Transformation of spatial and perturbation derivatives of travel time at a general interface between two general media

Luděk Klimeš

Department of Geophysics, Faculty of Mathematics and Physics, Charles University, Ke Karlovu 3, 121 16 Praha 2, Czech Republic, http://sw3d.cz/staff/klimes.htm

Summary

We consider the partial derivatives of travel time with respect to both spatial coordinates and perturbation parameters. We derive the explicit equations for transforming these travel–time derivatives of arbitrary orders at a general smooth interface between two general media. The equations are applicable to both real–valued and complex–valued travel time. The equations are expressed in terms of a general Hamiltonian function and are applicable to the transformation of travel–time derivatives in both isotropic and anisotropic media. The interface is specified by an implicit equation. No local coordinates are needed for the transformation.

Keywords

Ray theory, Hamilton–Jacobi equation, eikonal equation, travel time (action), spatial derivatives of travel time, perturbation derivatives of travel time, reflection or refraction at curved interfaces, anisotropy, heterogeneous media, paraxial approximation, Gaussian beams, wave propagation.

1. Introduction

Travel time (action) is a function of spatial coordinates and may also depend on one or more perturbation parameters. The spatial coordinates may be arbitrary, including curvilinear coordinates. The partial derivatives of travel time with respect to spatial coordinates are referred to as spatial derivatives, whereas the partial derivatives of travel time with respect to perturbation parameters are referred to as perturbation derivatives. Travel time satisfies a general partial differential equation of the first order, called the Hamilton–Jacobi equation. For example, various eikonal equations represent important special cases of the Hamilton–Jacobi equation.

In smooth media, travel time and its first–order spatial derivatives can be calculated by solving the non–linear ordinary differential equations for rays (geodesics) derived by Hamilton (1837) and called Hamilton's equations. The first–order spatial derivatives of travel time (the components of the slowness vector, called the components of normal slowness by Hamilton) can be transformed at smooth curved interfaces using Snell's law in the form derived by Hamilton (1837, eq. C^7).

In smooth media, the second–order spatial derivatives of travel time can be calculated along the rays by solving the linear ordinary differential equations derived by Červený (1972). The second–order spatial derivatives of travel time can be transformed at smooth curved interfaces using the equation derived by Hamilton (1837, eq. V^7).

In: Seismic Waves in Complex 3–D Structures, Report 20 (Department of Geophysics, Charles University, Prague, 2010), pp. 103–114

In smooth media, the third–order and higher–order spatial derivatives of travel time and all perturbation derivatives of travel time can be calculated along the rays by simple numerical quadratures using the equations derived by Klimeš (2002).

In this paper, we derive the explicit equations for transforming any spatial and perturbation derivatives of travel time at a general smooth curved interface between two arbitrary media. The equations are applicable to both real–valued and complex–valued travel time. The equations are expressed in terms of a general Hamiltonian function and are applicable to travel–time derivatives of arbitrary orders in generally heterogeneous media, both isotropic and generally anisotropic. The interface represents the surface at which the Hamiltonian function or its partial derivatives of an arbitrary order may be discontinuous. The interface is specified by an implicit equation. No local coordinates are needed for the transformation of travel–time derivatives.

We follow the approach of Hamilton (1837). Section 2 is devoted to the specification of the problem and to the description of the notation used hereinafter for partial derivatives. The derivation of equations for transforming the first–order and second–order derivatives of travel time in Sections 3 and 4 represents a cover version of the corresponding derivation by Hamilton (1837). The derivation is then extended to the transformation of the third–order and fourth–order derivatives of travel time in Sections 5 and 6. The equations for transforming any higher–order spatial and perturbation derivatives of travel time at a general interface between two general media are given in Section 7.

We use the componental notation for vectors and matrices. For example, p_i stands for the covariant vector with components p_i. The Einstein summation over repetitive indices is used throughout the paper.

2. Travel time at a smooth interface

Travel time is a function of spatial coordinates x^i, and may also depend on one or several perturbation parameters f^α. We consider a smooth interface implicitly described by equation

$$F(x^i, f^\alpha) = 0 \tag{1}$$

(Hamilton, 1837, eq. B^7). The interface represents the surface at which the Hamiltonian function or its partial derivatives of an arbitrary order may be discontinuous.

We denote the incident travel time by $\widetilde{\tau}(x^i, f^\alpha)$. It satisfies the Hamilton–Jacobi equation

$$\widetilde{H}\big(x^i, \widetilde{\tau}_{,j}(x^m, f^\mu), f^\alpha\big) = \widetilde{C} \quad , \tag{2}$$

but we do not need this equation, because the incident travel time and its spatial and perturbation derivatives are known along the interface.

The reflected or refracted travel time $\tau(x^i, f^\alpha)$ satisfies the Hamilton–Jacobi equation

$$H\big(x^i, \tau_{,j}(x^m, f^\mu), f^\alpha\big) = C \tag{3}$$

(Hamilton, 1837, eq. F^7), where Hamiltonian function $H = H(x^i, p_{,j}, f^\alpha)$ and constant C are given. The domains for solving equations (2) and (3) are situated on the same side of the interface for reflected travel time $\tau(x^i, f^\alpha)$, and on the opposite sides for refracted travel time $\tau(x^i, f^\alpha)$.

