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Discretization error for the superposition of Gaussian beams

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Summary. A high-frequency asymptotic integral expansion of a time-harmonic wavefield into Gaussian beams was derived in a previous paper by Klimeš. The discretization error caused by replacing this integral superposition by a discrete summation of Gaussian beams is estimated in this paper.

Key words: summation of Gaussian beams, discretization error

1 Introduction

A wavefield specified on an initial surface in terms of a complex-valued amplitude and phase was asymptotically expanded into Gaussian beams in Klimeš (1984). This expansion has the form of an integral superposition, which is commonly replaced by the summation of separate Gaussian beams in practical computations. In this paper we shall estimate the error caused by this replacement. The reader should be familiar with the Gaussian beam method. An ample list of references to Gaussian beams is given by Červený (1985).

The asymptotic integral expansion, briefly recalled in Section 2, was derived under the following assumptions: (a) the specified amplitude is approximately constant and the phase may be replaced by a Taylor expansion up to second order in the spot-ellipse of any beam; (b) the quantities describing Gaussian beams vary very slightly through the regions corresponding to the spot-ellipses of individual Gaussian beams and they may, therefore, be considered approximately constant. The spot-ellipse is understood to be the region, where the amplitude of a beam does not decrease more than by a factor of e^{-1} relative to its maximum value on the central ray.

Before studying the discretization error, we shall assume the above mentioned conditions are satisfied to guarantee sufficient accuracy in the integral expansion. In addition, we shall suppose that the steps in the ray parameters of the central rays of the beams are approximately constant in the spot-ellipses. There is no reason to use rapidly varying steps with slowly varying beam quantities.

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Note that the expansion assumptions need not be necessarily satisfied directly on the initial surface. Since the weighting function in the superposition is independent of the position along a fixed ray, the given wavefield may be extended by the ray method on to another surface on which the assumptions are fulfilled. Similarly, the existence of a surface, along which the integral superposition of beams may be replaced by the discrete with the estimated error, is a sufficient condition for the discrete superposition to be valid in all regions, where the Gaussian beams continue to be good approximations to the solutions of the elastodynamic equations. In view of these considerations, we shall not be surprised that the resulting expressions for the estimated error are independent of the position along a fixed ray.

In Section 3 we shall try to elucidate what may be considered 'slightly varying' or 'approximately constant' with regard to replacing an integral by a summation. Reading Section 3 is not important for understanding the following sections.

The discretization error will be studied first in Section 4 for the case of the one-parametric superposition of Gaussian beams (e.g. 2-D computations), because in this case the evaluation of the error is relatively simple and makes the proposed method of estimating the discretization error clear. Next the discretization error for the case of the two-parametric superposition will be estimated in Section 5. The principles of the method will be the same as for the one-parametric superposition, but additional considerations and approximations must be performed to obtain the results in very simple form, similar to that for the one-parametric case.

In this paper, we shall use the Einstein summation convention for subscripts. The capital letter subscripts will take the values 1, 2; lower-case indices will take the values 1, 2, 3. 2×2 matrices with components A_{AB} will be denoted either by $\mathbf{A}$ or A_{AB}. 3×3 matrices will always be described by means of their components. The symbol A^{-1}_{AB} will indicate the components of the matrix inverse to $\mathbf{A}$. An asterisk will denote the complex conjugacy.

2 Expansion of a high-frequency time-harmonic wavefield into Gaussian beams

2.1 TWO-PARAMETRIC INTEGRAL SUPERPOSITION

The two-parametric integral superposition of Gaussian beams may be expressed in Cartesian coordinates as (Klimeš 1984, equation (77))

$$u_i(x_l^0) = \frac{\omega}{2\pi}\iint d\gamma_1\, d\gamma_2\, A_i^{\mathrm{R}}\, |\det \mathbf{Q}^{\mathrm{R}}|\sqrt{-\det(\mathbf{M}-\mathbf{M}^{\mathrm{R}})}$$

$$\times \exp\left[\!\left[i\,\omega\left[\tau + (x_j^0 - x_j)\,p_j + \frac{1}{2}(x_j^0 - x_j)(x_k^0 - x_k)\,H_{jm}\,H_{kn}M_{mn} - t\right]\right]\!\right], \tag{1}$$

where the integration is performed over all ray parameters γ_1, γ_2 and the square root $\sqrt{-\det(\mathbf{M}-\mathbf{M}^{\mathrm{R}})}$ is taken with a positive real part. The Cartesian coordinates of the receiver are denoted by x_j^0, ω is a positive angular frequency and t is the time. All the following quantities are taken at the endpoint x_j of the ray specified by the ray parameters γ_A: A_i^{R} is the complex-valued vectorial ray amplitude including the phase shift due to caustics; τ is the travel time; p_j are Cartesian components of the slowness vector; $|\det \mathbf{Q}^{\mathrm{R}}|$ is the square of the geometrical spreading expressed in terms of the matrix

$$Q^{\mathrm{R}}_{AB} = \frac{\partial q_A}{\partial \gamma_B} \tag{2}$$

of geometrical spreading, which is the matrix of transformation from ray parameters γ_A to the ray-centred coordinates q_A. The matrix

$$H_{jm} = \frac{\partial x_j}{\partial q_m} = \frac{\partial q_m}{\partial x_j} \tag{3}$$

is a unitary 3×3 matrix, the columns of which constitute the local vector basis of the ray-centred coordinate system expressed in Cartesian coordinates x_i. $\mathbf{M}^{\mathrm{R}}$ is the matrix of the second derivatives of the ray travel-time field in the ray-centred coordinate system. The 3×3 matrix

$$M_{ij} = \begin{pmatrix} M_{11}, & M_{12}, & -V_1 v^{-2} \\ M_{12}, & M_{22}, & -V_2 v^{-2} \\ -V_1 v^{-2}, & -V_2 v^{-2}, & -V_3 v^{-2} \end{pmatrix} \tag{4}$$

is composed of an arbitrary complex-valued symmetric 2×2 matrix $\mathbf{M}$ with a positive-definite imaginary part and of the slowness gradient

$$-V_i v^{-2} = H_{ji} \frac{\partial (v^{-1})}{\partial x_j} \tag{5}$$

expressed in the ray-centred coordinate system. The matrix $\mathbf{M}$ controls the shape of the Gaussian beams and its proper choice will be discussed elsewhere.

2.2 TWO-PARAMETRIC DISCRETE SUMMATION

If integral (1) is discretized at points $(\gamma_1(k_1), \gamma_2(k_2))$ of the ray-parameter surface, the summation of the separate Gaussian beams

$$u_i^{\mathrm{D}}(x_l^0) = \frac{\omega}{2\pi} \sum_{k_1} \sum_{k_2} G A_i^{\mathrm{R}} \, |\det \mathbf{Q}^{\mathrm{R}}| \sqrt{-\det(\mathbf{M} - \mathbf{M}^{\mathrm{R}})}$$
$$\times \exp [\![i\omega [\tau + (x_j^0 - x_j) p_j + \tfrac{1}{2} (x_j^0 - x_j)(x_k^0 - x_k) H_{jm} H_{kn} M_{mn} - t]]\!] \tag{6}$$

is obtained instead of (1). Here

$$G = G(k_1, k_2) = \iint_{\Gamma(k_1, k_2)} d\gamma_1 \, d\gamma_2 \tag{7}$$

is the area of the element $\Gamma(k_1, k_2)$ of the ray-parameter surface, corresponding to the beam along the central ray with parameters $\gamma_1(k_1)$, $\gamma_2(k_2)$. The quantities A_i^{R}, τ, p_k, $\mathbf{Q}^{\mathrm{R}}$, $\mathbf{M}^{\mathrm{R}}$, M_{ij}, H_{ij} are taken at the endpoint x_i of the ray with ray parameters $\gamma_1(k_1)$, $\gamma_2(k_2)$.

