can be found in the mentioned reference.

A useful specification of the complex-valued slowness vector $\mathbf{p}$, called the mixed specification, was proposed by Červený and Pšenčík (2005a). It does not use the attenuation angle γ, and is written as

$$\mathbf{p} = \sigma\mathbf{n} + \mathrm{i}D\mathbf{m}, \quad \text{with } \mathbf{n}\cdot\mathbf{m} = 0. \tag{7}$$

Here $\mathbf{n}$ and $\mathbf{m}$ are two arbitrary, real-valued, mutually perpendicular, unit vectors, and D is a real-valued scalar $(-\infty < D < \infty)$, with a dimension of slowness (s/km), called the inhomogeneity parameter. The unit vectors $\mathbf{n}$ and $\mathbf{m}$ specify the propagation-attenuation plane Σ. Given $a_{ijk\ell}$, $\mathbf{n}$, $\mathbf{m}$, and D, the complex-valued scalar σ can be determined uniquely from constraint relation 6. The unit vector $\mathbf{n}$ defines the line along which the wave propagates, $\mathrm{Re}\,\mathbf{p} = \pm|\mathrm{Re}\,\mathbf{p}|\mathbf{n}$. The inhomogeneity parameter D controls the inhomogeneity of the plane wave under consideration. For $D = 0$, the vectors $\mathrm{Re}\,\mathbf{p}$ and $\mathrm{Im}\,\mathbf{p}$ are parallel, $\gamma = 0$, and the plane wave is homogeneous. For $D \neq 0$, the vectors $\mathrm{Re}\,\mathbf{p}$ and $\mathrm{Im}\,\mathbf{p}$ have the different directions, $\gamma \neq 0$, and the plane wave is inhomogeneous. For small $|D|$, we speak of weakly inhomogeneous waves, and for large $|D|$ of strongly inhomogeneous waves. A schematic picture of possible orientation of individual vectors in the propagation-attenuation plane Σ, perpendicular to the wavefront, is shown in Figure 1.

Let us consider a plane wave, homogeneous or inhomogeneous, defined by parameters $\mathbf{n}$, $\mathbf{m}$, and D, propagating in a viscoelastic anisotropic medium, described by the density-normalized viscoelastic moduli $a_{ijk\ell}$. We use the mixed specification of the complex-valued slowness vector $\mathbf{p}$ (see equation 7). Inserting the expression for $\mathbf{p}$ in the mixed specification from equation 7 into the constraint relation 6, the constraint relation 6 has the following form:

$$\det\left[a_{ijk\ell}\left(\sigma n_j + \mathrm{i}Dm_j\right)\left(\sigma n_\ell + \mathrm{i}Dm_\ell\right) - \delta_{ik}\right] = 0. \tag{8}$$

Here $\mathbf{n}$, $\mathbf{m}$, and D are known parameters of the plane wave; $a_{ijk\ell}$ are known parameters of the medium. Equation 8 is a sixth-degree polynomial equation in the unknown σ with complex-valued coefficients. Consequently, it **always** has six complex-valued roots corresponding to P, S1, and S2 plane waves propagating in the positive and negative $\mathbf{n}$ direction. It should be emphasized that the properties of inhomogeneous plane waves propagating in the $\mathbf{n}$ direction are generally different from the properties of plane waves propagating in the negative $\mathbf{n}$ direction in viscoelastic anisotropic media, if $\mathbf{m}$ and D remain unchanged. The quantity σ depends on the density-normalized viscoelastic moduli $a_{ijk\ell}$, the unit vectors $\mathbf{n}$ and $\mathbf{m}$, and the inhomogeneity parameter D.

The complex-valued quantity σ has several important properties. Assume that the density-normalized viscoelastic moduli $a_{ijk\ell}$ are known. Then σ is a function of the two mutually perpendicular unit vectors $\mathbf{n}$ and $\mathbf{m}$ and the inhomogeneity parameter D. It follows from equation 8 that σ satisfies the following important symmetry relation:

$$\sigma(\mathbf{n}, \mathbf{m}, D) = \sigma(-\mathbf{n}, -\mathbf{m}, D). \tag{9}$$

Thus the quantity σ is not changed if the signs of $\mathbf{n}$ and $\mathbf{m}$ are changed, with D fixed. It is easy to show that the quantity σ also satisfies the following relations

$$\sigma(\mathbf{n}, \mathbf{m}, D) = \sigma(\mathbf{n}, -\mathbf{m}, -D) = \sigma(-\mathbf{n}, \mathbf{m}, -D),$$

$$\sigma(-\mathbf{n}, \mathbf{m}, D) = \sigma(\mathbf{n}, -\mathbf{m}, D) = \sigma(\mathbf{n}, \mathbf{m}, -D). \tag{10}$$

If two of the three quantities $\mathbf{n}$, $\mathbf{m}$, and D are taken with the opposite sign, σ remains unchanged; σ is changed, however, if only one of the three mentioned quantities is taken with the opposite sign.

Keeping equations 9 and 10 in mind, the quantities $a_{ijk\ell}$, $\mathbf{n}$, $\mathbf{m}$, and D uniquely specify the slowness vector $\mathbf{p}$ of any plane wave, homogeneous or inhomogeneous, propagating in any isotropic or anisotropic, viscoelastic or perfectly elastic medium. Similarly, for the slowness vector of any plane wave, homogeneous or inhomogeneous, propagating in a medium specified by density-normalized viscoelastic or elastic moduli $a_{ijk\ell}$, we can find the quantities $\mathbf{n}$, $\mathbf{m}$, and D, which specify it uniquely.

The solution of the sixth-degree polynomial equation 8 with complex-valued coefficients is straightforward, and its six complex-valued roots σ **always** exist. In certain special cases, the polynomial equation can be factorized into two equations, one of fourth degree and one of second degree. For an isotropic viscoelastic medium or for SH waves propagating in a plane of symmetry of monoclinic,

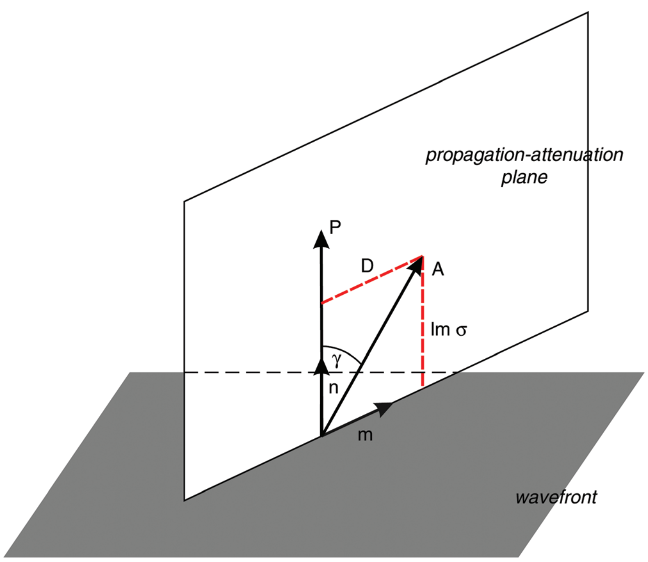

Figure 1. Schematic diagram showing an example of two real-valued, mutually orthogonal unit vectors $\mathbf{n}$ and $\mathbf{m}$, used to define complex-valued slowness vector $\mathbf{p}$. Vector $\mathbf{n}$ is perpendicular to the wavefront and specifies the direction of propagation vector $\mathbf{P}$. $\mathbf{P}$ and the attenuation vector $\mathbf{A}$ define the propagation-attenuation plane. Vectors $\mathbf{P}$ and $\mathbf{A}$ make a nonoriented attenuation (inhomogeneity) angle γ. Projections of $\mathbf{A}$ into $\mathbf{n}$ and $\mathbf{m}$ are $\mathrm{Im}\,\sigma$ and inhomogeneity parameter D, respectively (see equation 11).

orthorhombic, or hexagonal viscoelastic media, these factorized equations can be even solved analytically.

After having solved equation 8 for σ, we can determine important quantities characterizing the relevant plane wave. This applies to the propagation vector $\mathbf{P}$, the attenuation vector $\mathbf{A}$, the phase velocity $\mathcal{C}$, and the attenuation angle γ. They are given by relations:

$$\mathbf{P} = \mathbf{n}\,\mathrm{Re}\,\sigma, \quad \mathbf{A} = \mathbf{n}\,\mathrm{Im}\,\sigma + D\mathbf{m},$$
$$\mathcal{C} = 1/|\mathrm{Re}\,\sigma|, \quad \gamma = \arccos\{\epsilon_\sigma \,\mathrm{Im}\,\sigma/[(\mathrm{Im}\,\sigma)^2 + D^2]^{1/2}\}, \tag{11}$$

where $\arccos\{\ldots\}$ is represented by its principal value between 0° and 180°, and ϵ_σ is given by the relation

$$\epsilon_\sigma = \mathrm{Re}\,\sigma/|\mathrm{Re}\,\sigma|. \tag{12}$$

We could also compute other useful plane-wave quantities (Červený and Pšenčík, 2005a, 2006), which are not discussed in this paper.