The reflected or refracted travel time must be equal to the incident travel time along the interface,

$$\tau(x^i, f^\alpha) = \widetilde{\tau}(x^i, f^\alpha) \tag{4}$$

(Hamilton, 1837, eq. A^7) for x^i satisfying equation (1). Equation (4) thus represents the initial conditions for the Hamilton–Jacobi equation (3). We have to express the spatial and perturbation derivatives of reflected or refracted travel time τ in terms of the spatial and perturbation derivatives of incident travel time $\widetilde{\tau}$.

The Hamiltonian function $H = H(x^i, p_{,j}, f^\alpha)$ is a function of spatial coordinates x^i, of slowness–vector components p_i, and may also depend on one or several perturbation parameters f^α. We use notation

$$H^{,a...f}_{,i...n\alpha...\nu} = \frac{\partial}{\partial p^a} \cdots \frac{\partial}{\partial p^f} \frac{\partial}{\partial x^i} \cdots \frac{\partial}{\partial x^n} \frac{\partial}{\partial f^\alpha} \cdots \frac{\partial}{\partial f^\nu} H \tag{5}$$

for the partial derivatives of the Hamiltonian function, and the analogous notation for the partial derivatives of other functions such as $F(x^i, f^\alpha)$, $\widetilde{\tau}(x^i, f^\alpha)$, $\tau(x^i, f^\alpha)$.

Equations for the transformation of perturbation derivatives of travel time have the same form as the equations for the transformation of spatial derivatives of travel time. We thus unify spatial coordinates x^i with perturbation parameters f^α and denote these extended coordinates by $x^{\underline{i}}$,

$$x^{\underline{i}} = (x^i, f^\alpha) \quad . \tag{6}$$

Extended coordinate $x^{\underline{i}}$ represents both x^i and f^α. Derivative $H_{,\underline{i}}$ represents both $H_{,i}$ and $H_{,\alpha}$, derivative $H_{,\underline{ij}}$ represents $H_{,ij}$, $H_{,\alpha j}$, $H_{,i\beta}$ and $H_{,\alpha\beta}$, etc., and analogously for other functions.

3. Transformation of the first–order derivatives of travel time

We differentiate equation (4) with respect to x^i and f^α under constraint (1). We obtain Snell's law

$$\tau_{,i} = \widetilde{\tau}_{,i} + F_{,i}\,\lambda \tag{7}$$

(Hamilton, 1837, eq. C^7) and analogous law

$$\tau_{,\alpha} = \widetilde{\tau}_{,\alpha} + F_{,\alpha}\,\lambda \tag{8}$$

for perturbation derivatives. Here $\lambda = \lambda(x^m, f^\mu)$ is the Lagrange multiplier defined for x^i satisfying equation (1). Note that perturbation derivatives $F_{,\alpha}$ describe perturbations of the interface.

Since the reflected or refracted travel time τ must satisfy the Hamilton–Jacobi equation (3), the Lagrange multiplier can be calculated from non–linear algebraic equation

$$\boxed{H\big(x^i, \widetilde{\tau}_{,j} + F_{,j}\,\lambda, f^\mu\big) = C \qquad (9)}$$

(Hamilton, 1837, eq. F^7). Note that the Hamiltonian function used in equation (9) is often different from the Hamiltonian function used in Hamilton's equations for rays, especially in anisotropic media. In this way, the Hamiltonian function used in equation (9) is often different from the Hamiltonian function used in all other transformation equations for the spatial and perturbation derivatives of travel time. For example, in wave propagation in anisotropic media, various functions of the eigenvalues of the Christoffel matrix may be used as the Hamiltonian function in Hamilton's equations and in the transformation equations for the derivatives of travel time, whereas the determinant of the difference between the Christoffel and identity matrices is used as the Hamiltonian function in equation (9).

Equation (9) usually has several solutions λ, which correspond to different kinds of the reflected or refracted travel time. The Lagrange multiplier should be properly selected from the solutions of equation (9). Refer to Červený (2001, sec. 2.3.3) for a detailed discussion.

The selected Lagrange multiplier λ can then be inserted into equations (7) and (8), which may be expressed in common form

$$\boxed{\tau_{,\underline{i}} = \widetilde{\tau}_{,\underline{i}} + F_{,\underline{i}}\,\lambda \quad , \qquad (10)}$$

where the underlined lower–case subscripts represent both the lower–case and Greek subscripts.

4. Transformation of the second–order derivatives of travel time

For each subscript $\underline{i}$, we differentiate equation (10) with respect to $x^{\underline{j}}$ under constraint (1) and obtain equation

$$\tau_{,\underline{i}\underline{j}} = \widetilde{\tau}_{,\underline{i}\underline{j}} + F_{,\underline{i}\underline{j}}\,\lambda + F_{,\underline{i}}\,\lambda_{,\underline{j}} + F_{,\underline{j}}\,\lambda_{\underline{i}} \tag{11}$$

(Hamilton, 1837, eq. R^7), where $\lambda_{\underline{i}} = \lambda_{\underline{i}}(x^m, f^\mu)$ is the corresponding Lagrange multiplier. These Lagrange multipliers are defined for x^m satisfying equation (1).