2.3 ONE-PARAMETRIC INTEGRAL SUPERPOSITION

The one-parametric integral superposition of ribbon Gaussian beams may be expressed in Cartesian coordinates as

$$u_i(x_l^0) = \sqrt{\frac{\omega}{2\pi}} \int d\gamma_1 \, A_i^{\mathrm{R}} \sqrt{-im}$$
$$\times \exp [\![i\omega [\tau + (x_j^0 - x_j) p_j + \tfrac{1}{2} (x_j^0 - x_j)(x_k^0 - x_k) H_{jm} H_{kn} M_{mn} - t]]\!], \tag{8}$$

where the integration is performed over some curve in the ray-parameter surface, parametrized by γ_1. The square root $\sqrt{-im}$ is again taken with a positive real part. The matrix

$$M_{mn} = \begin{pmatrix} M_{11}, & M_{12}, & -V_1 v^{-2} \\ M_{12}, & M_{22}, & -V_2 v^{-2} \\ -V_1 v^{-2}, & -V_2 v^{-2}, & -V_3 v^{-2} \end{pmatrix}, \tag{9}$$

where

$$M_{AB} = M^{\mathrm{R}}_{AB} + m(Q^{\mathrm{R}})^{-1}_{1A}\,(Q^{\mathrm{R}})^{-1}_{1B}, \tag{10}$$

is specified by any given complex number m with positive imaginary part. The complex number m is used to describe the shape of the ribbon Gaussian beams instead of the matrix **M** because, from (10), the components of matrix **M** are not independent.

Equation (8) is a generalization of 2-D Gaussian beam summation. Matrices $\mathbf{M}^{\mathrm{R}}$ and $\mathbf{Q}^{\mathrm{R}}$ are diagonal in the 2-D case, and (10) then simplifies to

$$M_{11} = M^R_{11} + m(Q^R_{11})^{-2}. \tag{11}$$

The other components of **M** do not depend on m in this case and do not appear in (8) for $x_2 = x_2^0$. That is why the quantities m and M_{11} may be used as alternatives for describing the shape of the Gaussian beams in the 2-D case. However, m is more convenient than M_{11} for describing beams in a general one-parametric superposition.

Note that all quantities used in this section are independent of the choice of γ_2.

2.4 ONE-PARAMETRIC DISCRETE SUMMATION

If integral (8) is discretized at points $\gamma_1(k)$ of the curve on the ray-parameter surface, the summation

$$u_i^{\mathrm{D}}(x_l^0) = \sqrt{\frac{\omega}{2\pi}} \sum_k \Delta\gamma_1\, A_i^{\mathrm{R}} \sqrt{-im}$$
$$\times \exp\left[\!\left[i\omega\left[\tau + (x_j^0 - x_j)\,p_j + \frac{1}{2}(x_j^0 - x_j)(x_k^0 - x_k)\,H_{jm}\,H_{kn}M_{mn} - t\right]\right]\!\right] \tag{12}$$

of ribbon Gaussian beams is obtained.

Here

$$\Delta\gamma_1 = \Delta\gamma_1(k) = \int_{\Gamma'(k)} d\gamma_1 \tag{13}$$

is the length of the element $\Gamma'(k)$ of the ray-parameter curve, corresponding to the beam along the central ray with parameter $\gamma_1(k)$. The quantities A_i^{R}, τ, p_i, H_{ij}, m, M_{ij} are taken at the endpoint x_i of the ray.

2.5 DISCRETIZATION ERROR

The relative average square of the error, over some region Σ, caused by replacing the integral superposition $u_i(x_j)$ of Gaussian beams by their discrete summation $u_i^{\mathrm{D}}(x_j)$ is

$$\overline{\delta^2} = \frac{\displaystyle\int_\Sigma |u_i(x_j) - u_i^{\mathrm{D}}(x_j)|^2\, d\Sigma}{\displaystyle\int_\Sigma |u_k(x_l)|^2\, d\Sigma}$$

or

$$\overline{\delta^2} = \frac{\int_\Sigma |u_i(x_j) - u_i^{\mathrm{D}}(x_j)|^2 \, d\Sigma}{\int_\Sigma |u_k^{\mathrm{D}}(x_l)|^2 \, d\Sigma}. \tag{14}$$

The region Σ is taken in this paper to be the image of the element Γ of the ray-parameter surface corresponding to one beam. Considering our assumptions of approximately constant quantities, this error will vary only slightly from beam to beam.

If $|u_i(x_j)|$ does not vary too much inside Σ, the maximum relative error

$$\delta_{\max} = \max_{x_m \in \Sigma} \left[\frac{|u_i(x_j) - u_i^{\mathrm{D}}(x_j)|}{|u_k(x_l)|} \right]$$

or

$$\delta_{\max} = \max_{x_m \in \Sigma} \left[\frac{|u_i(x_j) - u_i^{\mathrm{D}}(x_j)|}{|u_k^{\mathrm{D}}(x_l)|} \right] \tag{15}$$

may also be introduced.

3 Comment on the term 'slightly varying'

Assume given an integral of function $f(q)$ that

$$F = \int f(q) \, dq, \tag{16}$$

which may be equivalently expressed in terms of another coordinate γ as

$$F = \int f[q(\gamma)] \; Q(\gamma) \, d\gamma, \tag{17}$$

where

$$Q(\gamma) = \frac{\partial q}{\partial \gamma}(\gamma). \tag{18}$$

Now take the function $f(q)$ to be digitized at the points

$$q_k = q(\gamma_k). \tag{19}$$

The discretization of (16) yields

$$F_1 = \sum_k f_k \, [q_{k+1/2} - q_{k-1/2}], \tag{20}$$

where

$$f_k = f(q_k). \tag{21}$$

Here

$$q_{k \pm 1/2} = q(\gamma_{k \pm 1/2}) \tag{22}$$

are the boundary points between the separate elementary intervals. The corresponding discretization of (17) yields

$$F_2 = \sum_k f_k \, Q_k \, [\gamma_{k+1/2} - \gamma_{k-1/2}] , \tag{23}$$

where

$$Q_k = Q(\gamma_k). \tag{24}$$

The sum F_2 is generally different from F_1. The relative difference in each elementary interval $\langle \gamma_{k-1/2}, \gamma_{k+1/2} \rangle$ is

$$\delta = \frac{q_{k+1/2} - q_{k-1/2} - Q_k [\gamma_{k+1/2} - \gamma_{k-1/2}]}{Q_k [\gamma_{k+1/2} - \gamma_{k-1/2}]} . \tag{25}$$

Now assume the function $Q(\gamma)$ to be monotonic inside each elementary interval. Then

$$|q_{k\pm 1/2} - [\![q_k + Q_k [\gamma_{k\pm 1/2} - \gamma_k]]\!] | \leqq |Q_{k\pm 1/2} - Q_k| \, |\gamma_{k\pm 1/2} - \gamma_k|, \tag{26}$$

$$(Q_{k+1/2} - Q_k)(Q_k - Q_{k-1/2}) \geqq 0 \tag{27}$$

and, consequently,

$$\delta \leqq \frac{|(Q_{k+1/2} - Q_k) - (Q_k - Q_{k-1/2})|}{2\,|Q_k|} = \frac{|\Delta Q_k|}{2|Q_k|} , \tag{28}$$

where $|\Delta Q_k| \, |Q_k|^{-1}$ are the relative changes of the quantity Q at distances corresponding to the discretization step, related to the non-linear trend in Q.