Equation 11 shows that the attenuation angle γ can be computed from $\mathrm{Im}\,\sigma$ if the inhomogeneity parameter D is used as a parameter specifying the relevant inhomogeneous plane wave. In seismological literature, however, the attenuation angle γ is often used as a parameter of the inhomogeneous plane wave, characterizing the inhomogeneity of the wave. This may lead to serious problems, mainly in anisotropic viscoelastic media. A major problem is that the specification of inhomogeneous plane wave propagating in a viscoelastic anisotropic medium by the attenuation angle γ may lead to the phenomenon of the so-called forbidden directions. This problem is completely avoided if the mixed specification of the slowness vector is used to specify the inhomogeneous plane wave. The attenuation angle γ can be then computed for any value of $\mathbf{n}$, $\mathbf{m}$, and D.

Equation 8 can be also expressed in a modified form, which is useful for inhomogeneous plane waves. We assume that $D \neq 0$ and introduce a complex-valued dimensionless scalar w through the relation

$$\sigma = wD. \tag{13}$$

Inserting this relation into the sixth-degree polynomial equation 8, we obtain a polynomial equation of the sixth degree for w:

$$\det\left[a_{ijk\ell}\left(wn_j + \mathrm{i}m_j\right)\left(wn_l + \mathrm{i}m_\ell\right) - D^{-2}\delta_{ik}\right] = 0. \tag{14}$$

The slowness vector can then be expressed in terms of w as

$$\mathbf{p} = D(w\mathbf{n} + \mathrm{i}\mathbf{m}). \tag{15}$$

For $\mathbf{P}$, $\mathbf{A}$, $\mathcal{C}$, and γ, we obtain:

$$\mathbf{P} = \mathbf{n}D\,\mathrm{Re}\,w, \quad \mathbf{A} = D(\mathbf{n}\,\mathrm{Im}\,w + \mathbf{m}),$$
$$\mathcal{C} = 1/|D\,\mathrm{Re}\,w|, \quad \gamma = \arccos\{\epsilon_w \,\mathrm{Im}\,w/[(\mathrm{Im}\,w)^2 + 1]^{1/2}\}, \tag{16}$$

where $\epsilon_w = \mathrm{Re}\,w/|\mathrm{Re}\,w|$. Equations 16 cannot be used for homogeneous plane waves ($D = 0$), but for $D \neq 0$ they are equivalent to equations 11. The function $\arccos\{\ldots\}$ is as defined after equation 11.

Equations 14 and 16 can be used to determine the boundary attenuation angles γ^*, corresponding to w^* obtained for $D \to \pm\infty$. In this limit, equation 14 yields

$$\det[a_{ijk\ell}(w^* n_j + \mathrm{i}m_j)(w^* n_\ell + \mathrm{i}m_\ell)] = 0. \tag{17}$$

This is still a sixth-degree polynomial equation for w^*, but w^* does not depend on D.

The limiting case $D \to \pm\infty$ yields the phase velocity $\mathcal{C} = 0$ (see equation 16), so that the relevant "slowness vector" does not correspond to a propagating plane wave. The equation for γ in equations 16 offers a formula for computing the boundary attenuation angle γ^*:

$$\gamma^* = \arccos\{\epsilon_w^* \,\mathrm{Im}\,w^*/[(\mathrm{Im}\,w^*)^2 + 1]^{1/2}\}, \tag{18}$$

where $\epsilon_w^* = \mathrm{Re}\,w^*/|\mathrm{Re}\,w^*|$. Thus, the boundary attenuation angles γ^* can be exactly determined for any inhomogeneous plane wave (including evanescent waves) propagating in any direction in any isotropic or anisotropic, dissipative or perfectly elastic media, if the mixed specification of the slowness vector is used.

P- AND S-WAVES IN ISOTROPIC DISSIPATIVE MEDIA

For isotropic media the polynomial equation 6 can be factorized and solved analytically. We express $a_{ijk\ell}$ in terms of complex-valued, density-normalized, viscoelastic Lame's moduli λ/ρ and μ/ρ as follows:

$$a_{ijk\ell} = (\lambda/\rho)\delta_{ij}\delta_{k\ell} + (\mu/\rho)(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}). \tag{19}$$

Using equation 19 reduces the constraint relation (equation 6) to:

$$(\alpha^2 p_i p_i - 1)(\beta^2 p_k p_k - 1)^2 = 0. \tag{20}$$

Here

$$\alpha^2 = \frac{\lambda + 2\mu}{\rho}, \quad \beta^2 = \frac{\mu}{\rho}. \tag{21}$$

Equation 20 shows that two types of plane waves can propagate in a homogeneous isotropic viscoelastic medium. The P-wave satisfies the relation $\alpha^2 p_i p_i = 1$, and the S-wave: $\beta^2 p_i p_i = 1$. The quantities α^2 and β^2 represent the squares of complex-valued velocities α and β of P and S waves, respectively. Each wave is controlled by the equation:

$$p_i p_i = 1/V^2, \tag{22}$$

where V is complex-valued and equals α for P-waves, and β for S-waves. The constraint relation (equation 22) can be used to determine the slowness vector. By inserting the expression for $\mathbf{p}$ in the mixed specification from equation 7 into constraint relation 22, a simple formula for σ is obtained:

$$\sigma = \pm(1/V^2 + D^2)^{1/2}. \tag{23}$$

The "$\pm$" signs correspond to the plane waves propagating in the $\pm\mathbf{n}$ directions. From equation 11, we then obtain

$$\mathbf{P} = \pm\mathbf{n}\,\mathrm{Re}(1/V^2 + D^2)^{1/2},$$
$$\mathbf{A} = \pm\mathbf{n}\,\mathrm{Im}(1/V^2 + D^2)^{1/2} + D\mathbf{m},$$
$$\mathcal{C} = 1/|\mathrm{Re}(1/V^2 + D^2)^{1/2}|,$$
$$\gamma = \arccos\{\mathrm{Im}(1/V^2 + D^2)^{1/2}/[(\mathrm{Im}(1/V^2 + D^2)^{1/2})^2 + D^2]^{1/2}\}. \tag{24}$$

Equations 24 are valid for homogeneous and inhomogeneous plane waves propagating in viscoelastic or perfectly elastic isotropic media (complex-valued or real-valued $1/V^2$). For perfectly elastic media (real-valued velocity V), we speak of evanescent waves. Note that $\mathcal{C}$ and γ are independent of the orientation of $\mathbf{n}$ and $\mathbf{m}$.

For homogeneous plane waves $(D=0)$, we obtain $\mathcal{C}_{hom} = 1/|\mathrm{Re}(1/V^2)^{1/2}|$ and $\gamma = 0°$, as expected. When the medium is perfectly elastic, $1/V^2$ is real-valued and the phase velocity $\mathcal{C} = V$.

For inhomogeneous plane waves $(D \neq 0)$, we can alternatively use equation 13 in equation 23, and compute w instead of σ

$$w = \pm[1/(V^2D^2) + 1]^{1/2}. \tag{25}$$

Then from equations 16 we obtain:

$$\mathcal{C} = 1/|D\,\mathrm{Re}[1/(V^2D^2) + 1]^{1/2}|,$$

$$\gamma = \arccos\Big(\mathrm{Im}[1/(V^2D^2) + 1]^{1/2}/\{[\mathrm{Im}(1/(V^2D^2) + 1)^{1/2}]^2 + 1\}^{1/2}\Big). \tag{26}$$

For $D \neq 0$, equations 26 are equivalent to those in equations 24.

Equations 25 and 26 do not contain any indeterminate expressions and can be used to compute exact values of $\mathcal{C}$ and γ even for $D \to \pm\infty$. For $D \to \pm\infty$, equation 25 yields the boundary value w^*:

$$w^* = \pm 1, \tag{27}$$

and equation 26 the boundary values $\mathcal{C}^*$ of the phase velocity and γ^* of the attenuation angle:

$$\mathcal{C}^* = 0, \quad \gamma^* = 90°. \tag{28}$$

In isotropic dissipative media, the boundary attenuation angle γ^* is 90°, and is independent of the wave type (P or S), the value of the complex-valued velocity V, the direction of propagation $\mathbf{n}$, and the orientation of the propagation-attenuation plane Σ. In all these situations, the boundary value $\mathcal{C}^*$ of the phase velocity is zero.

