Representation theorem for viscoelastic waves with a non-symmetric stiffness matrix

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Received: September 7, 2020; Revised: November 27, 2020; Accepted: December 31, 2020

ABSTRACT

In an elastic medium, it was proved that the stiffness tensor is symmetric with respect to the exchange of the first pair of indices and the second pair of indices, but the proof does not apply to a viscoelastic medium. In this paper, we thus derive the representation theorem for viscoelastic waves in a medium with a non–symmetric stiffness matrix. The representation theorem expresses the wave field at a receiver, situated inside a subset of the definition volume of the viscoelastodynamic equation, in terms of the volume integral over the subset and the surface integral over the boundary of the subset. For the given medium, we define the complementary medium corresponding to the transposed stiffness matrix. We define the frequency–domain complementary Green function as the frequency–domain Green function in the complementary medium. We then derive the provisional representation theorem as the relation between the frequency–domain wave field in the given medium and the frequency–domain complementary Green function. This provisional representation theorem yields the reciprocity relation between the frequency–domain Green function and the frequency–domain complementary Green function. The final version of the representation theorem is then obtained by inserting the reciprocity relation into the provisional representation theorem.

Keywords: anisotropic viscoelastic media, stiffness tensor, wave propagation, Green function, representation theorem, reciprocity relation

1. INTRODUCTION

The $3\times3\times3\times3$ complex–valued frequency–domain stiffness tensor (viscoelastic tensor, tensor of viscoelastic moduli) $c^{ijkl} = c^{ijkl}(x^m,\omega)$, projecting the strain tensor onto the stress tensor, depends on spatial coordinates x^m and circular frequency ω. It is symmetric with respect to the first pair of indices

$$c^{ijkl} = c^{jikl} \tag{1}$$

and with respect to the second pair of indices

$$c^{ijkl} = c^{ijlk} \quad . \tag{2}$$

It is thus frequently expressed in the form of the 6×6 stiffness matrix which lines correspond to the first pair of indices and columns to the second pair of indices.

In an elastic medium, it was proved that the stiffness tensor is symmetric with respect to the exchange of the first pair of indices and the second pair of indices,

$$c^{ijkl} = c^{klij} \quad . \tag{3}$$

The 6×6 stiffness matrix is thus symmetric in an elastic medium.

However, the above mentioned proof does not apply to a viscoelastic medium, and we do not know whether the symmetry of the stiffness matrix expressed by identity (3) holds in a viscoelastic medium. In a viscoelastic medium, identity (3) was proved in the low–frequency and high–frequency limits only (*Gurtin and Herrera, 1965, theorems 3.1 and 3.2*; *Christensen, 1971, Eqs 3.39–3.40*; *Fabrizio and Morro, 1988, corollaries 1 and 2; 1992, Eqs 3.28–3.29*; *Carcione, 2015, Eq. 2.24*), but not for finite frequencies (*Rogers and Pipkin, 1963*).

Analogously to *de Hoop (1995)* and *Thomson (1997)*, we thus consider viscoelastic waves with a *non–symmetric stiffness matrix*,

$$c^{ijkl} \neq c^{klij} \quad , \tag{4}$$

and derive the corresponding representation theorem. The *frequency–domain ray theory* in question has been proposed by *Klimeš (2018)*.

The lower–case Roman indices take values 1, 2 and 3. The Einstein summation over repetitive lower–case Roman indices is used throughout the paper.

2. GREEN FUNCTION

2.1. Elastodynamic equation in the time domain

The anisotropic elastodynamic equation for displacement $u_i(\boldsymbol{x},t)$ in the time domain reads

$$\left[c^{ijkl}(\boldsymbol{x})\, u_{k,l}(\boldsymbol{x},t)\right]_{,j} - \varrho(\boldsymbol{x})\, \ddot{u}_i(\boldsymbol{x},t) + f^i(\boldsymbol{x},t) = 0 \quad , \tag{5}$$

where $\varrho(\boldsymbol{x})$ is the density and $f^i(\boldsymbol{x},t)$ represents the force density at point $\boldsymbol{x}$. A subscript following a comma denotes the partial derivative with respect to the corresponding coordinate, a double dot denotes the second derivative with respect to time t. If the definition volume for elastodynamic equation (5) is not infinite, we assume homogeneous boundary conditions (*Aki and Richards, 1980, box 2.4*).