Since the difference between two discretizations of the same integral may be presumed to be connected with the discretization error, we may call the quantity Q 'approximately constant for discretization' (or 'approximately linear for discretization') if its relative changes (or non-linear relative changes) over the discretization step are not more than twice the desired maximum relative error. The terms 'approximately constant, linear or quadratic functions inside spot-ellipse' may be understood similarly if the Gaussian beams are approximately transformed from one coordinate system to another.

Note that if the step between two subsequent beams is as large as possible, but sufficiently small in order to keep the discretization error admissible, the size of the step can be shown not to be very different from the dimensions of the spot-ellipse in most cases. That is why the assumptions for deriving the integral asymptotic expansion into Gaussian beams and for deriving the discretization error may be conformal.

4 Discretization error for a one-parametric superposition

We assume that the Gaussian beams are sufficiently narrow to be able to replace the Cartesian coordinates by a paraxial ray approximation in the ray coordinates γ_1, τ. The integral superposition (8) of Gaussian beams may then be expressed in the ray coordinates as

$$u_i(\gamma_1^0, \tau) = \sqrt{\frac{\omega}{2\pi}} \int d\gamma_1 \, A_i^{\mathrm{R}} \sqrt{-im} \, \exp [\![i\omega [\tau - t + \tfrac{1}{2} (\gamma_1^0 - \gamma_1)^2 \, m]]\!], \tag{29}$$

where γ_1 is the parameter of the ray passing through the receiver. The quantity (see 10)

$$m = Q_{A1}^{\mathrm{R}} \, (M_{AB} - M_{AB}^{\mathrm{R}}) \, Q_{B1}^{\mathrm{R}} , \tag{30}$$

or simply (see 11)

$$m = Q_{11}^{\mathrm{R}}\,(M_{11} - M_{11}^{\mathrm{R}})\,Q_{11}^{\mathrm{R}} \tag{31}$$

in the 2-D case, controls the shape of the Gaussian beams.

The corresponding discrete sum (12) of Gaussian beams may be written in the form

$$u_i^{\mathrm{D}}(\gamma_1^0, \tau) = \sqrt{\frac{\omega}{2\pi}} \sum_{k=-\infty}^{+\infty} \Delta\gamma_1\, A_i^{\mathrm{R}} \sqrt{-im}\, \exp\, [\![i\omega[\tau - t + \tfrac{1}{2}(k-\xi)^2\, S]]\!], \tag{32}$$

where

$$\xi = \gamma^0/\Delta\gamma, \tag{33}$$

and

$$S = (\Delta\gamma)^2 m. \tag{34}$$

If we suppose that the quantities A_i^{R}, m are approximately constant in the effective regions of the Gaussian beams, we can evaluate the superposition (29) as

$$u_i(\gamma_1^0, \tau) = A_i^{\mathrm{R}} \exp\, [i\omega(\tau - t)] \tag{35}$$

and the sum (32), which is a periodic function of ξ, as

$$u_i^{\mathrm{D}}(\gamma_1^0, \tau) = \sum_{n=-\infty}^{+\infty} a_i(n) \exp\,(2\pi i n \xi), \tag{36}$$

where the Fourier coefficients (see 32 and 36)

$$a_i(n) = \int_0^1 d\xi \exp\,(-2\pi i n \xi) \sqrt{\frac{\omega}{2\pi}} \sum_{k=-\infty}^{+\infty} \Delta\gamma\, A_i^{\mathrm{R}} \sqrt{-im}\, \exp\, [\![i\omega[\tau - t + \tfrac{1}{2}(k-\xi)^2\, S]]\!] \tag{37}$$

may be evaluated analytically:

$$\begin{aligned} a_i(n) &= \sqrt{\frac{\omega}{2\pi}} A_i^{\mathrm{R}} \sqrt{-iS}\, \exp\, [i\omega(\tau - t)] \int_0^1 d\xi \sum_{k=-\infty}^{+\infty} \exp\left[\frac{i\omega}{2}(k-\xi)^2 S + 2\pi i n\,(k-\xi)\right] \\ &= A_i^{\mathrm{R}} \exp\, [i\omega(\tau - t)] \sqrt{\frac{-i\omega S}{2\pi}} \int_{-\infty}^{+\infty} d\xi \exp\left(\frac{i\omega}{2}\,\xi^2 S - 2\pi i n \xi\right) \\ &= A_i^{\mathrm{R}} \exp\, [i\omega(\tau - t)] \sqrt{\frac{-i\omega S}{2\pi}} \int_{-\infty}^{+\infty} d\xi \exp\left[\frac{i\omega S}{2}\left(\xi - \frac{2\pi n}{\omega S}\right)^2 - \frac{2\pi^2 i}{\omega S}\, n^2\right] \\ &= A_i^{\mathrm{R}} \exp\, [i\omega(\tau - t)] \exp\left[-\frac{2\pi^2 i}{\omega S}\, n^2\right]. \end{aligned} \tag{38}$$

Introducing the quantity

$$b = \exp\left(-\frac{2\pi^2 i}{\omega S}\right), \tag{39}$$

the difference between the integral and discrete superposition may be written in the form (see 35, 36, 38)

$$u_i^{\mathrm{D}} - u_i = u_i \left[\sum_{n=-\infty}^{+\infty} b^{n^2} \exp(2\pi i n \xi) - 1 \right] = u_i \, 2 \sum_{n=1}^{+\infty} b^{n^2} \cos(2\pi i n \xi). \tag{40}$$

Instead of (40) we may write the approximate form

$$u_i^{\mathrm{D}} - u_i = u_i \, 2b \cos(2\pi i \xi), \tag{41}$$

because b must be small, and the quantities b^4, b^9, b^{16}, ... then need not be taken into account.

The relative average quadratic error (14) then reads

$$\overline{\delta^2} = 2\delta^2, \tag{42}$$

and the maximum relative error (15) is

$$\delta_{\max} = 2\delta, \tag{43}$$

where

$$\delta = |b| = \exp\left[-\frac{\pi^2 i}{\omega} \left(\frac{1}{S} - \frac{1}{S^*} \right) \right] = \exp\left(-\frac{2\pi^2}{\omega} \frac{\operatorname{Im} S}{S^* S} \right). \tag{44}$$

Equation (44) may be rewritten as

$$\delta = \exp(-\pi^2 \kappa^{-2}) \tag{45}$$

where the coefficient

$$\kappa^2 = (\Delta\gamma_1)^2 Z_{11} \tag{46}$$

is expressed in terms of the quantity

$$Z_{11} = \frac{\omega}{2} \frac{m^* m}{\operatorname{Im} m}. \tag{47}$$

Here m is defined by (30). In the 2-D case, (47) simplifies to (see 31)

$$Z_{11} = \frac{\omega}{2} (Q_{11}^{\mathrm{R}})^2 (M_{11} - M_{11}^{\mathrm{R}})^* (\operatorname{Im} M_{11})^{-1} (M_{11} - M_{11}^{\mathrm{R}}). \tag{48}$$

Since we have considered the element Γ' of size $\Delta\gamma$ to be placed symmetrically about the central ray of a beam in this section, the square $\Delta\gamma^2$ of the step is equal to the 12 times enlarged specific moment

$$G_{11} = 12 \, \Delta\gamma^{-1} \int_{\Gamma'} (\gamma_1 - \gamma_1^0)^2 \, d\gamma_1 \tag{49}$$

of the element Γ' with respect to the central ray $\gamma_1 = \gamma_1^0$ of the beam. Then (46) may be rewritten as

$$\kappa^2 = G_{11} Z_{11}, \tag{50}$$

which is more convenient for practical use than (46) which contains no information about the mutual position of the ray and the corresponding element Γ'.