Let us now present simple examples of the attenuation angle γ and boundary attenuation angle γ^* for a plane wave (it may be P or S) propagating in an isotropic viscoelastic medium. The considered complex-valued velocities V of these waves are given by the relation $V^2 = 11.25(1 - \mathrm{i}Q^{-1})$, where $Q = 10, 20, 40$, and ∞. The symbol Q denotes the quality factor (see equation 33).

We introduce Cartesian coordinates x_1, x_3 arbitrarily in the Σ plane and express the components of the real-valued unit vectors $\mathbf{n}$ and $\mathbf{m}$ in terms of the propagation angle i as follows:

$$n_1 = \sin i, \quad n_3 = \cos i,$$

$$m_1 = \cos i, \quad m_3 = -\sin i. \tag{29}$$

We consider i in the range $0° \leq i \leq 360°$.

Figure 2 shows the behavior of the phase velocity $\mathcal{C}$ and the attenuation angle γ as a function of D. The black curve in Figure 2 corresponds to $Q = 10$, the red curve to $Q = 20$, the green curve to $Q = 40$, and the blue curve to perfectly elastic medium, $Q = \infty$. The phase velocity $\mathcal{C}$ is only slightly dependent on Q for D fixed. The dependence of $\mathcal{C}$ on D, for fixed Q, is more pronounced. The phase velocity $\mathcal{C}$ is largest for homogeneous plane wave $(D=0)$, and decreases with increasing $|D|$. As $D \to \pm\infty$, the phase velocity approaches zero.

The attenuation angle γ in isotropic media is always in the range $0 \leq \gamma \leq 90°$, and for perfectly elastic isotropic media (blue curve) $\gamma = 90°$. This is an important difference between perfectly elastic and viscoelastic isotropic media. In perfectly elastic isotropic media, only inhomogeneous plane waves with an attenuation angle $\gamma = 90°$ may exist (evanescent waves). In viscoelastic isotropic media, the inhomogeneous plane waves may be characterized by any value of γ between 0° and 90°. The behavior of the attenuation angle for $|D|$ small is of particular interest. In perfectly elastic isotropic media, homogeneous plane waves with $|\mathbf{A}| \neq 0$ do not exist. In weakly viscoelastic isotropic media the attenuation angle γ increases very fast with increasing $|D|$, particularly for small $|D|$. For $|D|$ further increasing, the attenuation angle γ approaches asymptotically 90°. The curves in Figure 2 are symmetric about $D = 0$. This means that the phase velocity $\mathcal{C}$ and the attenuation angle γ do not depend on the orientation of vectors $\mathbf{n}$ and $\mathbf{m}$ (see equations 24). This distinguishes propagation of inhomogeneous plane waves in isotropic and anisotropic viscoelastic media.

Figure 3 shows the dependence of the phase velocity $\mathcal{C}$ and the attenuation angle γ on the propagation angle i, for constant D. Four very different values of D are considered: $D = 0.001$ (blue), $D = 0.01$ (green), $D = 0.1$ (red), and $D = 1$ (black). Because we consider an isotropic medium, the phase velocity $\mathcal{C}$ and the attenuation angle γ are independent of the propagation angle i for D fixed. The curves of the phase velocity for $D = 0.001$ and $D = 0.01$ coincide effectively and show that the phase velocity is close to that of a homogeneous wave. The phase velocity $\mathcal{C}$ decreases strongly for $D > 0.1$, and deviates from the phase velocity of homogeneous waves. For $D \sim 1$, the

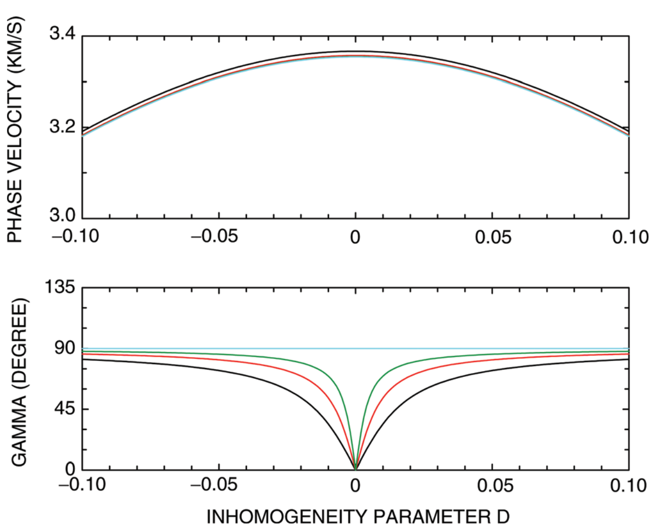

Figure 2. Variations with D of the phase velocity $\mathcal{C}$ (top) and the attenuation angle γ (bottom) of inhomogeneous P or S plane waves in isotropic viscoelastic media using mixed specification. The colored curves correspond to the complex velocity $V^2 = 11.25(1 - \mathrm{i}Q^{-1})$ with selected values of Q factor: black — $Q = 10$, red — $Q = 20$, green — $Q = 40$, and blue — $Q = \infty$ (perfectly elastic).

phase velocities decrease to about 30% of the phase velocity of the homogeneous wave. For $D \to \pm\infty$, the phase velocity approaches zero. The attenuation angles γ for small $|D|$ (0.01, 0.001) are less than 45°, for larger D (0.1, 1), they are close to 90°. This indicates that, for $D \geq 1$, the attenuation angle γ can be used as an estimate of the boundary attenuation angle γ^*.

Let us now compare the approach based on the mixed specification of the slowness vector $\mathbf{p}$ using equation 7 with the classical approach often used in seismic literature. In the classical approach, γ is used as the parameter specifying the inhomogeneity of the plane wave. Let us separate the real and imaginary parts of $1/V^2$:

$$1/V^2 = \mathrm{Re}(1/V^2) + \mathrm{i}\,\mathrm{Im}(1/V^2). \qquad (30)$$

The phase velocity $\mathcal{C}$ is then determined as a function of γ, $\mathrm{Re}(V^{-2})$, and $\mathrm{Im}(V^{-2})$ and can be easily derived. Using equations 1 and 30 in equation 22, and separating real and imaginary parts, we obtain two equations:

$$|\mathbf{P}|^2 - |\mathbf{A}|^2 = \mathrm{Re}(1/V^2), \quad 2|\mathbf{A}||\mathbf{P}|\cos\gamma = \mathrm{Im}(1/V^2). \qquad (31)$$

Solving these two equations for $|\mathbf{P}|$ and $|\mathbf{A}|$, we obtain

$$|\mathbf{P}| = \sqrt{\mathrm{Re}(1/V^2)/2}\,\sqrt{1 + [1 + 1/Q^2\cos^2\gamma]^{1/2}},$$

$$|\mathbf{A}| = \sqrt{\mathrm{Re}(1/V^2)/2}\,\sqrt{-1 + [1 + 1/Q^2\cos^2\gamma]^{1/2}}, \qquad (32)$$

where

$$Q^{-1} = \mathrm{Im}(1/V^2)/\mathrm{Re}(1/V^2) = -\mathrm{Im}\,V^2/\mathrm{Re}\,V^2 \qquad (33)$$

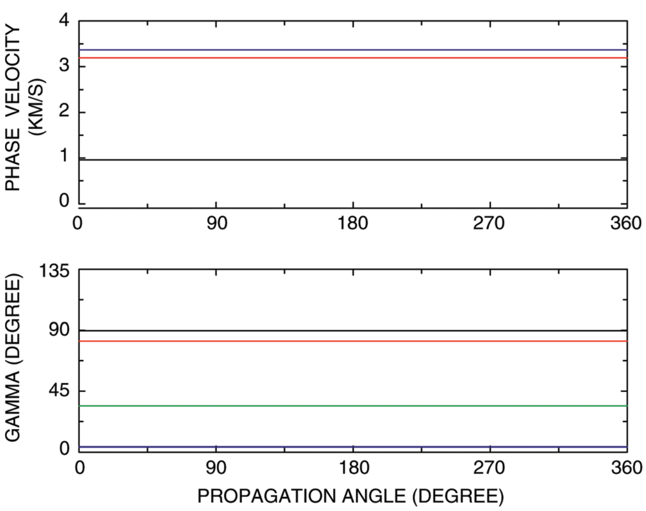

Figure 3. Variations with propagation angle i of the phase velocity $\mathcal{C}$ (top) and the attenuation angle γ (bottom) of inhomogeneous P or S plane waves in isotropic viscoelastic medium with complex velocity $V^2 = 11.25(1 - \mathrm{i}/10)$, using mixed specification. The colored curves correspond to selected D, $D = 1$ — black, $D = 0.1$ — red, $D = 0.01$ — green, and $D = 0.001$ — blue. The phase-velocity curves coincide effectively for $D = 0.001$ and $D = 0.01$ and correspond to the phase-velocity curve of a homogeneous wave. The γ curve for $D = 1$ coincides effectively with the γ curve for $D \to \infty$. The black curve thus represents the boundary attenuation angle $\gamma^* = 90°$.