We introduce projection operator

$$N_{\underline{i}\underline{j}} = (F_{,\underline{r}}F_{,\underline{r}})^{-1}F_{,\underline{i}}F_{,\underline{j}} \tag{12}$$

onto the normal to the interface, and projection operator

$$E_{\underline{i}\underline{j}} = \delta_{\underline{i}\underline{j}} - N_{\underline{i}\underline{j}} \tag{13}$$

onto the plane tangent to the interface. Since $\tau_{,\underline{i}\underline{j}}$ is symmetric with respect to its subscripts,

$$N_{\underline{a}\underline{i}}E_{\underline{b}\underline{j}}\tau_{,\underline{i}\underline{j}} = N_{\underline{a}\underline{i}}E_{\underline{b}\underline{j}}\tau_{,\underline{j}\underline{i}} \quad . \tag{14}$$

We insert (11) into (14), and arrive at

$$E_{\underline{a}\underline{j}}\lambda_{,\underline{j}} = E_{\underline{a}\underline{j}}\lambda_{\underline{j}} \quad . \tag{15}$$

Since $N_{\underline{a}\underline{j}}\lambda_{,\underline{j}}$ is not defined by equation (9), we may put

$$N_{\underline{a}\underline{j}}\lambda_{,\underline{j}} = N_{\underline{a}\underline{j}}\lambda_{\underline{j}} \quad . \tag{16}$$

The summation of (15) and (16) yields

$$\lambda_{,\underline{i}} = \lambda_{\underline{i}} \quad , \tag{17}$$

and equation (11) reads

$$\boxed{\tau_{,\underline{i}\underline{j}} = T_{\underline{i}\underline{j}} + F_{,\underline{i}}\,\lambda_{\underline{j}} + F_{,\underline{j}}\,\lambda_{\underline{i}} \quad , \qquad (18)}$$

where the values of

$$\boxed{T_{\underline{i}\underline{j}} = \widetilde{\tau}_{,\underline{i}\underline{j}} + F_{,\underline{i}\underline{j}}\,\lambda \qquad (19)}$$

are known. Lagrange multiplier λ is determined by algebraic equation (3), and is used in transformation equation (10) for the first–order derivatives.

We differentiate the Hamilton–Jacobi equation (3) with respect to $x^{\underline{i}}$,

$$H_{,\underline{i}} + H^{,r}\tau_{,r\underline{i}} = 0 \tag{20}$$

(Hamilton, 1837, eq. T[7]). We insert (18) into (20), and arrive at

$$H_{,\underline{i}} + H^{,r}T_{r\underline{i}} + F_{,\underline{i}}H^{,r}\lambda_r + H^{,r}F_{,r}\,\lambda_{\underline{i}} = 0 \tag{21}$$

(Hamilton, 1837, eq. U[7]). We formally define

$$H^{,\alpha} = 0 \quad , \tag{22}$$

and express equation (21) as

$$\big(H^{,r}F_{,r}\delta^{\underline{a}}_{\underline{i}} + F_{,\underline{i}}H^{,\underline{a}}\big)\lambda_{\underline{a}} = -S_{\underline{i}} \quad , \tag{23}$$

where

$$\boxed{S_{\underline{i}} = H_{,\underline{i}} + H^{,r}T_{r\underline{i}} \quad .} \tag{24}$$

The inverse matrix to

$$\big(H^{,r}F_{,r}\delta^{\underline{a}}_{\underline{i}} + F_{,\underline{i}}H^{,\underline{a}}\big) \tag{25}$$

is

$$(H^{,q}F_{,q})^{-1}\big[\delta^{\underline{i}}_{\underline{a}} - (2H^{,s}F_{,s})^{-1}F_{,\underline{a}}H^{,\underline{i}}\big] \quad . \tag{26}$$

Then

$$\lambda_{\underline{a}} = -(H^{,q}F_{,q})^{-1}\big[\delta^{\underline{i}}_{\underline{a}} - (2H^{,s}F_{,s})^{-1}F_{,\underline{a}}H^{,\underline{i}}\big]S_{\underline{i}} \quad , \tag{27}$$

which may also be expressed as

$$\lambda_{\underline{i}} = -(H^{,q}F_{,q})^{-1}S_{\underline{i}} + \tfrac{1}{2}(H^{,q}F_{,q})^{-2}F_{,\underline{i}}\,H^{,r}S_r \quad . \tag{28}$$

We normalize $F_{,\underline{i}}$ so that its scalar product with $H^{,i}$ is unit,

$$\boxed{N_{\underline{i}} = (H^{,q}F_{,q})^{-1}F_{,\underline{i}} \quad .} \tag{29}$$

Equation (28) then reads

$$\boxed{\lambda_{\underline{i}} = (H^{,q}F_{,q})^{-1}\left(-\,S_{\underline{i}} + \tfrac{1}{2}N_{\underline{i}}\,H^{,r}S_r\right) \quad .} \tag{30}$$

Equation (18) with (19) and (28) represents the explicit equation for the transformation of the second–order derivatives of travel time at a general interface between two general media.