The 3×3 time–domain Green function $G_{im}(\boldsymbol{x},\boldsymbol{x}',t-t')$ for an elastic medium is the solution of equation

$$\left[c^{ijkl}(\boldsymbol{x})\, G_{km,l}(\boldsymbol{x},\boldsymbol{x}',t-t')\right]_{,j} - \varrho(\boldsymbol{x})\, \ddot{G}_{im}(\boldsymbol{x},\boldsymbol{x}',t-t') + \delta^i_m\, \delta(\boldsymbol{x}-\boldsymbol{x}')\, \delta(t-t') = 0 \tag{6}$$

with zero initial conditions for $t < t'$. The partial derivatives are related to variables $\boldsymbol{x}$ and t. Symbol δ^i_m is the Kronecker delta, $\delta(\boldsymbol{x}-\boldsymbol{x}')$ and $\delta(t-t')$ represent the 3–D and 1–D Dirac distributions, respectively.

Taking the scalar product of the equation for the Green function with $f^m(\boldsymbol{x}',t')$ and integrating over the subset V of the definition volume for elastodynamic equation (5) containing the support of $f^m(\boldsymbol{x}',t')$, we see that

$$u_i(\boldsymbol{x},t) = \int_V \mathrm{d}^3\boldsymbol{x}'\,\mathrm{d}t'\, G_{im}(\boldsymbol{x},\boldsymbol{x}',t{-}t')\, f^m(\boldsymbol{x}',t') \tag{7}$$

is the solution of the time–domain elastodynamic equation.

2.2. Viscoelastodynamic equation in the frequency domain

In order to avoid coefficient $(2\pi)^{-\frac{1}{2}}$ at the force density in the frequency–domain, we consider here Fourier transform (*Červený, 2001, Eq. A.1.2*)

$$u(\omega) = \int_{-\infty}^{+\infty} \mathrm{d}t\; u(t)\,\exp(\mathrm{i}\omega t) \tag{8}$$

without coefficient $(2\pi)^{-\frac{1}{2}}$.

The anisotropic viscoelastodynamic equation for the displacement in the frequency domain reads

$$\left[c^{ijkl}(\boldsymbol{x},\omega)\,u_{k,l}(\boldsymbol{x},\omega)\right]_{,j} + \omega^2\varrho(\boldsymbol{x})\,u_i(\boldsymbol{x},\omega) + f^i(\boldsymbol{x},\omega) = 0 \quad . \tag{9}$$

If the definition volume for viscoelastodynamic equation (9) is not infinite, we assume homogeneous boundary conditions (*Aki and Richards, 1980, box 2.4*).

The frequency–domain Green function for a viscoelastic medium is the solution of equation

$$\left[c^{ijkl}(\boldsymbol{x},\omega)\,G_{km,l}(\boldsymbol{x},\boldsymbol{x}',\omega)\right]_{,j} + \omega^2\varrho(\boldsymbol{x})\,G_{im}(\boldsymbol{x},\boldsymbol{x}',\omega) + \delta^i_m\,\delta(\boldsymbol{x}{-}\boldsymbol{x}') = 0 \quad , \tag{10}$$

analytical with respect to the inverse Fourier transform. The partial derivatives are related to variable $\boldsymbol{x}$.

Taking the scalar product of Eq. (10) for the frequency–domain Green function with $f^m(\boldsymbol{x}',\omega)$ and integrating over the subset V of the definition volume for viscoelastodynamic equation (9) containing the support of $f^m(\boldsymbol{x}',\omega)$, we see that

$$u_i(\boldsymbol{x},\omega) = \int_V \mathrm{d}^3\boldsymbol{x}'\; G_{im}(\boldsymbol{x},\boldsymbol{x}',\omega)\, f^m(\boldsymbol{x}',\omega) \tag{11}$$

is the solution of the frequency–domain viscoelastodynamic equation.

3. REPRESENTATION THEOREM

Analogously to *Kamenetskii (2001, Eq. 12)*, we define *complementary medium* $\tilde{c}^{ijkl}(\boldsymbol{x},\omega)$ as

$$\tilde{c}^{ijkl}(\boldsymbol{x},\omega) = c^{klij}(\boldsymbol{x},\omega) \quad . \tag{12}$$

We define the frequency–domain *complementary Green function* $\tilde{G}_{km}(\boldsymbol{x},\boldsymbol{x}',\omega)$ as the frequency–domain Green function in the complementary medium,

$$\left[c^{klij}(\boldsymbol{x},\omega)\,\tilde{G}_{km,l}(\boldsymbol{x},\boldsymbol{x}',\omega)\right]_{,j} + \omega^2\varrho(\boldsymbol{x})\,\tilde{G}_{im}(\boldsymbol{x},\boldsymbol{x}',\omega) + \delta^i_m\,\delta(\boldsymbol{x}{-}\boldsymbol{x}') = 0 \quad . \tag{13}$$