is the quality factor for homogeneous plane waves (see, for example, Borcherdt, 2009, or Behura and Tsvankin, 2009). Consequently, the phase velocity $\mathcal{C}$ of an inhomogeneous plane wave propagating in a homogeneous, viscoelastic, isotropic medium is given by

$$\mathcal{C} = 1/|\mathbf{P}| = \sqrt{2/\mathrm{Re}(1/V^2)}/\sqrt{1 + [1 + 1/Q^2\cos^2\gamma]^{1/2}}. \qquad (34)$$

Note that for homogeneous waves in isotropic viscoelastic media, equation 34 becomes

$$\mathcal{C}_{hom} = \sqrt{2/\mathrm{Re}(1/V^2)}/\sqrt{1 + [1 + Q^{-2}]^{1/2}}. \qquad (35)$$

In perfectly elastic media, equations 31 yield $|\mathbf{P}|^2 - |\mathbf{A}|^2 = V^{-2}$ and $|\mathbf{A}||\mathbf{P}|\cos\gamma = 0$. The latter equation shows that $\mathbf{A}$ is always perpendicular to $\mathbf{P}$ ($\cos\gamma = 0$). Consequently, the product $Q^2\cos^2\gamma$ is indeterminate ($\infty \times 0$) in this case and equation 34 cannot be used to compute the phase velocity $\mathcal{C}$ of the evanescent wave. Equations $|\mathbf{P}|^2 - |\mathbf{A}|^2 = V^{-2}$ and $|\mathbf{A}||\mathbf{P}|\cos\gamma = 0$ are, however, not sufficient to calculate $|\mathbf{P}|$ and $|\mathbf{A}|$ and $\mathcal{C}$, unless some other condition is used (see Borcherdt, 2009). However, equations 26, derived using the mixed specification, can be used without any problem in viscoelastic as well as perfectly elastic media.

A reader may say: "This problem is of no interest for me; I am not interested in evanescent waves, but in inhomogeneous plane waves propagating in viscoelastic isotropic media." However, except for very weakly inhomogeneous plane waves, inhomogeneous plane waves propagating in weakly dissipative media are very similar to evanescent waves. In weakly dissipative media, which play an important role in the attenuation analysis in seismic exploration, equation 34 gives, in fact, inaccurate results, and is extremely sensitive to small variations in Q and γ (Figure 2).

The classical equations 32–34 lead to the conclusion that a unique solution does not exist for the propagation $\mathbf{P}$ and attenuation $\mathbf{A}$ vectors, for $\gamma = 90°$ (Borcherdt, 2009, p. 37). This conclusion, however, is true only when the slowness vector $\mathbf{p}$ is specified by γ. If we use the mixed specification of inhomogeneous plane waves, in which D is used as the parameter specifying inhomogeneity of the plane wave, propagation $\mathbf{P}$ and attenuation $\mathbf{A}$ vectors are determined uniquely as functions of D, no matter if the medium is dissipative or perfectly elastic. Thus the mixed specification of the slowness vector provides a smooth and continuous transition from inhomogeneous plane waves propagating in weakly dissipative media to evanescent waves propagating in perfectly elastic media and vice versa. This is another important advantage of the use of the mixed specification.

SH WAVES IN PLANES OF SYMMETRY OF MONOCLINIC VISCOELASTIC MEDIA

The simplest case of viscoelastic anisotropic media, which allows for the expression of all equations discussed previously in a simple analytical form, corresponds to the SH plane waves propagating in a symmetry plane Σ^S of monoclinic viscoelastic media. These equations may also be used in planes of symmetry of hexagonal or orthorhombic media, and, of course, in isotropic media. We assume that the plane Σ^S corresponds to the Cartesian coordinate plane (x_1, x_3), and that the vectors $\mathbf{n}$ and $\mathbf{m}$ are given by equations 29, i.e., these vectors, and the slowness vector as well, are situated in the plane (x_1, x_3). The constraint relation (equation 6) then factorizes into two polynomial equations, one of the fourth, the other of the second degree. The latter equation is the equation for plane SH waves, polarized in the direction perpendicular to Σ^S. The equation reads

$$A_{66}p_1^2 + A_{44}p_3^2 + 2A_{46}p_1p_3 = 1. \tag{36}$$

Here A_{66}, A_{44}, and A_{46} are the complex-valued density-normalized viscoelastic moduli in Voigt notation. Inserting the expression for $\mathbf{p}$ in the mixed specification from equation equation 7 into equation 36, we obtain a quadratic equation for σ, which has two complex-valued roots $\sigma_{1,2}$:

$$\sigma_{1,2} = -\mathrm{i}D\Lambda/\Gamma_{22} \pm \sqrt{1/\Gamma_{22} + D^2\Delta/\Gamma_{22}^2}, \tag{37}$$

where

$$\Gamma_{22} = A_{66}n_1^2 + A_{44}n_3^2 + 2A_{46}n_1n_3,$$

$$\Lambda = (A_{66} - A_{44})n_1n_3 + A_{46}(n_3^2 - n_1^2),$$

$$\Delta = A_{44}A_{66} - A_{46}^2. \tag{38}$$

Using equations 11, we can then compute the propagation vector $\mathbf{P}$, the attenuation vector $\mathbf{A}$, the phase velocity $\mathcal{C}$, and the attenuation angle γ. For $\sigma = \sigma_1$, we obtain these quantities for plane waves propagating in the positive $\mathbf{n}$ direction; for $\sigma = \sigma_2$, we obtain them for plane waves propagating in the negative $\mathbf{n}$ direction.

Alternatively, for $D \neq 0$ we can use relation 13 in equation 37 to compute w. For $w_{1,2}$ we obtain:

$$w_{1,2} = -\mathrm{i}\Lambda/\Gamma_{22} \pm \sqrt{1/(D^2\Gamma_{22}) + \Delta/\Gamma_{22}^2}. \tag{39}$$

Using equation 39 in equations 16, we can calculate the phase velocity $\mathcal{C}$ and the attenuation angle γ.

For $D \to \pm\infty$, the boundary values of $w_{1,2}$ are obtained from equation 39 as

$$w_{1,2}^* = -\mathrm{i}\Lambda/\Gamma_{22} \pm \sqrt{\Delta/\Gamma_{22}^2}. \tag{40}$$

The boundary phase velocity $\mathcal{C}^*$ is zero and the boundary attenuation angles γ_1^* and γ_2^* are given by equation 18 in terms of $\operatorname{Im} w_{1,2}^*$.

For weakly dissipative media, we can approximate the density-normalized viscoelastic moduli A_{44}, A_{46}, and A_{66} by their real parts A_{44}^R, A_{46}^R, and A_{66}^R. Quantities defined in equations 38 are then real valued and equation 40 can be used for an estimate of $\operatorname{Im} w_{1,2}^*$:

$$\operatorname{Im} w_{1,2}^* \doteq -\Lambda^R/\Gamma_{22}^R. \tag{41}$$

Here $\Lambda^R = (A_{66}^R - A_{44}^R)n_1n_3 + A_{46}^R(n_3^2 - n_1^2)$ and $\Gamma_{22}^R = A_{66}^R n_1^2 + A_{44}^R n_3^2 + 2A_{46}^R n_1 n_3$. Inserting equation 41 into equation 18, an approximate formula for computing the boundary attenuation angle is obtained:

$$\gamma_{1,2}^* \doteq \arccos\left\{-\epsilon_w^* \frac{\Lambda^R/\Gamma_{22}^R}{[(\Lambda^R/\Gamma_{22}^R)^2 + 1]^{1/2}}\right\}. \tag{42}$$

Equation 42 gives an "evanescent wave approximation" for $\gamma_{1,2}^*$. Note that the evanescent wave approximation does not depend on the dissipative properties of the medium. Consequently, the boundary attenuation angles that differ from 90° are typically the effect of anisotropy, not of dissipation.