Equation (18) with inserted Lagrange multipliers (28) reads

$$\tau_{,\underline{i}\underline{j}} = T_{\underline{i}\underline{j}} - (H^{,q}F_{,q})^{-1}\left[F_{,\underline{i}}S_{\underline{j}} + F_{,\underline{j}}S_{\underline{i}} - (H^{,p}F_{,p})^{-1}F_{,\underline{i}}F_{,\underline{j}}\,H^{,r}S_r\right] \tag{31}$$

(Hamilton, 1837, eq. V[7]), where $S_{\underline{i}}$ is given by (24). Equation (31) may be expressed more concisely in terms of normalized gradient (29),

$$\tau_{,\underline{i}\underline{j}} = T_{\underline{i}\underline{j}} - N_{\underline{i}}S_{\underline{j}} - N_{\underline{j}}S_{\underline{i}} + N_{\underline{i}}N_{\underline{j}}\,H^{,r}S_r \quad . \tag{32}$$

This equation could also be obtained directly from equation (18) with (29) and (30).

5. Transformation of the third–order derivatives of travel time

We differentiate equation (11) with respect to $x^{\underline{k}}$ under constraint (1) and obtain equation

$$\tau_{,\underline{i}\underline{j}\underline{k}} = \widetilde{\tau}_{,\underline{i}\underline{j}\underline{k}} + F_{,\underline{i}\underline{j}\underline{k}}\,\lambda + F_{,\underline{i}\underline{j}}\,\lambda_{,\underline{k}} + F_{,\underline{i}\underline{k}}\,\lambda_{,\underline{j}} + F_{,\underline{j}\underline{k}}\,\lambda_{\underline{i}} + F_{,\underline{i}}\,\lambda_{,\underline{j}\underline{k}} + F_{,\underline{j}}\,\lambda_{\underline{i},\underline{k}} + F_{,\underline{k}}\,\lambda_{\underline{i}\underline{j}} \ , \tag{33}$$

where $\lambda_{\underline{i}\underline{j}} = \lambda_{\underline{i}\underline{j}}(x^m, f^\mu)$ is the corresponding Lagrange multiplier. These Lagrange multipliers are defined for x^m satisfying equation (1).

Since $\tau_{,\underline{i}\underline{j}\underline{k}}$ is symmetric with respect to its subscripts,

$$N_{\underline{a}\underline{j}}E_{\underline{b}\underline{k}}\tau_{,\underline{i}\underline{j}\underline{k}} = N_{\underline{a}\underline{j}}E_{\underline{b}\underline{k}}\tau_{,\underline{i}\underline{k}\underline{j}} \quad . \tag{34}$$

We insert (33) into (34), and arrive at

$$E_{\underline{a}\underline{k}}\lambda_{\underline{i},\underline{k}} = E_{\underline{a}\underline{k}}\lambda_{\underline{i}\underline{k}} \quad . \tag{35}$$

Since $N_{\underline{a}\underline{k}}\lambda_{\underline{i},\underline{k}}$ is not defined, we may put

$$N_{\underline{a}\underline{k}}\lambda_{\underline{i},\underline{k}} = N_{\underline{a}\underline{k}}\lambda_{\underline{i}\underline{k}} \quad . \tag{36}$$

The summation of (35) and (36) yields

$$\lambda_{\underline{i},\underline{j}} = \lambda_{\underline{i}\underline{j}} \quad . \tag{37}$$

We insert (17) and (37) into equation (33),

$$\boxed{\tau_{,\underline{i}\underline{j}\underline{k}} = T_{\underline{i}\underline{j}\underline{k}} + F_{,\underline{i}}\lambda_{\underline{j}\underline{k}} + F_{,\underline{j}}\lambda_{\underline{i}\underline{k}} + F_{,\underline{k}}\lambda_{\underline{i}\underline{j}} \quad ,} \tag{38}$$

where the values of

$$\boxed{T_{\underline{i}\underline{j}\underline{k}} = \widetilde{\tau}_{,\underline{i}\underline{j}\underline{k}} + F_{,\underline{i}\underline{j}\underline{k}}\,\lambda + F_{,\underline{i}\underline{j}}\,\lambda_{\underline{k}} + F_{,\underline{i}\underline{k}}\,\lambda_{\underline{j}} + F_{,\underline{j}\underline{k}}\,\lambda_{\underline{i}}} \tag{39}$$

are known. Lagrange multiplier λ is determined by algebraic equation (3), and is used in transformation equation (10) for the first–order derivatives. Lagrange multipliers $\lambda_{\underline{i}}$ are given by equation (30), and are used in transformation equation (18) for the second–order derivatives.