Equations (42), (43), (45), (47), (49) and (50) yield an estimate of the discretization error for the one-parametric superposition of Gaussian beams. Note that the first term in

$$\kappa^2 = \frac{\omega}{2} G_{11} \; [\mathrm{Im}\, m + (\mathrm{Re}\, m)^2 \; (\mathrm{Im}\; m)^{-1}] \tag{51}$$

corresponds to Gaussian decay and the second to destructive interference. The discretization error caused by Gaussian decay is greater for larger $\mathrm{Im}\, m$, i.e. for narrower beams. On the contrary, the discretization error caused by the oscillations appearing instead of the desired destructive interference is reduced for larger $\mathrm{Im}\, m$, i.e. narrower beams. If there is no destructive interference ($\mathrm{Re}\, m = 0$), the coefficient κ has the meaning of the ratio of the step between two subsequent beams to their characteristic half-width, i.e. to the distance, where Gaussian decay reaches e^{-1}. In other words, without destructive interference, Gaussian decay over the step between subsequent beams is $\exp(-\kappa^2)$.

5 Discretization error for a two-parametric superposition

We assume that the Gaussian beams are sufficiently narrow to be able to replace the Cartesian coordinates by a paraxial ray approximation in the ray coordinates γ_1, γ_2, τ. The integral superposition (1) may be expressed in ray coordinates as

$$u_i(\gamma_A^0) = \frac{\omega}{2\pi} \iint d\gamma_1 \, d\gamma_2 \, A_i^{\mathrm{R}} \, |\det \mathbf{Q}^{\mathrm{R}}| \sqrt{-\det(\mathbf{M} - \mathbf{M}^{\mathrm{R}})}$$

$$\times \exp [\![i\omega \, [\tau + \tfrac{1}{2} (\gamma_K^0 - \gamma_K)(\gamma_L^0 - \gamma_L) \, Q_{MK}^{\mathrm{R}} \, Q_{NL}^{\mathrm{R}} \, (M_{MN} - M_{MN}^{\mathrm{R}}) - t]]\!], \tag{52}$$

where γ_A^0 are the parameters of the ray passing through the receiver. All other quantities are defined in Section 2.1.

The corresponding discrete sum (6) of Gaussian beams over a homogeneous rectangular net in the ray-parameter surface generated with the steps $\Delta\gamma_1, \Delta\gamma_2$ may be written in the form

$$u_i^{\mathrm{D}}(\gamma_A^0) = \frac{\omega}{2\pi} \sum_{k_1=-\infty}^{+\infty} \sum_{k_2=-\infty}^{+\infty} G A_i^{\mathrm{R}} \, |\det \mathbf{Q}^{\mathrm{R}}| \sqrt{-\det(\mathbf{M} - \mathbf{M}^{\mathrm{R}})}$$

$$\times \exp [\![i\omega \, [\tau + \tfrac{1}{2} (\xi_A - k_A) S_{AB} (\xi_B - k_B) - t]]\!], \tag{53}$$

where

$$\xi_1 = \gamma_1^0 / \Delta\gamma_1 ; \quad \xi_2 = \gamma_2^0 / \Delta\gamma_2 , \tag{54}$$

$$G = \Delta\gamma_1 \, \Delta\gamma_2 \tag{55}$$

and

$$\mathbf{S} = \mathbf{\Gamma}^{\mathrm{T}} (\mathbf{Q}^{\mathrm{R}})^{\mathrm{T}} \, (\mathbf{M} - \mathbf{M}^{\mathrm{R}}) \, \mathbf{Q}^{\mathrm{R}} \mathbf{\Gamma}. \tag{56}$$

Here

$$\mathbf{\Gamma} = \begin{pmatrix} \Delta\gamma_1 & 0 \\ 0 \; , & \Delta\gamma_2 \end{pmatrix} \tag{57}$$

is the matrix of the steps in the ray parameters.

If we suppose that the quantities A_i^{R}, $\mathbf{Q}^{\mathrm{R}}$, $\mathbf{M}^{\mathrm{R}}$, $\mathbf{M}$ are approximately constant in the effective regions (or spot-ellipses) of the Gaussian beams, we can evaluate superposition (52) as

$$u_i(\gamma_A^0) = A_i^{\mathrm{R}} \exp\,[i\omega(\tau - t)]\,. \tag{58}$$

We assume *matrix (56) to be constant for all Gaussian beams.* The sum (53) is then a periodic function of coordinates ξ_A and may be expressed as

$$u_i^{\mathrm{D}}(\gamma_A^0) = \sum_{n_1=-\infty}^{+\infty} \sum_{n_2=-\infty}^{+\infty} a_i(n_1, n_2) \exp\,(2\pi i n_A\, \xi_A), \tag{59}$$

where the Fourier coefficients

$$a_i(n_1, n_2) = \int_0^1 d\xi_1 \int_0^1 d\xi_2\, \frac{\omega}{2\pi} \exp\,(-2\pi i n_A\, \xi_A)\, A_i^{\mathrm{R}}\, G\, |\det \mathbf{Q}^{\mathrm{R}}|\, \sqrt{-\det\,(\mathbf{M} - \mathbf{M}^{\mathrm{R}})}$$

$$\times \sum_{k_1=-\infty}^{+\infty} \sum_{k_2=-\infty}^{+\infty} \exp\,[\![i\omega\,[\tau + \tfrac{1}{2}\,(k_B - \xi_B)\,(k_C - \xi_C) S_{BC} - t]]\!] \tag{60}$$

may be again evaluated analytically

$$a_i(n_1, n_2) = \frac{\omega}{2\pi} A_i^{\mathrm{R}} \exp\,[i\omega(\tau - t)]\, \sqrt{-\det \mathbf{S}} \int_0^1 d\xi_1 \int_0^1 d\xi_2 \sum_{k_1=-\infty}^{+\infty} \sum_{k_2=-\infty}^{+\infty}$$

$$\times \exp\left[\frac{i\omega}{2}\,(k_A - \xi_A)\,(k_B - \xi_B) S_{AB} + 2\pi i n_A\,(k_A - \xi_A)\right]$$

$$= \frac{\omega}{2\pi} A_i^{\mathrm{R}} \exp\,[i\omega(\tau - t)]\, \sqrt{-\det \mathbf{S}}$$

$$\times \int_{-\infty}^{+\infty} d\xi_1 \int_{-\infty}^{+\infty} d\xi_2 \exp\left(\frac{i\omega}{2}\, \xi_A S_{AB} \xi_B + 2\pi i n_A\, \xi_A\right)$$

$$= \frac{\omega}{2\pi} A_i^{\mathrm{R}} \exp\,[i\omega(\tau - t)]\, \sqrt{-\det \mathbf{S}} \int_{-\infty}^{+\infty} d\xi_1 \int_{-\infty}^{+\infty} d\xi_2$$

$$\times \exp\left[\frac{i\omega}{2}\left(\xi_A + \frac{2\pi}{\omega} S_{AC}^{-1} n_C\right) S_{AB} \left(\xi_B + \frac{2\pi}{\omega} S_{BD}^{-1}\, n_D\right) - \frac{2\pi^2 i}{\omega}\, n_C S_{CD}^{-1} n_D\right]$$

$$= A_i^{\mathrm{R}} \exp\,[i\omega(\tau - t)] \exp\left(-\frac{2\pi^2 i}{\omega}\, n_C S_{CD}^{-1} n_D\right). \tag{61}$$

For the matrix S_{CD} see (56).