Now we shall present several numerical examples. Four models of viscoelastic anisotropic media used in the following tests are derived from the model used by Carcione and Cavallini (1995). We consider SH-wave propagation in a symmetry plane of a monoclinic medium, described by the three complex-valued density-normalized viscoelastic moduli: $A_{44} = A_{44}^R - \mathrm{i}A_{44}^I$, $A_{66} = A_{66}^R - \mathrm{i}A_{66}^I$, $A_{46} = A_{46}^R - \mathrm{i}A_{46}^I$. In the following tests, we form models by independently varying real and imaginary parts of these density-normalized viscoelastic moduli. These parts are given in Table 1. They have dimensions of $(\mathrm{km/s})^2$. From the left part of Table 1 we can see that anisotropy of the model R1 is rather strong. Anisotropy of models R2, R3, and R4 successively decreases. The model R4 is isotropic. Similarly, from the right part of Table 1, we can see that dissipation of the model I1 is rather strong. Dissipation of models I1, I2, and I3 successively decreases. The model with imaginary parts I4 is perfectly elastic. Consequently, the model R4I4 corresponds to the perfectly elastic isotropic medium.

In the following figures we show behaviors of attenuation angles γ, boundary attenuation angles γ^*, and phase velocities $\mathcal{C}$ as a function of either the inhomogeneity parameter D or the propagation angle i. The propagation angle i was defined by equation 29. All plots contain colored curves corresponding to different choices of D or model parameters.

Figures 4 and 5 show dependence of the attenuation angle γ (bottom) and the phase velocity $\mathcal{C}$ (top) on the inhomogeneity factor D, for propagation angles $i = 45°$ (Figure 4) and $i = 135°$ (Figure 5). Individual curves correspond to different strengths of dissipation in the medium. The black curves correspond to the model with the strongest dissipation (R1I1), red to R1I2, green to R1I3, and blue to perfectly elastic model R1I4. Model R1 with strongest anisotropy is considered. An important characteristic in both figures is that γ and $\mathcal{C}$ are nonsymmetric with respect to $D = 0$. The plots of the attenuation angle γ show important relations between two parameters used for the specification of inhomogeneity of plane waves, γ and D. We can see that γ varies strongly with $|D|$ when inhomogeneity of the wave is weak. The interval in which γ strongly varies with $|D|$ is narrower the weaker the dissipation of the medium. For a perfectly elastic medium, the interval shrinks to zero. The attenuation angle γ varies only a little with varying $|D|$ for strongly inhomogeneous waves. As $D \to \pm\infty$, $\gamma \to \gamma_{1,2}^*$. The boundary attenuation angles γ_1^* and γ_2^* are different and generally differ from 90°. The difference between γ_1^* and γ_2^* depends on the propagation angle i. Compare Figures 4 and 5.

Let us test the accuracy of the evanescent wave approximation given by equation 42 for the computation of boundary attenuation

Table 1. Specification of complex-valued density-normalized viscoelastic moduli describing the SH-wave propagation in the plane of symmetry of the monoclinic medium.

	A_{44}^R	A_{66}^R	A_{46}^R		Λ_{44}^T	Λ_{66}^T	Λ_{46}^T
Rl	5	11.5	2.5	**I1**	1	1.125	0
R2	6.625	9.875	1.25	**I2**	0.5	0.5625	0
R3	7.4375	9.0625	0.625	**I3**	0.25	0.28125	0
R4	8.25	8.25	0	**I4**	0	0	0

angles $\gamma^*_{1,2}$. We consider $i = 135°$, corresponding to Figure 5. In this case, the evanescent wave approximation given by equation 42 yields $\gamma^*_1 = 60,9°$ and $\gamma^*_2 = 119,1°$. This corresponds very well to the boundary attenuation angles shown in Figure 5.

The phase velocity depends on changes of dissipation only weakly. This is expected because the phase velocity is controlled mostly by the real parts of density-normalized viscoelastic moduli and depends only weakly on their imaginary parts. As $D \to \pm\infty$, $\mathcal{C} \to 0$. Also of importance is the shift of the maxima of phase velocities from $D = 0$, which means that maximum phase velocities are associated with inhomogeneous waves, not with homogeneous waves. This is different when compared to isotropic media, where the maximum phase velocity is always associated with the homogeneous plane wave.

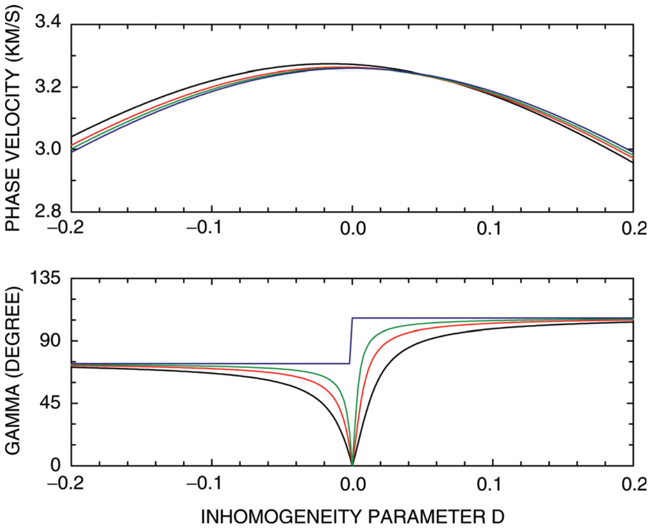

Figure 4. Variations with D of the phase velocity $\mathcal{C}$ (top) and the attenuation angle γ (bottom) of inhomogeneous SH plane waves in symmetry planes of viscoelastic monoclinic media using mixed specification. The colored curves correspond to models with strongest anisotropy and varying dissipation, black — R1I1, red — R1I2, green — R1I3, and blue — R1I4 (perfectly elastic); see Table 1. The propagation angle $i = 45°$.

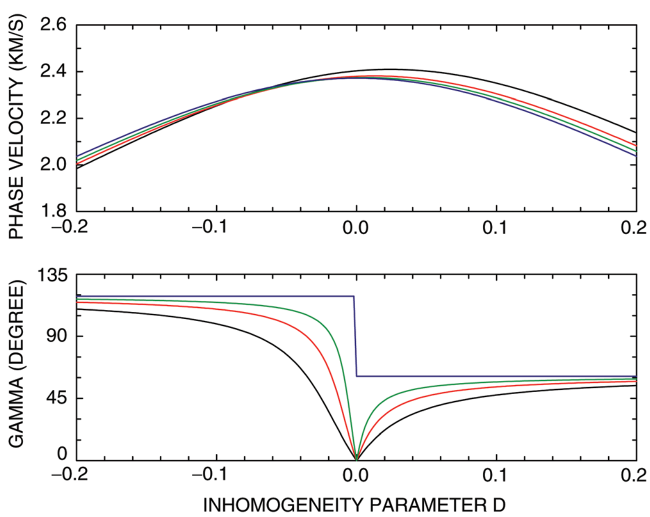

Figure 5. Variations with D of the phase velocity $\mathcal{C}$ (top) and the attenuation angle γ (bottom) of inhomogeneous SH plane waves in symmetry planes of viscoelastic monoclinic media using mixed specification. The colored curves correspond to models with strongest anisotropy and varying dissipation, black — red — R1I2, green — R1I3, and blue — R1I4 (perfectly elastic); see Table 1. The propagation angle $i = 135°$.

In Figures 6 and 7, we show again the attenuation angle γ (bottom) and the phase velocity $\mathcal{C}$ (top), but now as functions of propagation angle i for $D > 0$ (Figure 6) and $D < 0$ (Figure 7). The results are shown for the model R1I2 (see Table 1), i.e., for strong anisotropy and relatively strong dissipation. Individual curves in Figures 6 and 7 are calculated for $|D| = 10$ — black, $|D| = 1$ — red, $|D| = 0.03$ — green, and $|D| = 0.01$ — blue. We can clearly see angular dependence of attenuation angles as well as of phase velocities. In the plot of the attenuation angles we can see the rapid increase of values of γ for a small change of $|D|$ when the inhomogeneity of waves is weak (blue and green curves). For larger values of $|D|$, the γ curves are practically identical (black and red). A further increase of $|D|$ would yield curves coinciding effectively with the black and red curve. This means that the black and red curves in Figures 6 and 7 can be considered as representing the boundary attenuation angles $\gamma^*_{1,2}$.

The phase velocity of weakly inhomogeneous waves remains practically the same (coinciding green and blue curves in the upper plots of Figures 6 and 7). For larger values of $|D|$, the phase velocities decrease, for $|D| = 10$ (black curve) being negligibly small. While behavior of phase velocities is practically identical for positive and negative D, the behavior of the attenuation angles γ differs considerably.