We differentiate equation (20) with respect to $x^{\underline{j}}$,

$$H_{,\underline{i}\underline{j}} + H_{,\underline{i}}^{\ ,r}\tau_{,r\underline{j}} + H_{,\underline{j}}^{\ ,r}\tau_{,r\underline{i}} + H^{,rs}\tau_{,r\underline{i}}\tau_{,s\underline{j}} + H^{,r}\tau_{,r\underline{i}\underline{j}} = 0 \quad . \tag{40}$$

We insert (38) into (40), and arrive at

$$H_{,\underline{i}\underline{j}} + H_{,\underline{i}}^{\ ,r}\tau_{,r\underline{j}} + H_{,\underline{j}}^{\ ,r}\tau_{,r\underline{i}} + H^{,rs}\tau_{,r\underline{i}}\tau_{,s\underline{j}} + H^{,r}T_{r\underline{i}\underline{j}} + F_{,\underline{i}}H^{,r}\,\lambda_{r\underline{j}} + F_{,\underline{j}}H^{,r}\,\lambda_{r\underline{i}} + H^{,r}F_{,r}\,\lambda_{\underline{i}\underline{j}} = 0 \ . \tag{41}$$

We express equation (41) as

$$H^{,r}F_{,r}\,\lambda_{\underline{i}\underline{j}} + F_{,\underline{i}}\,H^{,r}\lambda_{r\underline{j}} + F_{,\underline{j}}\,H^{,r}\lambda_{r\underline{i}} = -S_{\underline{i}\underline{j}} \quad , \tag{42}$$

where

$$\boxed{S_{\underline{i}\underline{j}} = H_{,\underline{i}\underline{j}} + H_{,\underline{i}}^{\ ,r}\tau_{,r\underline{j}} + H_{,\underline{j}}^{\ ,r}\tau_{,r\underline{i}} + H^{,rs}\tau_{,r\underline{i}}\tau_{,s\underline{j}} + H^{,r}T_{r\underline{i}\underline{j}} \quad .} \tag{43}$$

Analogously to solution (28) of equation (23), which has the form of

$$\lambda_{\underline{i}} = -(H^{,q}F_{,q})^{-1}S_{\underline{i}} + F_{,\underline{i}}A \quad , \tag{44}$$

we shall search for the solution of equation (42) in the form

$$\lambda_{\underline{i}\underline{j}} = -(H^{,q}F_{,q})^{-1}S_{\underline{i}\underline{j}} + F_{,\underline{i}}A_{\underline{j}} + F_{,\underline{j}}A_{\underline{i}} + F_{,\underline{i}}F_{,\underline{j}}A \quad . \tag{45}$$

Here we use unknowns A and $A_{\underline{i}}$ in expression (45) only locally, and A has no relation to A in expression (44). We insert (45) into (42),

$$\begin{aligned}-S_{\underline{i}\underline{j}} - (H^{,q}F_{,q})^{-1}\big(F_{,\underline{i}}\,H^{,r}S_{r\underline{j}} + F_{,\underline{j}}\,H^{,r}S_{r\underline{i}}\big)&\\ + 2H^{,r}F_{,r}\big(F_{,\underline{i}}A_{\underline{j}} + F_{,\underline{j}}A_{\underline{i}}\big) + F_{,\underline{i}}F_{,j}\,H^{,r}A_r + 3H^{,r}F_{,r}\,F_{,\underline{i}}F_{,j}A &= -S_{\underline{i}\underline{j}} \quad .\end{aligned} \tag{46}$$

We fit the terms with single gradient $F_{,i}$ by

$$A_{\underline{i}} = \tfrac{1}{2}(H^{,q}F_{,q})^{-2}H^{,r}S_{r\underline{i}} \quad , \tag{47}$$

and then the terms with dyadic product $F_{,\underline{i}}F_{,\underline{j}}$ by

$$A = -\tfrac{2}{3}(H^{,q}F_{,q})^{-1}H^{,r}A_r = -\tfrac{1}{3}(H^{,q}F_{,q})^{-3}H^{,r}S_{rs}H^{,s} \quad . \tag{48}$$

We insert (47) and (48) into (45) and obtain solution

$$\begin{aligned}\lambda_{\underline{i}\underline{j}} = -(H^{,q}F_{,q})^{-1}S_{\underline{i}\underline{j}} + \tfrac{1}{2}(H^{,s}F_{,s})^{-2}\big(F_{,\underline{i}}\,H^{,r}S_{r\underline{j}} + F_{,\underline{j}}\,H^{,r}S_{r\underline{i}}\big)&\\ - \tfrac{1}{3}(H^{,q}F_{,q})^{-3}F_{,\underline{i}}F_{,\underline{j}}\,H^{,r}S_{rs}H^{,s}&\end{aligned} \tag{49}$$

of equation (42). Equation (49) may be expressed more concisely in terms of normalized gradient (29),

$$\boxed{\lambda_{\underline{i}\underline{j}} = (H^{,q}F_{,q})^{-1}\left[-S_{\underline{i}\underline{j}} + \tfrac{1}{2}\big(N_{\underline{i}}\,H^{,r}S_{r\underline{j}} + N_{\underline{j}}\,H^{,r}S_{r\underline{i}}\big) - \tfrac{1}{3}N_{\underline{i}}N_{\underline{j}}\,H^{,r}S_{rs}H^{,s}\right] \quad ,} \tag{50}$$

where $S_{\underline{i}\underline{j}}$ is given by (43). Equation (38) with (39) and (50) represents the explicit equation for the transformation of the third–order derivatives of travel time at a general interface between two general media.