The difference between the discrete and integral superpositions may be expressed (see 58, 59, 61) as

$$u_i^{\mathrm{D}} - u_i = u_i \sum_{n_1=-\infty}^{+\infty} \sum_{n_2=-\infty}^{+\infty} b(n_1, n_2) \exp\,(2\pi i n_A\, \xi_A) - u_i, \tag{62}$$

where

$$b(n_1, n_2) = \exp\left(-\frac{2\pi^2 i}{\omega} n_C S^{-1}_{CD} n_D\right) . \tag{63}$$

The relative average square of the discretization error (see 14)

$$\overline{\delta^2} = |A_j^R|^{-2} \int_0^1 d\xi_1 \int_0^1 d\xi_2 \; |u_i^D - u_i|^2 \tag{64}$$

may be evaluated in the form

$$\overline{\delta^2} = \sum_{n_1=-\infty}^{+\infty} \sum_{n_2=-\infty}^{+\infty} \delta^2(n_1, n_2) - 1 = \sum_{n_1=-\infty}^{+\infty} \sum_{\substack{n_2=-\infty \\ n_1^2+n_2^2>0}}^{+\infty} \delta^2(n_1, n_2), \tag{65}$$

where

$$\delta(n_1, n_2) = \sqrt{b^*(n_1, n_2)\, b(n_1, n_2)}. \tag{66}$$

The maximum relative error (15) is less or equal to

$$\delta_{\max} = \sum_{n_1=-\infty}^{+\infty} \sum_{n_2=-\infty}^{+\infty} \delta(n_1, n_2) - 1 = \sum_{n_1=-\infty}^{+\infty} \sum_{\substack{n_2=-\infty \\ n_1^2+n_2^2>0}}^{+\infty} \delta(n_1, n_2). \tag{67}$$

Substituting (63) into (66) we obtain the explicit definition of the particular errors

$$\delta(n_1, n_2) = \exp\left[-\frac{\pi^2 i}{\omega} (S^{-1} - S^{-1\,*})_{CD}\, n_C n_D\right]. \tag{68}$$

5.1 GEOMETRICAL INTERPRETATION OF THE PARTICULAR ERRORS

Introducing the matrix

$$\mathbf{K} = \left[\frac{i}{\omega}(\mathbf{S}^{-1} - \mathbf{S}^{-1\,*})\right]^{-1} = \left[\frac{i}{\omega}\mathbf{S}^{-1}(\mathbf{S}^* - \mathbf{S})\,\mathbf{S}^{-1\,*}\right]^{-1} = \omega\,\mathbf{S}^*\,[i(\mathbf{S}^* - \mathbf{S})]^{-1}\,\mathbf{S}, \tag{69}$$

which has the form (see 56)

$$\mathbf{K} = \boldsymbol{\Gamma}\mathbf{Z}\boldsymbol{\Gamma}, \tag{70}$$

with

$$\mathbf{Z} = \frac{\omega}{2}\,(\mathbf{Q}^R)^T\,(\mathbf{M} - \mathbf{M}^R)^*\,(\operatorname{Im}\mathbf{M})^{-1}\,(\mathbf{M} - \mathbf{M}^R)\,\mathbf{Q}^R, \tag{71}$$

we may rewrite (68) as

$$\delta(n_1, n_2) = \exp(-\pi^2\, n_A K^{-1}_{AB}\, n_B). \tag{72}$$

The matrix $\boldsymbol{\Gamma}$ of the steps in ray parameters is given by (57).

Since the matrix $\mathbf{K}$ is more physically meaningful to the author than the matrix $\mathbf{K}^{-1}$, we shall rewrite (72) directly in terms of matrix $\mathbf{K}$. The summations in (65) and (67) are

performed symmetrically over all integer values of n_1 and n_2. Thus we may replace the particular errors $\delta(n_1, n_2)$ by

$$\delta'(n_1, n_2) = \delta(-n_2, n_1) = \exp(-\pi^2 n_A \epsilon_{AB} K^{-1}_{BC} \epsilon_{DC} n_D), \tag{73}$$

where

$$\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{74}$$

Since

$$\boldsymbol{\epsilon}\mathbf{K}^{-1}\boldsymbol{\epsilon}^{\mathrm{T}} = \mathbf{K}\,[\det \mathbf{K}]^{-1}, \tag{75}$$

equation (73) reads

$$\delta'(n_1, n_2) = \exp\,[-\pi^2\,(\det \mathbf{K})^{-1}\, n_A K_{AB} n_B]. \tag{76}$$

Let us use a graphical interpretation of (76). We define vectors

$$\mathbf{L}_1 = \begin{pmatrix} L_{11} \\ L_{21} \end{pmatrix}, \quad \mathbf{L}_2 = \begin{pmatrix} L_{12} \\ L_{22} \end{pmatrix} \tag{77}$$

as the columns of matrix

$$\mathbf{L} = \mathbf{X}\boldsymbol{\Gamma} \tag{78}$$

where the transformation matrix $\mathbf{X}$ is given by

$$\mathbf{X} = \mathbf{U}\mathbf{Z}^{1/2}. \tag{79}$$

Here $\mathbf{Z}$ is defined by (71) and $\mathbf{U}$ is an arbitrary 2×2 unitary matrix. Equation (76) then reads

$$\delta'(n_1, n_2) = \exp\,[-\pi^2\,(\det \mathbf{K})^{-1}\,(\mathbf{L}_A n_A, \mathbf{L}_B n_B)], \tag{80}$$

which may be rewritten in a more telling way as

$$\delta'(n_1, n_2) = \exp\,[-\pi^2\,(\det \mathbf{K})^{-1}\, d^2(n_1, n_2)], \tag{81}$$

where

$$d(n_1, n_2) = |\mathbf{L}_1 n_1 + \mathbf{L}_2 n_2| \tag{82}$$

is the distance between the origin and point (n_1, n_2) of the homogeneous (but generally not isotropic) net generated by vectors $\mathbf{L}_1$ and $\mathbf{L}_2$ (see Fig. 1).

We denote by d_1 the size of the two smallest distances $d(n_1, n_2)$ in Fig. 1. Similarly we denote by $d_2, d_3, \ldots$ the sizes of the second, third and other pairs of smallest distances $d(n_1, n_2)$. We also denote by $\delta_1, \delta_2, \delta_3, \ldots$ the corresponding particular errors (81)

$$\delta_\alpha = \exp\,[-\pi^2\,(\det \mathbf{K})^{-1}\, d_\alpha^2]. \tag{83}$$

Since the distance d_4 is large in comparison with d_1 and d_2,

$$d_4^2 \geqq \min(d_1^2 + d_2^2, 4d_1^2) \geqq 2d_1^2, \tag{84}$$

the corresponding particular error

$$\delta_4 \leqq \max(\delta_1\delta_2, \delta_1^4) \leqq \delta_1^2 \tag{85}$$

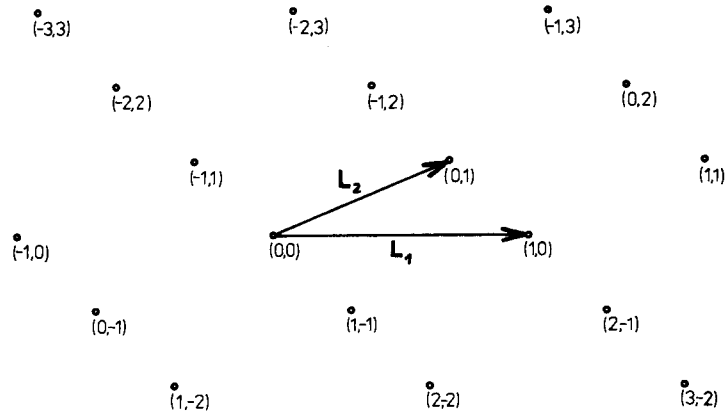

Figure 1. The net generated by vectors (77) is the image of the take-off parameters of rays in mapping (79).

is very small in comparison with δ_1 because the particular error δ_1 must be a small number. This makes the particular errors $\delta_4, \delta_5, \ldots$ negligible and the sum forming the average quadratic error (65) may be restricted to

$$\overline{\delta^2} = 2\delta_1^2 + 2\delta_2^2 + 2\delta_3^2. \tag{86}$$

The same is valid for the maximum error (67)

$$\delta_{max} = 2\delta_1 + 2\delta_2 + 2\delta_3. \tag{87}$$

5.2 ELEMENTARY SURFACES CORRESPONDING TO BEAMS

It may be rather cumbersome to construct the net generated by the vectors $\mathbf{L}_1, \mathbf{L}_2$ (see Fig. 1) and to determine the smallest distances d_1, d_2, d_3 between the nodes. We shall express the distances d_1, d_2, d_3 in terms of the properties of the element of the ray-parameter surface corresponding to one beam.