In Figure 8, we study the effects of changing anisotropy (real parts of density-normalized viscoelastic moduli) on the boundary attenuation angle γ^* (bottom) and the phase velocity $\mathcal{C}$ (top).

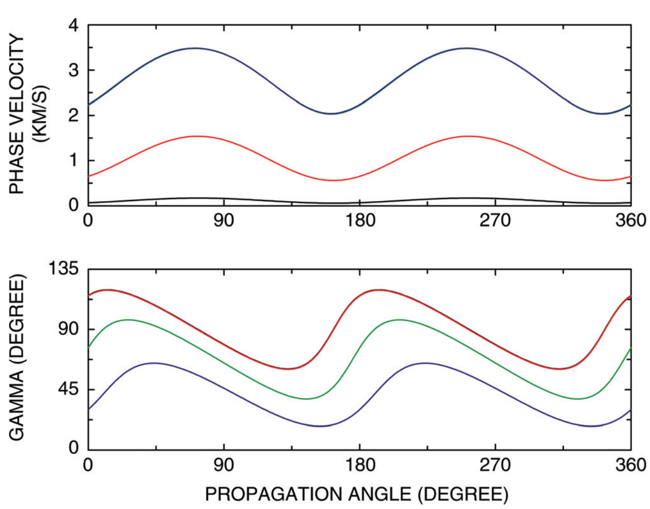

Figure 6. Variations with propagation angle of the phase velocity $\mathcal{C}$ (top) and the attenuation angle γ (bottom) of inhomogeneous SH plane waves in symmetry planes of the viscoelastic monoclinic medium R1I2 (see Table 1) using mixed specification. The colored curves correspond to varying D, $D = 10$ — black, $D = 1$ — red, $D = 0.03$ — green, and $D = 0.01$ — blue. The phase-velocity curves coincide effectively for $D = 0.01$ and $D = 0.03$ and correspond to the phase-velocity curve of a homogeneous wave. The γ curves coincide effectively for $D = 1$ and $D = 10$ and correspond to the γ curve of $D \to \infty$. The red curve thus represents the boundary attenuation angle γ^*.

The colored curves in Figure 8 correspond to the model R1I1 — black, R2I1 — red, R3I1 green, and R4I1 (isotropic) — blue. The $\mathcal{C}$ curves are plotted for $D = 0.02$, the γ^* curves are independent of D. We can see that changing strength of anisotropy has a pronounced effect on the boundary attenuation angle as well as the phase velocity. As the strength of anisotropy increases, the intensity of variations of $\mathcal{C}$ and γ^* with respect to the propagation angle i increases too.

In comparison with anisotropy, the effect of dissipation (imaginary parts of density-normalized viscoelastic moduli) on the boundary attenuation angle γ^* and the phase velocity $\mathcal{C}$ is negligible. This is shown in Figure 9, in the case of phase velocities (top) for $D = 0.02$. The γ^* curves are again independent of D. The colored curves in Figure 9 correspond to the model R1I1 — black, R1I2 — red, R1I3 — green, and R1I4 (perfectly elastic) — blue. We can see that the boundary attenuation angles of inhomogeneous plane waves propagating in dissipative media do not differ much from the boundary attenuation angles of evanescent plane waves. This indicates the approximate applicability of the "evanescent wave approximation" (see equation 42), for the estimation of $\gamma^*_{1,2}$.

Figure 10 illustrates the phenomenon of forbidden directions. In contrast to previous figures, the directional specification of the slowness vector is used in Figure 10 instead of the mixed specification. In this case, the attenuation angles γ (bottom) are not computed, but chosen. Four angles γ are considered (see the horizontal lines in the bottom plot of Figure 10), $\gamma = 25^\circ$ — black, $\gamma = 58^\circ$ — red, $\gamma = 62^\circ$ — green, and $\gamma = 66^\circ$ — blue. The oscillating blue curve in the same plot is the boundary attenuation angle γ^* calculated by the mixed specification of the slowness vector. The curves in the upper plot represent squares of the phase velocities calculated by directional specification of the slowness vector for the four considered values of γ. We can see that except for the black curve (25°), the curves are distorted between propagation angles 90° and 180°, and 270° and 360°. The green and blue γ curves (62° and 66°) intersect zero, the corresponding squares of phase velocity thus becoming negative. Carcione and Cavallini (1995) called this phenomenon forbidden directions; see also Krebes and Le (1994) and Carcione (2007). In the mentioned intervals of the propagation angles, blue and green lines corresponding to $\gamma = 62^\circ$ and 66° intersect the curve representing the boundary attenuation angle

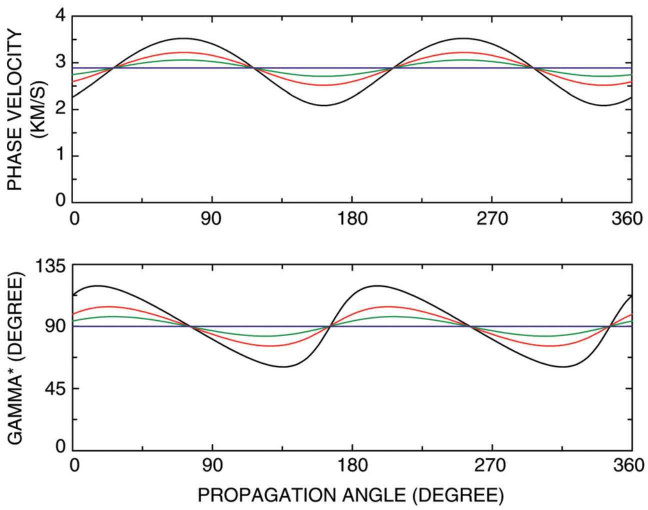

Figure 8. Effects of anisotropy on the phase velocity $\mathcal{C}$ (top) and the boundary attenuation angle γ^* (bottom) of inhomogeneous SH plane waves in symmetry planes of viscoelastic monoclinic media using mixed specification. The $\mathcal{C}$ curves are plotted for $D = 0.2$; the γ^* curve does not depend on D. The colored curves correspond to models with strongest dissipation and decreasing anisotropy, black — R1I1, red — R2I1, green — R3I1, and blue — R4I1 (isotropic); see Table 1.

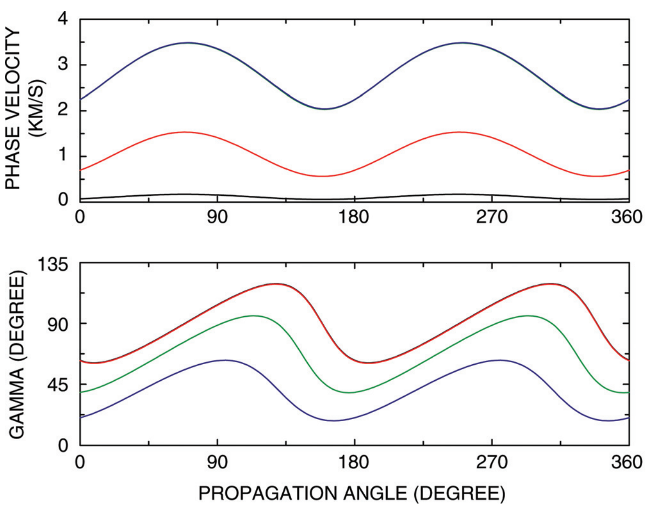

Figure 7. Variations with propagation angle of the phase velocity $\mathcal{C}$ (top) and the attenuation angle γ (bottom) of inhomogeneous SH plane waves in symmetry planes of the viscoelastic monoclinic medium R1I2 (see Table 1) using mixed specification. The colored curves correspond to negative values of D, $D = -10$ — black, $D = -1$ — red, $D = -0.03$ — green, and $D = -0.01$ — blue. Variation of γ and γ^* with propagation angle has a different character than in Fig. 6.

