REPRESENTATION THEOREM FOR VISCOELASTIC WAVES WITH A NON-SYMMETRIC STIFFNESS MATRIX

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The $3\times3\times3\times3$ complex–valued frequency–domain stiffness tensor (elastic tensor, tensor of elastic moduli) $c^{ijkl} = c^{ijkl}(x^m,\omega)$ is symmetric with respect to the first pair of indices, $c^{ijkl} = c^{jikl}$, and with respect to the second pair of indices $c^{ijkl} = c^{ijlk}$. It is thus frequently expressed in the form of the 6×6 stiffness matrix which lines correspond to the first pair of indices and columns to the second pair of indices.

In an elastic medium, it was proved that the stiffness tensor is symmetric with respect to the exchange of the first pair of indices and the second pair of indices, $c^{ijkl} = c^{klij}$. The 6×6 stiffness matrix is thus symmetric in an elastic medium.

However, the above mentioned proof does not apply to a viscoelastic medium. Analogously to [5], we thus consider viscoelastic waves with a *non–symmetric stiffness matrix*, $c^{ijkl} \neq c^{klij}$. The *frequency–domain ray theory* in question is described in [4]. Here we derive the corresponding representation theorem. Refer to [3] for more details. For the sake of better correspondence, the equations are labelled here according to [3].

Viscoelastodynamic equation in the frequency domain

The anisotropic viscoelastodynamic equation for the displacement in the frequency domain reads

$$\left[c^{ijkl}(\mathbf{x},\omega)\, u_{k,l}(\mathbf{x},\omega)\right]_{,j} + \omega^2 \varrho(\mathbf{x})\, u_i(\mathbf{x},\omega) + f^i(\mathbf{x},\omega) = 0 \quad . \tag{9}$$

The Einstein summation over repetitive lower–case Roman indices is used hereinafter. If the definition volume for viscoelastodynamic equation (9) is not infinite, we assume homogeneous boundary conditions according to [1], box 2.4.

The frequency–domain Green function for a viscoelastic medium is the solution of equation

$$\left[c^{ijkl}(\mathbf{x},\omega)\, G_{km,l}(\mathbf{x},\mathbf{x}',\omega)\right]_{,j} + \omega^2 \varrho(\mathbf{x})\, G_{im}(\mathbf{x},\mathbf{x}',\omega) + \delta^i_m\, \delta(\mathbf{x}-\mathbf{x}') = 0 \quad , \tag{10}$$

analytical with respect to the inverse Fourier transform. The partial derivatives are related to variable $\mathbf{x}$.

Representation theorem

Analogously to [2], eq. 12, we define *complementary medium* $\tilde{c}^{ijkl}(\mathbf{x},\omega) = c^{klij}(\mathbf{x},\omega)$ corresponding to the transposed stiffness matrix.

We define the frequency–domain *complementary Green function* $\tilde{G}_{km}(\mathbf{x},\mathbf{x}',\omega)$ as the frequency–domain Green function in the complementary medium,

$$\left[c^{klij}(\mathbf{x},\omega)\, \tilde{G}_{km,l}(\mathbf{x},\mathbf{x}',\omega)\right]_{,j} + \omega^2 \varrho(\mathbf{x})\, \tilde{G}_{im}(\mathbf{x},\mathbf{x}',\omega) + \delta^i_m\, \delta(\mathbf{x}-\mathbf{x}') = 0 \quad . \tag{13}$$

We first derive the *provisional representation theorem* as the relation between the frequency–domain wave field $u_i(\mathbf{x},\omega)$ in the given medium and the frequency–domain complementary Green function $\tilde{G}_{im}(\mathbf{x},\mathbf{x}',\omega)$. We consider volume V which is the subset of the definition volume for viscoelastodynamic equation (9) and need not contain the support of force density $f^i(\mathbf{x},\omega)$. We multiply equation

(13) for the frequency–domain complementary Green function by $u_i(\mathbf{x},\omega)$, subtract the product of the frequency–domain viscoelastodynamic equation (9) with $\tilde{G}_{im}(\mathbf{x},\mathbf{x}',\omega)$, integrate over volume V, and after simple conversions arrive at [3]

$$u_m(\mathbf{x}',\omega) = \int_V \mathrm{d}^3\mathbf{x} \left\{ \tilde{G}_{im}(\mathbf{x},\mathbf{x}',\omega)\, f^i(\mathbf{x},\omega) - \left[\tilde{G}_{im,j}(\mathbf{x},\mathbf{x}',\omega)\, c^{ijkl}(\mathbf{x},\omega)\, u_k(\mathbf{x},\omega)\right]_{,l} + \left[\tilde{G}_{im}(\mathbf{x},\mathbf{x}',\omega)\, c^{ijkl}(\mathbf{x},\omega)\, u_{k,l}(\mathbf{x},\omega)\right]_{,j} \right\} \quad . \tag{16}$$

We apply the divergence theorem to the integral of the gradients and obtain the provisional representation theorem [3],

$$u_m(\mathbf{x}',\omega) = \int_V \mathrm{d}^3\mathbf{x}\; \tilde{G}_{im}(\mathbf{x},\mathbf{x}',\omega)\, f^i(\mathbf{x},\omega) + \oint_{\partial V} \mathrm{d}^2\mathbf{x} \left[\tilde{G}_{im}(\mathbf{x},\mathbf{x}',\omega)\, n_j(\mathbf{x})\, c^{ijkl}(\mathbf{x},\omega)\, u_{k,l}(\mathbf{x},\omega) - \tilde{G}_{im,j}(\mathbf{x},\mathbf{x}',\omega)\, c^{ijkl}(\mathbf{x},\omega)\, u_k(\mathbf{x},\omega)\, n_l(\mathbf{x}) \right] , \tag{17}$$

where $n_i(\mathbf{x})$ is the unit normal to the surface ∂V of volume V pointing outside volume V.

For $f^i(\mathbf{x},\omega) = \delta^i_n\, \delta(\mathbf{x}-\mathbf{x}'')$, the above equation yields $u_m(\mathbf{x}',\omega) = G_{mn}(\mathbf{x}',\mathbf{x}'',\omega)$. Integrating over the whole definition volume, we obtain *reciprocity relation*

$$G_{mn}(\mathbf{x}',\mathbf{x}'',\omega) = \tilde{G}_{nm}(\mathbf{x}'',\mathbf{x}',\omega) \tag{18}$$

between the frequency–domain Green function and the frequency–domain complementary Green function.

We now insert reciprocity relation (18) into provisional representation theorem (17), and obtain the final version of the *representation theorem*:

$$u_m(\mathbf{x}',\omega) = \int_V \mathrm{d}^3\mathbf{x}\; G_{mi}(\mathbf{x}',\mathbf{x},\omega)\, f^i(\mathbf{x},\omega) + \oint_{\partial V} \mathrm{d}^2\mathbf{x} \left[G_{mi}(\mathbf{x}',\mathbf{x},\omega)\, n_j(\mathbf{x})\, c^{ijkl}(\mathbf{x},\omega)\, u_{k,l}(\mathbf{x},\omega) - G_{mi,j}(\mathbf{x}',\mathbf{x},\omega)\, c^{ijkl}(\mathbf{x},\omega)\, u_k(\mathbf{x},\omega)\, n_l(\mathbf{x}) \right] . \tag{19}$$

The integral over volume V represents the wave field corresponding to the sources situated inside volume V. The integral over the surface ∂V of volume V represents the wave field corresponding to the sources situated outside volume V, and is zero if all sources are situated inside volume V.

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