It seems to be natural to couple the node of the net in Fig. 1 to the elementary surface Δ of the points, the distance of which from the node is smaller than the distances from other nodes (see Fig. 2). The elementary surface Δ has the shape of a centrally symmetric hexagon inscribed in a circle of radius r. The distances of the sides of the hexagon from the node in its centre are

$$a_1 = \tfrac{1}{2}d_1, \quad a_2 = \tfrac{1}{2}d_2, \quad a_3 = \tfrac{1}{2}d_3. \tag{88}$$

Note that the described hexagons involve rectangles as the limiting cases for $d_3^2 = d_1^2 + d_2^2$.

If we introduce the quantities

$$s = d_1^2 + d_2^2 + d_3^2, \quad t = d_1^2 d_2^2 + d_2^2 d_3^2 + d_3^2 d_1^2, \quad u = d_1^2 d_2^2 d_3^2, \tag{89}$$

we may express the area

$$D = \iint_\Delta d\zeta_1 \, d\zeta_2 \tag{90}$$

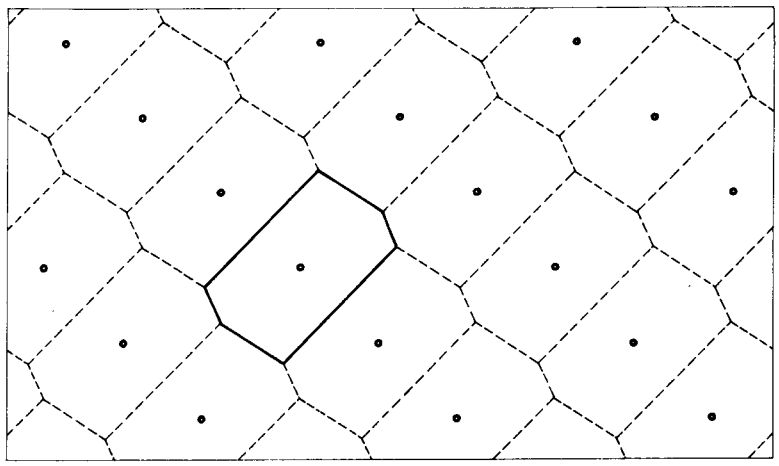

Figure 2. The images Δ of the elementary surfaces Γ in mapping (79).

of the hexagon Δ in the following way

$$D = \sqrt{t - (\tfrac{1}{2}s)^2}. \tag{91}$$

Here, ζ_1, ζ_2 are any Cartesian coordinates introduced into Figs 1 and 2 with their origin in the considered node (see 77 and 78). The radius of the circumscribed circle of the hexagon is

$$r = \tfrac{1}{2}D^{-1} d_1 d_2 d_3 = \tfrac{1}{2} D^{-1}\sqrt{u}. \tag{92}$$

We define the 12 times enlarged specific moment of the elementary surface Δ as

$$D_{AB} = 12D^{-1} \iint_\Delta \zeta_A \zeta_B \, d\zeta_1 \, d\zeta_2 . \tag{93}$$

The trace and the determinant of the matrix **D** are

$$\operatorname{tr} \mathbf{D} = s - 4r^2 , \tag{94}$$

$$\det \mathbf{D} = D^2 - \tfrac{1}{2}(s - 8r^2)^2 . \tag{95}$$

On the other hand, the quantities D, tr **D** and det **D** may be specified and quantities (89) may be evaluated. Since the triangle with sides d_1, d_2 and d_3 is acute-angled,

$$|d_1^2 - d_2^2| \leqq d_3^2 \leqq d_1^2 + d_2^2 , \tag{96}$$

the inequality

$$s \geqq 8r^2 \tag{97}$$

is valid. Considering (97), equation (95) reads

$$s - 8r^2 = \sqrt{2(D^2 - \det \mathbf{D})}. \tag{98}$$

Equations (94) and (98) yield

$$4r^2 = \operatorname{tr} \mathbf{D} - \sqrt{2(D^2 - \det \mathbf{D})}, \tag{99}$$

$$s = 2 \operatorname{tr} \mathbf{D} - \sqrt{2(D^2 - \det \mathbf{D})}. \tag{100}$$

Using in addition (91) and (92), the relations

$$t = D^2 + [\operatorname{tr} \mathbf{D} - \tfrac{1}{2}\sqrt{2\,(D^2 - \det \mathbf{D})}]^2, \tag{101}$$

$$u = D^2\ [\operatorname{tr} \mathbf{D} - \sqrt{2(D^2 - \det \mathbf{D})}] \tag{102}$$

are obtained.

The squares of distances d_1, d_2 and d_3 may be obtained from the quantities D, tr **D** and det **D** as the roots of the cubic equation

$$(d^2)^3 - s(d^2)^2 + t(d^2) - u = 0, \tag{103}$$

where s, t and u are given by (100)–(102). The relative error of replacing the integral superposition by the discrete in the discussed regular case is then given by (83) and (86) or (87). Note that the equivalence

$$\det \mathbf{K} = D^2 \tag{104}$$

may be used for substitution into (83).

Let us now express the relation between the quantities D, tr **D**, det **D** and the similar quantities describing the element Γ of the ray-parameter surface parameterized by ray parameters γ_1, γ_2. Assume that the elementary surface Δ in Fig. 2 is the image of the element Γ in the transformation (see 79),

$$\zeta_A = X_{AB}(\gamma_B - \gamma_B^0), \tag{105}$$

where γ_B^0 are the parameters of the central ray of the considered beam and the transformation matrix **X** is given by (79). Inserting (105) into (90) and (93) we obtain

$$D = \det \mathbf{X}\, G \tag{106}$$

and

$$\mathbf{D} = \mathbf{X}\mathbf{G}\mathbf{X}^{\mathrm{T}}, \tag{107}$$

where

$$G = \iint_\Gamma d\gamma_1\, d\gamma_2 \tag{108}$$

is the area of element Γ of the ray-parameter surface corresponding to the considered beam and

$$G_{AB} = 12\, G^{-1} \iint_\Gamma d\gamma_1\, d\gamma_2\, (\gamma_A - \gamma_A^0)(\gamma_B - \gamma_B^0) \tag{109}$$

is the 12 times enlarged specific moment of element Γ. Considering (79), equations (106) and (107) yield

$$D = (\det \mathbf{Z})^{1/2} G, \tag{110}$$

$$\operatorname{tr} \mathbf{D} = \operatorname{tr} (\mathbf{Z}\mathbf{G}), \tag{111}$$

$$\det \mathbf{D} = \det \mathbf{Z} \det \mathbf{G}. \tag{112}$$

5.3 APPROXIMATE SOLUTION OF THE CUBIC EQUATION (103)

The expressions for the discretization error work with the solutions of cubic equation (103). Considering that our assumption of the constant matrix (56) will be satisfied only approximately during practical computations, the exact solutions of (103) may be replaced by approximate ones in most cases.