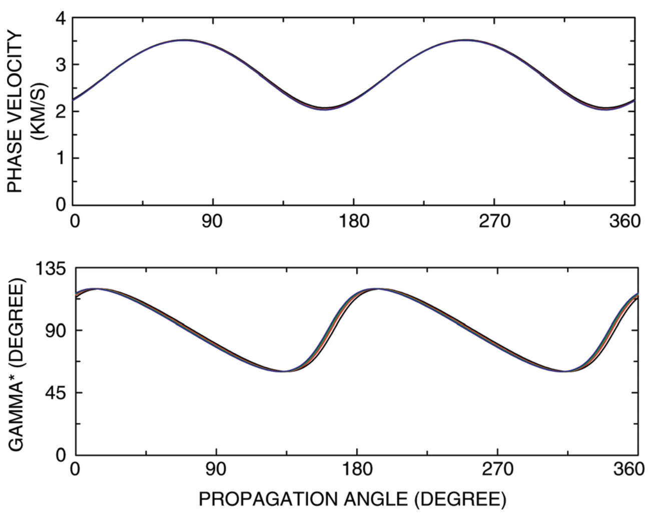

Figure 9. Effects of dissipation on the phase velocity $\mathcal{C}$ (top) and the boundary attenuation angle γ^* (bottom) of inhomogeneous SH plane waves in symmetry planes of viscoelastic monoclinic media using mixed specification. The $\mathcal{C}$ curves are plotted for $D = 0.2$, the γ^* curve does not depend on D. The colored curves correspond to models with strongest anisotropy and decreasing dissipation, black — R1I1, red — R1I2, green — R1I3, and blue — R1I4 (perfectly elastic); see Table 1.

γ^* and enter the region $\gamma > \gamma^*$, in which the attenuation angles should not be chosen. When $\gamma = \gamma^*$, the phase velocity is zero, and for $\gamma > \gamma^*$ the phase velocity $\mathcal{C}$ is complex-valued. It should, however, be emphasized that the phase velocity $\mathcal{C}$ behaves anomalously not only for $\gamma > \gamma^*$, but also for $\gamma < \gamma^*$, for which $\gamma \sim \gamma^*$. All this indicates that the use of the attenuation angle γ as a parameter, which can be freely chosen when specifying the slowness vector, is not desirable. The above situation can be avoided if the mixed specification of the slowness vector is used. The mixed specification yields automatically only angles γ, which always satisfy $\gamma \leq \gamma^*$.

ENERGY-ATTENUATION ANGLE AND ENERGY ANGLE

In anisotropic viscoelastic media, an important role is played by the time-harmonic, complex valued energy flux vector, also called the complex-valued Poynting vector. The real part of it represents the time-averaged energy flux vector **S**. The direction of **S** corresponds to the direction of the ray. Suitable expressions for the time-averaged energy-flux vector **S** are given in Červený and Pšenčík (2006). They are valid for anisotropic or isotropic, viscoelastic or perfectly elastic media. Here we are interested in the nonoriented and nonnegative angle γ_{en} between the energy flux vector **S** and the attenuation vector **A**, called the energy-attenuation angle. We define it by the relation

$$\cos\gamma_{en} = \mathbf{A}\cdot\mathbf{S}/|\mathbf{A}||\mathbf{S}|. \quad (43)$$

It was shown by Červený and Pšenčík (2006) that $\cos\gamma_{en}$ is always nonnegative,

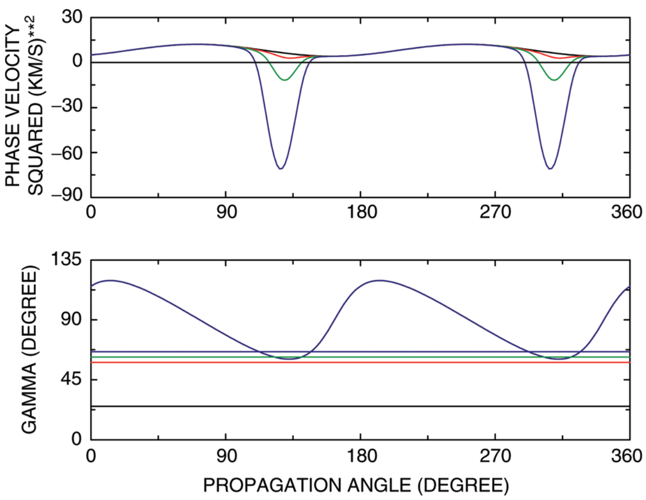

Figure 10. Variations with propagation angle i of the squares of the phase velocity $\mathcal{C}$ (top) of inhomogeneous SH plane waves in symmetry planes of viscoelastic monoclinic media, calculated by directional specification. The attenuation angles γ in the bottom plot, independent of i ($\gamma = 25°$ — black, $\gamma = 58°$ — red, $\gamma = 62°$ — green, and $\gamma = 66°$ — blue) are the attenuation angles, for which $\mathcal{C}^2$ in the upper plot are calculated. The oscillating blue curve in the bottom plot is the boundary attenuation angle γ^* calculated by mixed specification. Note that green and blue $\mathcal{C}^2$ curves intersect zero, defining thus forbidden directions. The forbidden directions in the top plot correspond to the regions where $\gamma > \gamma^*$ in the bottom plot.

$$\cos\gamma_{en} = \frac{1}{2}W_d/|\mathbf{A}||\mathbf{S}| \geq 0. \quad (44)$$

Here $W_d \geq 0$ is the time-averaged dissipated energy density. The inhomogeneous plane wave, for which **S** and **A** are parallel ($\gamma_{en} = 0°$), is called the intrinsic inhomogeneous plane wave (Deschamps and Assouline, 2000). It is of interest to note that the attenuation is minimal for an intrinsic inhomogeneous plane wave, not for a homogeneous wave.

In Figure 11, we show the plots of the energy-attenuation angle γ_{en} for the models used for plots of the attenuation angle γ and the phase velocity $\mathcal{C}$ in Figure 5. Note that for SH waves propagating in a plane of symmetry of a monoclinic viscoelastic medium, the time-averaged energy-flux vector **S** is also situated in the plane of symmetry. This is, of course, not true generally. For general anisotropy, the vector **S** may point out of the propagation-attenuation plane. There are several important differences between plots of the energy-attenuation angle γ_{en}, shown in Figure 11, and plots of the attenuation angle γ, shown in Figure 5. First, the energy-attenuation angle γ_{en} never exceeds 90°. This clearly explains the variability of the boundary attenuation angle γ^* in anisotropic dissipative media. As the boundary energy-attenuation angle γ^*_{en} is always 90°, the attenuation vector **A** is perpendicular to the energy-flux vector **S** for $\gamma_{en} = \gamma^*_{en}$. However, the direction of the vector **S** is generally different from the direction of the propagation vector **P**. Consequently, the relevant attenuation vector **A**, perpendicular to the vector **S**, cannot be simultaneously perpendicular to the propagation vector **P**, and thus γ^* must be different from 90°. This also implies that the amplitudes never increase exponentially in the direction of energy flux, but they may increase exponentially in the direction of the propagation vector **P** (Červený and Pšenčík, 2006). Second, the energy-attenuation angle γ_{en}, as a function of the inhomogeneity parameter D, vanishes for $D = D_{att}$, where $D_{att} \neq 0$ generally. This inhomogeneity parameter D_{att} corresponds to the intrinsic inhomogeneous wave. Third, for $D = 0$, the energy-attenuation angles γ_{en} in Figure 11 are the same for all three models R1I1, R1I2, and R1I3 (with an accuracy of the curve thickness). Thus, the energy-attenuation angle for $D = 0$, $\gamma_{en}(D = 0)$, can be used to estimate the boundary attenuation angles $\gamma^*_{1,2}$: $\gamma^*_{1,2} = 90° \pm \gamma_{en}(D = 0)$. In fact, $\gamma_{en}(D = 0)$ represents an angle between energy-flux vector **S** and the propagation vector **P** because $\mathbf{P} \parallel \mathbf{A}$ for $D = 0$. This can be shown in Figures 5 and 11. Figure 11 shows that $\gamma_{en}(D = 0) \sim 29°$ so that $\gamma^*_1 \sim 61°$ and $\gamma^*_2 \sim 119°$ (see Figure 5).

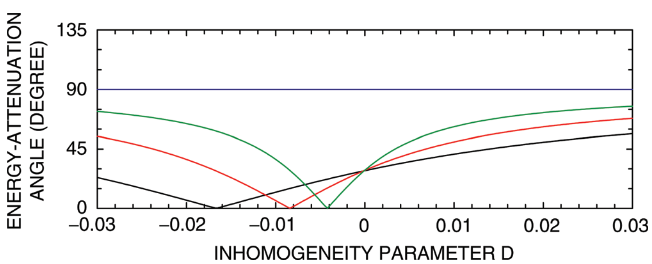

Figure 11. Variations with D $(-0.03 \leq D \leq 0.03)$ of energy-attenuation angle γ_{en} (degrees) of the SH inhomogeneous/homogeneous plane waves. The colored curves correspond to models with strongest anisotropy and varying dissipation, black — R1I1, red — R1I2, green — R1I3, and blue — R1I4 (perfectly elastic); see Table 1. Propagation angle $i = 135°$.