Note that equation (38) with inserted Lagrange multipliers (50) reads

$$\begin{aligned}\tau_{,\underline{i}\underline{j}\underline{k}} = T_{\underline{i}\underline{j}\underline{k}} &- N_{\underline{i}}S_{\underline{j}\underline{k}} - N_{\underline{j}}S_{\underline{i}\underline{k}} - N_{\underline{k}}S_{\underline{i}\underline{j}}\\ &+ N_{\underline{i}}N_{\underline{j}}H^{,r}S_{r\underline{k}} + N_{\underline{i}}N_{\underline{k}}H^{,r}S_{r\underline{j}} + N_{\underline{j}}N_{\underline{k}}H^{,r}S_{r\underline{i}}\\ &- N_{\underline{i}}N_{\underline{j}}N_{\underline{k}}\,H^{,r}S_{rs}H^{,s} \quad .\end{aligned} \tag{51}$$

6. Transformation of the fourth–order derivatives of travel time

We differentiate equation (33) with respect to $x^{\underline{l}}$ under constraint (1) and obtain equation

$$\begin{aligned}\tau_{,\underline{i}\underline{j}\underline{k}\underline{l}} = \widetilde{\tau}_{,\underline{i}\underline{j}\underline{k}\underline{l}} &+ F_{,\underline{i}\underline{j}\underline{k}\underline{l}}\,\lambda\\ &+ F_{,\underline{i}\underline{j}\underline{k}}\,\lambda_{,\underline{l}} + F_{,\underline{i}\underline{j}\underline{l}}\,\lambda_{,\underline{k}} + F_{,\underline{i}\underline{k}\underline{l}}\,\lambda_{,\underline{j}} + F_{,\underline{j}\underline{k}\underline{l}}\,\lambda_{\underline{i}}\\ &+ F_{,\underline{i}\underline{j}}\,\lambda_{,\underline{k}\underline{l}} + F_{,\underline{i}\underline{k}}\,\lambda_{,\underline{j}\underline{l}} + F_{,\underline{j}\underline{k}}\,\lambda_{\underline{i},\underline{l}} + F_{,\underline{i}\underline{l}}\,\lambda_{,\underline{j}\underline{k}} + F_{,\underline{j}\underline{l}}\,\lambda_{\underline{i},\underline{k}} + F_{,\underline{k}\underline{l}}\,\lambda_{\underline{i}\underline{j}}\\ &+ F_{,\underline{i}}\,\lambda_{,\underline{j}\underline{k}\underline{l}} + F_{,\underline{j}}\,\lambda_{\underline{i},\underline{k}\underline{l}} + F_{,\underline{k}}\,\lambda_{\underline{i}\underline{j},\underline{l}} + F_{,\underline{l}}\,\lambda_{\underline{i}\underline{j}\underline{k}} \quad ,\end{aligned} \tag{52}$$

where $\lambda_{\underline{i}\underline{j}\underline{k}} = \lambda_{\underline{i}\underline{j}\underline{k}}(x^m, f^\mu)$ is the corresponding Lagrange multiplier. These Lagrange multipliers are defined for x^m satisfying equation (1).

Since $\tau_{,\underline{i}\underline{j}\underline{k}\underline{l}}$ is symmetric with respect to its subscripts,

$$N_{\underline{a}\underline{k}}E_{\underline{b}\underline{l}}\tau_{,\underline{i}\underline{j}\underline{k}\underline{l}} = N_{\underline{a}\underline{k}}E_{\underline{b}\underline{l}}\tau_{,\underline{i}\underline{j}\underline{l}\underline{k}} \quad . \tag{53}$$

We insert (52) into (53), and arrive at

$$E_{\underline{a}\underline{l}}\lambda_{\underline{i}\underline{j},\underline{l}} = E_{\underline{a}\underline{l}}\lambda_{\underline{i}\underline{j}\underline{l}} \quad . \tag{54}$$

Since $N_{\underline{a}\underline{l}}\lambda_{\underline{i}\underline{j},\underline{l}}$ is not defined, we may put

$$N_{\underline{a}\underline{l}}\lambda_{\underline{i}\underline{j},\underline{l}} = N_{\underline{a}\underline{l}}\lambda_{\underline{i}\underline{j}\underline{l}} \quad . \tag{55}$$

The summation of (54) and (55) yields

$$\lambda_{\underline{i}\underline{j},\underline{k}} = \lambda_{\underline{i}\underline{j}\underline{k}} \quad . \tag{56}$$