The square of the area D of a centrally symmetric hexagon inscribed in a circle may be proved to belong to the interval

$$D^2 \in \left\langle \det \mathbf{D}, \frac{27}{25} \det \mathbf{D} \right\rangle. \tag{113}$$

The minimum value $D^2 = \det \mathbf{D}$ is achieved for any rectangle, the maximum value $D^2 = (27/25) \det \mathbf{D}$ is achieved only in the case of an equilateral hexagon. In the latter case matrix $\mathbf{D}$ is a multiple of the unit matrix.

In the case of a rectangle, $D = \det \mathbf{D}$, the coefficients (100)–(102) of (103) have the form

$$s = 2 \operatorname{tr} \mathbf{D}, \quad t = \det \mathbf{D} + [\operatorname{tr} \mathbf{D}]^2, \quad u = [\det \mathbf{D}]^2 \operatorname{tr} \mathbf{D}, \tag{114}$$

and (103) has solutions

$$d_1^2 = D_1, \quad d_2^2 = D_2, \quad d_3^2 = D_1 + D_2 = \operatorname{tr} \mathbf{D}, \tag{115}$$

where

$$D_1 = \tfrac{1}{2} \operatorname{tr} \mathbf{D} - \sqrt{(\tfrac{1}{2} \operatorname{tr} \mathbf{D})^2 - \det \mathbf{D}} \tag{116}$$

and

$$D_2 = \tfrac{1}{2} \operatorname{tr} \mathbf{D} + \sqrt{(\tfrac{1}{2} \operatorname{tr} \mathbf{D})^2 - \det \mathbf{D}} \tag{117}$$

are the eigenvalues of matrix $\mathbf{D}$.

In the case of an equilateral hexagon, $D_1 = D_2$, $D^2 = (27/25) D_1^2$, and the coefficients (100)–(102) take the values

$$s = \frac{18}{5} D_1, \quad t = \frac{108}{25} D_1^2, \quad u = \frac{27}{25} \frac{8}{5} D_1^3, \tag{118}$$

and (103) has the solution

$$d_1^2 = d_2^2 = d_3^2 = \frac{6}{5} D_1 = \frac{6}{5} D_2 = \frac{6}{10} (D_1 + D_2). \tag{119}$$

The solution (115) for a rectangle prompts the use of an approximate solution

$$d_1^2 = D_1, \quad d_2^2 = D_2, \quad d_3^2 = D_1 + D_2 - \sqrt{2(D^2 - \det \mathbf{D})} \tag{120}$$

of (103). The relations

$$s = (d_1^2 + d_2^2 + d_3^2), \tag{121}$$

$$t = (d_1^2 d_2^2 + d_2^2 d_3^2 + d_3^2 d_1^2) + \tfrac{3}{2} (D^2 - \det \mathbf{D}), \tag{122}$$

$$u = (d_1^2 d_2^2 d_3^2) D^2 / \det \mathbf{D} \tag{123}$$

are valid for the approximate roots (120). The relative error is 17 per cent for d_1^2 and d_2^2 and 33 per cent for d_3^2 in the case of the equilateral hexagon.

The relations (122) and (123) may be improved considerably with only a small adverse effect on (121) by the choice

$$d_1^2 = D_1 D^2/\det \mathbf{D}, \tag{124}$$

$$d_2^2 = D_2 D^2/\det \mathbf{D}, \tag{125}$$

$$d_3^2 = (D_1 + D_2)\det \mathbf{D}/D^2 - \sqrt{2(D^2 - \det \mathbf{D})}. \tag{126}$$

The relations

$$s = (d_1^2 + d_2^2 + d_3^2) - \operatorname{tr} \mathbf{D}\,(D^2 - \det \mathbf{D})^2/(D^2 \det \mathbf{D}), \tag{127}$$

$$t = (d_1^2 d_2^2 + d_2^2 d_3^2 + d_3^2 d_1^2) + (D^2 - \det \mathbf{D})\,[\operatorname{tr} \mathbf{D}\sqrt{2(D^2 - \det \mathbf{D})} + \tfrac{1}{2}\det \mathbf{D} - D^2]/\det \mathbf{D}, \tag{128}$$

$$u = (d_1^2 d_2^2 d_3^2) + (D^2 - \det \mathbf{D})\,\frac{D^2}{\det \mathbf{D}}\sqrt{2(D^2 - \det \mathbf{D})} \tag{129}$$

hold for the approximate roots (124)–(126). The approximate formulae (124)–(126) are exact for a rectangle. In the case of equilateral hexagon they yield

$$d_1^2 = \frac{54}{50} D_1 = 1.08\, D_1, \quad d_2^2 = \frac{54}{50} D_1 = 1.08\, D_1, \quad d_3^2 = \frac{392}{270} D_1 \doteq 1.45\, D_1, \tag{130}$$

and the relative error is 10 per cent for d_1^2 and d_2^2 (see 119) and 21 per cent for d_3^2. Since

$$\sqrt{2(D^2 - \det \mathbf{D})} \leqq \tfrac{2}{5}\sqrt{\det \mathbf{D}} \leqq \tfrac{1}{5}\operatorname{tr} \mathbf{D}, \tag{131}$$

the relation

$$d_3^2 \geqslant \frac{196}{270}\operatorname{tr} \mathbf{D} \geqslant \frac{196}{270}\,\frac{25}{27}(d_1^2 + d_2^2) \geqslant \frac{980}{27^2} d_1^2 \doteq 1.344\, d_1^2 \tag{132}$$

is valid for the approximate roots (124)–(126). Considering (83), it is clear that the contribution of the third approximate root (126) to expression (86) or (87) may be neglected in comparison with the contribution of the approximate roots (124) and (125). This neglect may, to some extent, even improve the relative error of 10 per cent corresponding to the approximate estimates (124) and (125) of roots d_1^2 and d_2^2. After this restriction, the average quadratic and the maximum discretization errors (86) and (87) have the form

$$\overline{\delta^2} \doteq 2\delta_1^2 + 2\delta_2^2 \tag{133}$$

and

$$\delta_{\max} \doteq 2\delta_1 + 2\delta_2. \tag{134}$$

The approximate particular errors δ_1 and δ_2 are obtained by inserting (124) and (125) into (83) and using (104)

$$\delta_1 = \exp(-\pi^2/\kappa_1^2), \quad \delta_2 = \exp(-\pi^2/\kappa_2^2). \tag{135}$$

Here

$$\kappa_1^2 = D_2\,, \quad \kappa_2^2 = D_1 \tag{136}$$

may be evaluated using (71), (111), (112), (116) and (117), or alternatively,

$$\begin{aligned} \kappa_1^2 &= \tfrac{1}{2}\mathrm{tr}\,(\mathbf{ZG}) + \sqrt{\tfrac{1}{2}\mathrm{tr}\,(\mathbf{ZG}) - \det \mathbf{Z} \det \mathbf{G}}, \\ \kappa_2^2 &= \tfrac{1}{2}\mathrm{tr}\,(\mathbf{ZG}) - \sqrt{\tfrac{1}{2}\mathrm{tr}\,(\mathbf{ZG}) - \det \mathbf{Z} \det \mathbf{G}}, \end{aligned} \tag{137}$$

are the eigenvalues of the matrix

$$\sqrt{\mathbf{G}}\mathbf{Z}\sqrt{\mathbf{G}} \tag{138}$$

where $\mathbf{Z}$ is given by (71) and $\mathbf{G}$ is given by (109).

6 Discussion

6.1 CONCLUSIONS

If the assumptions of slowly varying quantities are satisfied, (133)–(137) (or 42, 43, 45, 47, 49, 50 for the one-parametric superposition) may be used for estimating the discretization error for a given ray diagram and a given shape of Gaussian beams at a specified frequency.