The most straightforward estimate of the difference between the boundary attenuation angle γ^* and 90°, can be obtained from the energy angle i_{en} (Červený and Pšenčík, 2006):

$$\cos i_{en} = \mathbf{P} \cdot \mathbf{S} / |\mathbf{P}||\mathbf{S}|. \quad (45)$$

The angle i_{en} between $\mathbf{P}$ and $\mathbf{S}$ is the angle by which γ^* differs from 90°. Obviously $i_{en} = \gamma_{en}(D=0)$ and we can estimate the boundary attenuation angles using the relation:

$$\gamma^*_{1,2} = 90^\circ \pm i_{en}. \quad (46)$$

Because $i_{en} = \gamma_{en}(D=0)$, we have $i_{en} \sim 29^\circ$ in Figure 11.

CONCLUSIONS

Most of the expressions derived in this paper are exact and valid for inhomogeneous and homogeneous plane waves, propagating in homogeneous isotropic or anisotropic, viscoelastic or perfectly elastic media. Thus, they are valid, without any change, even for evanescent waves, propagating in perfectly elastic media. They allow us to compute the propagation and attenuation vectors $\mathbf{P}$ and $\mathbf{A}$, the phase velocity $\mathcal{C}$, the attenuation angle γ, and the boundary attenuation angle γ^*. The formulae make possible a smooth and continuous transition from inhomogeneous plane waves propagating in weakly dissipative anisotropic media to evanescent waves propagating in perfectly elastic anisotropic media.

It is shown that the choice of the attenuation angle γ as a free parameter of inhomogeneous plane waves is not suitable. In anisotropic viscoelastic media, it may yield the phenomenon of forbidden directions, for which the square of the phase velocity is negative. Inaccurate results for the phase velocity are, however, obtained not only in the forbidden direction region (for $\gamma > \gamma^*$), but also for $\gamma < \gamma^*$ with $\gamma \sim \gamma^*$. In isotropic media, the cases of perfectly elastic and viscoelastic media must be treated separately. The separate treatment leads to the conclusion that the plane waves with the attenuation angle 90° cannot propagate in viscoelastic media, but can propagate in perfectly elastic media. The choice of the mixed specification of the slowness vector, in which the inhomogeneity parameter D is used to control the inhomogeneity of the plane wave under consideration (and not the attenuation angle γ), avoids all these problems. We show that the plane waves propagating in viscoelastic or perfectly elastic media can be treated by the same universal algorithm, and that the plane wave always represents a nonpropagating mode ($\mathcal{C}=0$) for the boundary attenuation angle $\gamma^* = 90^\circ$.

The boundary attenuation angle γ^* in isotropic media is always 90°, independent of viscoelastic moduli and of the choice of vectors $\mathbf{n}$ and $\mathbf{m}$. In anisotropic viscoelastic media, however, the situation is different. The boundary attenuation angle γ^* may be greater than, equal to, or less than 90°. It depends on the viscoelastic moduli (mostly on their real parts) and on the choice of vectors $\mathbf{n}$ and $\mathbf{m}$. For fixed orientation of the vector $\mathbf{m}$ and fixed D, it is different for inhomogeneous plane waves propagating in the positive and negative $\mathbf{n}$ direction.

The phase velocity $\mathcal{C}$ of an inhomogeneous plane wave propagating in an isotropic viscoelastic medium depends on the inhomogeneity parameter D. It is always less than the phase velocity of the homogeneous plane wave ($D=0$) propagating in the same medium. It is the same for the wave propagating in the positive and negative $\mathbf{n}$ directions. In anisotropic viscoelastic media, however, the phase velocity of an inhomogeneous plane wave may be greater than the phase velocity of the relevant homogeneous plane wave. The phase velocity $\mathcal{C}$ is different for the plane waves propagating in the positive and negative $\mathbf{n}$ direction. In isotropic and anisotropic viscoelastic media, the phase velocity decreases from its maximum value and approaches zero for $D \to \pm\infty$. Thus, the plane wave loses the character of a propagating wave for $D \to \pm\infty$.

It was shown that the deviations of boundary attenuation angles from 90° are caused by the deviations of energy-flux vectors from propagation vectors in anisotropic media. Maximum deviation of attenuation vector $\mathbf{A}$ from the energy-flux vector $\mathbf{S}$ is 90°. But the deviation of the attenuation vector $\mathbf{A}$ from the propagation vector $\mathbf{P}$ may be less or greater than 90°.

This paper addresses only plane waves propagating in homogeneous media. The waves generated by point sources in homogeneous viscoelastic media are not considered here. The waves generated by point sources are always homogeneous in homogeneous isotropic viscoelastic media, but may be inhomogeneous in homogeneous anisotropic viscoelastic media.

For inhomogeneous anisotropic viscoelastic media, the equations for plane waves propagating in homogeneous media may be applied only locally, for high-frequency seismic waves. Moreover, these equations are only approximate. Gajewski and Pšenčík (1992) used an approach based on a modification of the ray-series method to compute high-frequency seismic body waves propagating in inhomogeneous anisotropic weakly dissipative media, in which the dissipation is considered as a perturbation. It follows from their treatment that the attenuation coefficient along the ray is independent of the attenuation angle γ, at least in the first-order approximation. Ray perturbation theory gives an analogous result (Červený, 2001, section 5.5.3). For more detailed discussions see also Červený and Pšenčík (2008), and Behura and Tsvankin (2009). A powerful method to investigate the properties of high-frequency seismic body waves propagating in an inhomogeneous anisotropic dissipative media is based on the so-called *perturbation Hamiltonians* (Červený and Pšenčík, 2009). The latter paper also presents many other relevant references.

ACKNOWLEDGMENTS

The authors are grateful to Pat Daley and anonymous reviewers for their useful comments. The research was supported by the Seismic Waves in Complex 3D Structures Consortium Project, by research projects P210/11/0117 and P210/10/0736 of the Grant Agency of the Czech Republic, and by research project MSM 0021620860 of the Ministry of Education of the Czech Republic.

REFERENCES

Behura, I., and I. Tsvankin, 2009, Role of the inhomogeneity angle in anisotropic attenuation analysis: Geophysics, **74**, no. 5, WB177–WB191, doi:10.1190/1.3148439.

Borcherdt, R. D., 2009. Viscoelastic waves in layered media: Cambridge University Press.

Carcione, J. M., 2007, Wave fields in real media: Wave propagation in anisotropic, anelastic, porous and electromagnetic media: Elsevier Science.

Carcione, J. M., and F. Cavallini, 1995, Forbidden directions for inhomogeneous pure shear waves in dissipative anisotropic media: Geophysics, **60**, 522–530, doi:10.1190/1.1443789.

Červený, V., 2001, Seismic ray theory: Cambridge University Press.

Červený, V., and I. Pšenčík, 2005a, Plane waves in viscoelastic anisotropic media. I—Theory: Geophysical Journal International, **161**, no. 1, 197–212, doi:10.1111/j.1365-246X.2005.02589.x.
Červený, V., and I. Pšenčík, 2005b, Plane waves in viscoelastic anisotropic media: II—Numerical examples: Geophysical Journal International, **161**, no. 1, 213–229, doi:10.1111/j.1365-246X.2005.02590.x.
Červený, V., and I. Pšenčík, 2006, Energy flux in viscoelastic anisotropic media: Geophysical Journal International, **166**, no. 3, 1299–1317, doi:10.1111/j.1365-246X.2006.03057.x.
Červený, V., and I. Pšenčík, 2008, Quality factor Q in dissipative anisotropic media: Geophysics, **73**, no. 4, T63–T75, doi:10.1190/1.2937173.
Červený, V., and I. Pšenčík, 2009, Perturbation Hamiltonians in heterogeneous anisotropic weakly dissipative media: Geophysical Journal International, **178**, no. 2, 939–949, doi:10.1111/j.1365-246X.2009.04218.x.
Deschamps, M., and F. Assouline, 2000, Attenuation along the Poynting vector direction of inhomogeneous plane waves in absorbing and anisotropic solids: Acta Acustica united with Acustica, **86**, 295–302.
Gajewski, D., and I. Pšenčík, 1992, Vector wavefield for weakly attenuating anisotropic media by the ray method: Geophysics, **57**, 27–38, doi:10.1190/1.1443186.
Krebes, E. S., and L. H. T. Le, 1994, Inhomogeneous plane waves and cylindrical waves in anisotropic anelastic media: Journal of Geophysical Research, **99**, B12, 23899–23919, doi:10.1029/94JB02126.