We consider volume V which is the subset of the definition volume for viscoelastodynamic equation (9) containing point $\boldsymbol{x}'$, and need not contain the support of force density $f^i(\boldsymbol{x},\omega)$, see Fig. 1. We multiply Eq. (13) for the frequency–domain complementary Green function by $u_i(\boldsymbol{x},\omega)$, subtract the product of the frequency–domain viscoelastodynamic equation (9) with $\tilde{G}_{im}(\boldsymbol{x},\boldsymbol{x}',\omega)$, and integrate over volume V containing point $\boldsymbol{x}'$,

$$\begin{aligned} u_m(\boldsymbol{x}',\omega) = \int_V \mathrm{d}^3\boldsymbol{x} \Big\{ & f^i(\boldsymbol{x},\omega)\, \tilde{G}_{im}(\boldsymbol{x},\boldsymbol{x}',\omega) \\ & - \left[c^{klij}(\boldsymbol{x},\omega)\, \tilde{G}_{km,l}(\boldsymbol{x},\boldsymbol{x}',\omega)\right]_{,j} u_i(\boldsymbol{x},\omega) \\ & + \left[c^{ijkl}(\boldsymbol{x},\omega)\, u_{k,l}(\boldsymbol{x},\omega)\right]_{,j} \tilde{G}_{im}(\boldsymbol{x},\boldsymbol{x}',\omega) \Big\} \quad , \end{aligned} \tag{14}$$

which can be expressed as

$$\begin{aligned} u_m(\boldsymbol{x}',\omega) = \int_V \mathrm{d}^3\boldsymbol{x} \Big\{ & \tilde{G}_{im}(\boldsymbol{x},\boldsymbol{x}',\omega)\, f^i(\boldsymbol{x},\omega) \\ & - \left[\tilde{G}_{im,j}(\boldsymbol{x},\boldsymbol{x}',\omega)\, c^{ijkl}(\boldsymbol{x},\omega)\right]_{,l} u_k(\boldsymbol{x},\omega) \\ & + \tilde{G}_{im}(\boldsymbol{x},\boldsymbol{x}',\omega) \left[c^{ijkl}(\boldsymbol{x},\omega)\, u_{k,l}(\boldsymbol{x},\omega)\right]_{,j} \Big\} \quad , \end{aligned} \tag{15}$$

and finally as

$$\begin{aligned} u_m(\boldsymbol{x}',\omega) = \int_V \mathrm{d}^3\boldsymbol{x} \Big\{ & \tilde{G}_{im}(\boldsymbol{x},\boldsymbol{x}',\omega)\, f^i(\boldsymbol{x},\omega) \\ & - \left[\tilde{G}_{im,j}(\boldsymbol{x},\boldsymbol{x}',\omega)\, c^{ijkl}(\boldsymbol{x},\omega)\, u_k(\boldsymbol{x},\omega)\right]_{,l} \\ & + \left[\tilde{G}_{im}(\boldsymbol{x},\boldsymbol{x}',\omega)\, c^{ijkl}(\boldsymbol{x},\omega)\, u_{k,l}(\boldsymbol{x},\omega)\right]_{,j} \Big\} \quad . \end{aligned} \tag{16}$$

We apply the divergence theorem to the integral of the gradients and obtain the representation theorem in its provisional form

$$\begin{aligned} u_m(\boldsymbol{x}',\omega) = & \int_V \mathrm{d}^3\boldsymbol{x}\; \tilde{G}_{im}(\boldsymbol{x},\boldsymbol{x}',\omega)\, f^i(\boldsymbol{x},\omega) \\ & + \oint_{\partial V} \mathrm{d}^2\boldsymbol{x} \left[\tilde{G}_{im}(\boldsymbol{x},\boldsymbol{x}',\omega)\, n_j(\boldsymbol{x})\, c^{ijkl}(\boldsymbol{x},\omega)\, u_{k,l}(\boldsymbol{x},\omega) \right. \\ & \qquad \left. - \tilde{G}_{im,j}(\boldsymbol{x},\boldsymbol{x}',\omega)\, c^{ijkl}(\boldsymbol{x},\omega)\, u_k(\boldsymbol{x},\omega)\, n_l(\boldsymbol{x})\right] \quad , \end{aligned} \tag{17}$$

where $n_i(\boldsymbol{x})$ are the components of the unit normal to the boundary ∂V of volume V pointing outside volume V.