Especially, if the relation (see 51)

$$\frac{\omega}{2} m^* \,(\mathrm{Im}\, m)^{-1}\, m\, G_{11} \leqq \kappa^2 \tag{139}$$

is valid for all beams in the one-parametric superposition, the average quadratic discretization error is less or equal to (see 42)

$$\sqrt{\overline{\delta^2}} = \sqrt{2}\exp\left(-\pi^2/\kappa^2\right) \tag{140}$$

and the maximum discretization error is less or equal to (see 43)

$$\delta_{\max} = 2\exp\left(-\pi^2/\kappa^2\right). \tag{141}$$

Values (140) and (141) are reached for equality in (139). For m and G_{11} refer to (30) and (49). The physical sense of the quantity κ^2 is shortly discussed at the end of Section 4. For functions (140) and (141) see Table 1.

Similarly, if the relation (see 138 and 71)

$$\frac{\omega}{2} (\mathbf{Q}^{\mathrm{R}})^{\mathrm{T}} (\mathbf{M} - \mathbf{M}^{\mathrm{R}})^* \,(\mathrm{Im}\,\mathbf{M})^{-1}\, (\mathbf{M} - \mathbf{M}^{\mathrm{R}})\, \mathbf{Q}^{\mathrm{R}} \leqq \kappa^2\, \mathbf{G}^{-1} \tag{142}$$

holds true for all beams in the two-parametric superposition, the average quadratic discretization error does not exceed the value of

$$\sqrt{\overline{\delta^2}} = 2\exp\left(-\pi^2/\kappa^2\right), \tag{143}$$

and the maximum discretization error does not exceed the value of

$$\delta_{\max} = 4\exp\left(-\pi^2/\kappa^2\right). \tag{144}$$

Table 1. Relative discretization error

κ^2	One-parametric superposition $\sqrt{\overline{\delta^2}}$	One-parametric superposition δ_{max}	Two-parametric superposition Range of $\sqrt{\overline{\delta^2}}$	Two-parametric superposition Range of δ_{max}		
	$\sqrt{2}\exp\left(-\frac{\pi^2}{\kappa^2}\right)$	$2\exp\left(-\frac{\pi^2}{\kappa^2}\right)$	$\sqrt{6}\exp\left(-\frac{10}{9}\frac{\pi^2}{\kappa^2}\right)$	$2\exp\left(-\frac{\pi^2}{\kappa^2}\right)$	$6\exp\left(-\frac{10}{9}\frac{\pi^2}{\kappa^2}\right)$	$4\exp\left(-\frac{\pi^2}{\kappa^2}\right)$
0.9	0.000024	0.000035	0.000013	0.000035	0.000031	0.000069
1.0	0.000073	0.000103	0.000042	0.000103	0.000104	0.000207
1.1	0.000179	0.000254	0.000115	0.000254	0.000281	0.000507
1.2	0.000379	0.000536	0.000263	0.000536	0.000645	0.001072
1.3	0.00071	0.00101	0.00053	0.00101	0.00130	0.00202
1.4	0.00123	0.00174	0.00097	0.00174	0.00238	0.00347
1.5	0.00196	0.00278	0.00164	0.00278	0.00401	0.00555
1.6	0.00296	0.00419	0.00259	0.00419	0.00633	0.0084
1.7	0.0043	0.0060	0.0039	0.0060	0.0095	0.0120
1.8	0.0059	0.0083	0.0055	0.0083	0.0136	0.0166
1.9	0.0078	0.0111	0.0076	0.0111	0.0187	0.0222
2.0	0.0102	0.0144	0.0102	0.0144	0.0249	0.029
2.1	0.0129	0.0182	0.0132	0.0182	0.0324	0.036
2.2	0.0159	0.0225	0.0168	0.0225	0.0411	0.045
2.3	0.019	0.027	0.021	0.027	0.051	0.055
2.4	0.023	0.033	0.025	0.033	0.062	0.07
2.5	0.027	0.039	0.030	0.039	0.075	0.08
2.6	0.032	0.045	0.036	0.045	0.088	0.09
2.7	0.037	0.052	0.042	0.052	0.103	0.10
2.8	0.042	0.059	0.049	0.059	0.119	0.12
2.9	0.047	0.067	0.056	0.067	0.137	0.13
3.0	0.053	0.075	0.063	0.075	0.155	0.15
Neglected terms	$\frac{\sqrt{2}}{2}\exp^7\left(-\frac{\pi^2}{\kappa^2}\right)$	$2\exp^4\left(-\frac{\pi^2}{\kappa^2}\right)$	$\frac{\sqrt{6}}{2}\exp^5\left(-\frac{10}{9}\frac{\pi^2}{\kappa^2}\right)$	$\exp^3\left(-\frac{\pi^2}{\kappa^2}\right)$	$6\exp^3\left(-\frac{10}{9}\frac{\pi^2}{\kappa^2}\right)$	$4\exp^2\left(-\frac{\pi^2}{\kappa^2}\right)$

For the quantities used in (142) refer to Section 2.1 and (109). Note that the discretization errors slightly depend on the shape of the elementary surfaces Γ, which are described in terms of $\mathbf{G}$ in (142). It means that the upper bounds of the discretization errors are not given exactly by (143) and (144) but lie within the intervals (see 119 and 130)

$$\sqrt{\overline{\delta^2}} \in \left\langle \sqrt{6} \exp\left(-\frac{10}{9}\pi^2\kappa^{-2}\right), \quad 2\exp(-\pi^2\kappa^{-2}) \right\rangle \tag{145}$$

and

$$\delta_{\max} \in \left\langle 6 \exp\left(-\frac{10}{9}\pi^2\kappa^{-2}\right), \quad 4\exp(-\pi^2\kappa^{-2}) \right\rangle, \tag{146}$$

depending on the shape of the elementary surfaces Γ. The left endpoints in (145) and (146) correspond to the shape of an equilateral hexagon, the right ones to a rectangle. However, the differences between the left and right endpoints of intervals (145) and (146) do not seem to be very important for practical computations.

The dependence on κ^2 of functions (140), (141), (145) and (146) is shown in Table 1. It can be seen that a good choice of κ^2 can be made from the interval $\langle 1, 2 \rangle$. The expressions at the bottom of the table are the highest neglected terms at estimating the discretization errors.

If the shape of the Gaussian beams is specified, equations (139) or (142) may be used for determining the steps in the ray parameters in ray shooting.

If G_{11} in (139) or $\mathbf{G}$ in (142) is specified (i.e. the ray diagram is given), equations (139) or (142) may be used for modifying (or possibly even determining) the shape of Gaussian beams in order to keep the discretization error in some limits. A simple example is given in the following section.

6.2 WEIGHTING BETWEEN PARAXIAL RAY APPROXIMATIONS

The limiting case of superpositions (1) or (8) of the Gaussian beams for extremely narrow beams at receivers is the ray method. If the ray diagram is specified, the narrowest beams are obtained from (139) by choosing

$$m = i\frac{2\kappa^2}{\omega}(G_{11})^{-1}, \tag{147}$$

see also (10), or from (142) by choosing

$$\mathbf{M} = \mathbf{M}^{\mathrm{R}} + i\frac{2\kappa^2}{\omega}[\mathbf{Q}^{\mathrm{R}}\mathbf{G}(\mathbf{Q}^{\mathrm{R}})^{\mathrm{T}}]^{-1}. \tag{148}$$

Choices (147) or (148) give the Gaussian weighting between the paraxial ray approximations from the closest endpoints of rays.

Similarly, equations (139) or (142) may be used to determine the broadest Gaussian windows applied to the plane-wave decomposition of the wave field in order to avoid the oscillations caused by the discretization.

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