We insert (17), (37) and (56) into equation (52),

$$\boxed{\tau_{,\underline{i}\underline{j}\underline{k}\underline{l}} = T_{\underline{i}\underline{j}\underline{k}\underline{l}} + F_{,\underline{i}}\,\lambda_{\underline{j}\underline{k}\underline{l}} + F_{,\underline{j}}\,\lambda_{\underline{i}\underline{k}\underline{l}} + F_{,\underline{k}}\,\lambda_{\underline{i}\underline{j}\underline{l}} + F_{,\underline{l}}\,\lambda_{\underline{i}\underline{j}\underline{k}} \quad ,} \tag{57}$$

where the values of

$$\boxed{\begin{aligned} T_{\underline{i}\underline{j}\underline{k}\underline{l}} = \widetilde{\tau}_{,\underline{i}\underline{j}\underline{k}\underline{l}} &+ F_{,\underline{i}\underline{j}\underline{k}\underline{l}}\,\lambda \\ &+ F_{,\underline{i}\underline{j}\underline{k}}\,\lambda_{\underline{l}} + F_{,\underline{i}\underline{j}\underline{l}}\,\lambda_{\underline{k}} + F_{,\underline{i}\underline{k}\underline{l}}\,\lambda_{\underline{j}} + F_{,\underline{j}\underline{k}\underline{l}}\,\lambda_{\underline{i}} \\ &+ F_{,\underline{i}\underline{j}}\,\lambda_{\underline{k}\underline{l}} + F_{,\underline{i}\underline{k}}\,\lambda_{\underline{j}\underline{l}} + F_{,\underline{j}\underline{k}}\,\lambda_{\underline{i}\underline{l}} + F_{,\underline{i}\underline{l}}\,\lambda_{\underline{j}\underline{k}} + F_{,\underline{j}\underline{l}}\,\lambda_{\underline{i}\underline{k}} + F_{,\underline{k}\underline{l}}\,\lambda_{\underline{i}\underline{j}} \end{aligned}} \tag{58}$$

are known. Lagrange multiplier λ is determined by algebraic equation (3), and is used in transformation equation (10) for the first–order derivatives. Lagrange multipliers $\lambda_{\underline{i}}$ are given by equation (30), and are used in transformation equation (18) for the second–order derivatives. Lagrange multipliers $\lambda_{\underline{i}\underline{j}}$ are given by equation (50), and are used in transformation equation (38) for the third–order derivatives.

We differentiate equation (40) with respect to $x^{\underline{k}}$,

$$\begin{aligned} H_{,\underline{i}\underline{j}\underline{k}} &+ H_{\underline{i}\underline{j}}^{;r}\tau_{,r\underline{k}} + H_{\underline{i}\underline{k}}^{;r}\tau_{,r\underline{j}} + H_{\underline{j}\underline{k}}^{;r}\tau_{,r\underline{i}} \\ &+ H_{\underline{i}}^{;rs}\tau_{,r\underline{j}}\tau_{,s\underline{k}} + H_{\underline{j}}^{;rs}\tau_{,r\underline{i}}\tau_{,s\underline{k}} + H_{\underline{k}}^{;rs}\tau_{,r\underline{i}}\tau_{,s\underline{j}} + H^{,rst}\tau_{,r\underline{i}}\tau_{,s\underline{j}}\tau_{,t\underline{k}} \\ &+ H_{\underline{i}}^{;r}\tau_{,r\underline{j}\underline{k}} + H_{\underline{j}}^{;r}\tau_{,r\underline{i}\underline{k}} + H_{\underline{k}}^{;r}\tau_{,r\underline{i}\underline{j}} \\ &+ H^{,rs}\tau_{,r\underline{i}}\tau_{,s\underline{j}\underline{k}} + H^{,rs}\tau_{,r\underline{j}}\tau_{,s\underline{i}\underline{k}} + H^{,rs}\tau_{,r\underline{k}}\tau_{,s\underline{i}\underline{j}} + H^{,r}\tau_{,r\underline{i}\underline{j}\underline{k}} = 0 \quad . \end{aligned} \tag{59}$$

We insert (57) into (59), and arrive at

$$H^{,r}F_{,r}\,\lambda_{\underline{i}\underline{j}\underline{k}} + F_{,\underline{i}}\,H^{,r}\lambda_{r\underline{j}\underline{k}} + F_{,\underline{j}}\,H^{,r}\lambda_{r\underline{i}\underline{k}} + F_{,\underline{k}}\,H^{,r}\lambda_{r\underline{i}\underline{j}} = -S_{\underline{i}\underline{j}\underline{k}} \quad , \tag{60}$$

where

$$\boxed{\begin{aligned} S_{\underline{i}\underline{j}\underline{k}} = H_{,\underline{i}\underline{j}\underline{k}} &+ H_{\underline{i}\underline{j}}^{;r}\tau_{,r\underline{k}} + H_{\underline{i}\underline{k}}^{;r}\tau_{,r\underline{j}} + H_{\underline{j}\underline{k}}^{;r}\tau_{,r\underline{i}} \\ &+ H_{\underline{i}}^{;rs}\tau_{,r\underline{j}}\tau_{,s\underline{k}} + H_{\underline{j}}^{;rs}\tau_{,r\underline{i}}\tau_{,s\underline{k}} + H_{\underline{k}}^{;rs}\tau_{,r\underline{i}}\tau_{,s\underline{j}} + H^{,rst}\tau_{,r\underline{i}}\tau_{,s\underline{j}}\tau_{,t\underline{k}} \\ &+ H_{\underline{i}}^{;r}\tau_{,r\underline{j}\underline{k}} + H_{\underline{j}}^{;r}\tau_{,r\underline{i}\underline{k}} + H_{\underline{k}}^{;r}\tau_{,r\underline{i}\underline{j}} \\ &+ H^{,rs}\tau_{,r\underline{i}}\tau_{,s\underline{j}\underline{k}} + H^{,rs}\tau_{,r\underline{j}}\tau_{,s\underline{i}\underline{k}} + H^{,rs}\tau_{,r\underline{k}}\tau_{,s\underline{i}\underline{j}} + H^{,r}T_{,r\underline{i}\underline{j}\underline{k}} \quad . \end{aligned}} \tag{61}$$