For $f^i(\boldsymbol{x},\omega) = \delta^i_n\, \delta(\boldsymbol{x}–\boldsymbol{x}'')$, the above equation yields $u_m(\boldsymbol{x}',\omega) = G_{mn}(\boldsymbol{x}',\boldsymbol{x}'',\omega)$. Integrating over the whole definition volume, we obtain *reciprocity relation*

$$G_{mn}(\boldsymbol{x}',\boldsymbol{x}'',\omega) = \tilde{G}_{nm}(\boldsymbol{x}'',\boldsymbol{x}',\omega) \quad . \tag{18}$$

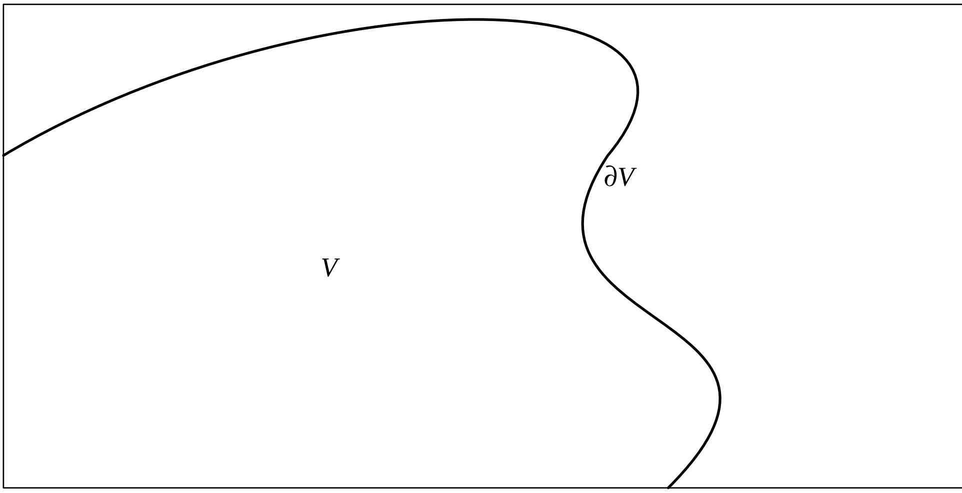

Fig. 1. Receiver point x' is situated inside volume V, which is a finite or infinite subset of the definition volume and has possible boundary ∂V. Sources may be located both inside or outside volume V. The integral over volume V in representation theorem (19) represents the wave field generated by the sources situated inside V, and vanish for the sources situated outside V. The integral over boundary ∂V in representation theorem (19) represents the wave field generated by the sources situated outside V, and vanish for the sources situated inside V. For example, if a ray from a source situated outside V intersects boundary ∂V three times, it contributes to the integral over boundary ∂V twice with a positive sign and once with a negative sign.

We insert this reciprocity relation into the above provisional form (17) of the representation theorem and obtain the final version of the *representation theorem*:

$$\boxed{\begin{aligned} u_m(\boldsymbol{x}',\omega) = & \int_V \mathrm{d}^3\boldsymbol{x}\; G_{mi}(\boldsymbol{x}',\boldsymbol{x},\omega)\, f^i(\boldsymbol{x},\omega) \\ & + \oint_{\partial V} \mathrm{d}^2\boldsymbol{x}\, \Big[G_{mi}(\boldsymbol{x}',\boldsymbol{x},\omega)\, n_j(\boldsymbol{x})\, c^{ijkl}(\boldsymbol{x},\omega)\, u_{k,l}(\boldsymbol{x},\omega) \\ & \qquad - G_{mi,j}(\boldsymbol{x}',\boldsymbol{x},\omega)\, c^{ijkl}(\boldsymbol{x},\omega)\, u_k(\boldsymbol{x},\omega)\, n_l(\boldsymbol{x}) \Big] \quad . \end{aligned}} \tag{19}$$

The integral over volume V represents the wave field corresponding to the sources situated inside volume V. The integral over the boundary ∂V of volume V represents the wave field corresponding to the sources situated outside volume V, and is zero if all sources are situated inside volume V, see Fig. 1.

Acknowledgements: The suggestions by Alexey Stovas and an anonymous reviewer made it possible for me to improve the paper. The research has been supported by the Czech Science Foundation under contract 20-06887S, and by the members of the consortium “Seismic Waves in Complex 3–D Structures” (see “http://sw3d.cz”).

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