Analogously to (45), we shall search for the solution of equation (60) in the form

$$\begin{aligned} \lambda_{\underline{i}\underline{j}\underline{k}} = -(H^{,q}F_{,q})^{-1}S_{\underline{i}\underline{j}\underline{k}} &+ F_{,\underline{i}}A_{\underline{j}\underline{k}} + F_{,\underline{j}}A_{\underline{i}\underline{k}} + F_{,\underline{k}}A_{\underline{i}\underline{j}} \\ &+ F_{,\underline{i}}F_{,\underline{j}}A_{\underline{k}} + F_{,\underline{i}}F_{,\underline{k}}A_{\underline{j}} + F_{,\underline{j}}F_{,\underline{k}}A_{\underline{i}} + F_{,\underline{i}}F_{,\underline{j}}F_{,\underline{k}}A \quad . \end{aligned} \tag{62}$$

Here we use unknowns A, $A_{\underline{i}}$ and $A_{\underline{i}\underline{j}}$ in expression (62) only locally: they have no relation to A and $A_{\underline{i}}$ used locally in the previous section. We insert (62) into (60),

$$\begin{aligned}
-S_{\underline{i}\underline{j}\underline{k}} - (H^{,q}F_{,q})^{-1}\big(&F_{,\underline{i}}\,H^{,r}S_{r\underline{j}\underline{k}} + F_{,\underline{j}}\,H^{,r}S_{r\underline{i}\underline{k}} + F_{,\underline{k}}\,H^{,r}S_{r\underline{i}\underline{j}}\big)\\
&+ 2H^{,r}F_{,r}\big(F_{,\underline{i}}A_{\underline{j}\underline{k}} + F_{,\underline{j}}A_{\underline{i}\underline{k}} + F_{,\underline{k}}A_{\underline{i}\underline{j}}\big)\\
&+ 2\big(F_{,\underline{i}}F_{,\underline{j}}H^{,r}A_{r\underline{k}} + F_{,\underline{i}}F_{,\underline{k}}H^{,r}A_{r\underline{j}} + F_{,\underline{j}}F_{,\underline{k}}H^{,r}A_{r\underline{i}}\big)\\
&+ 3H^{,r}F_{,r}\big(F_{,\underline{i}}F_{,\underline{j}}A_{\underline{k}} + F_{,\underline{i}}F_{,\underline{k}}A_{\underline{j}} + F_{,\underline{j}}F_{,\underline{k}}A_{\underline{i}}\big)\\
&+ 3F_{,\underline{i}}F_{,\underline{j}}F_{,\underline{k}}\,H^{,r}A_{r} + 4H^{,r}F_{,r}\,F_{,\underline{i}}F_{,\underline{j}}F_{,\underline{k}}A = -S_{\underline{i}\underline{j}\underline{k}} \quad . \qquad (63)
\end{aligned}$$

We fit the terms with single gradient $F_{,i}$ by

$$A_{\underline{i}\underline{j}} = \tfrac{1}{2}(H^{,q}F_{,q})^{-2}H^{,r}S_{r\underline{i}\underline{j}} \quad , \qquad (64)$$

then the terms with dyadic product $F_{,\underline{i}}F_{,\underline{j}}$ by

$$A_{\underline{i}} = -\tfrac{2}{3}(H^{,q}F_{,q})^{-1}H^{,r}A_{r\underline{i}} = -\tfrac{1}{3}(H^{,q}F_{,q})^{-3}H^{,r}H^{,s}S_{rs\underline{i}} \quad , \qquad (65)$$

and then the terms with triadic product $F_{,\underline{i}}F_{,\underline{j}}F_{,\underline{k}}$ by

$$A = -\tfrac{3}{4}(H^{,q}F_{,q})^{-1}H^{,r}A_{r} = \tfrac{1}{4}(H^{,q}F_{,q})^{-4}H^{,r}H^{,s}H^{,t}S_{rst} \quad . \qquad (66)$$

We insert (64), (65) and (66) into (62) and obtain solution

$$\begin{aligned}
\lambda_{\underline{i}\underline{j}\underline{k}} = -(H^{,q}F_{,q})^{-1}S_{\underline{i}\underline{j}\underline{k}} &+ \tfrac{1}{2}(H^{,q}F_{,q})^{-2}\big(F_{,\underline{i}}\,H^{,r}S_{r\underline{j}\underline{k}} + F_{,\underline{j}}\,H^{,r}S_{r\underline{i}\underline{k}} + F_{,\underline{k}}\,H^{,r}S_{r\underline{j}\underline{k}}\big)\\
&- \tfrac{1}{3}(H^{,q}F_{,q})^{-3}H^{,r}H^{,s}\big(F_{,\underline{i}}F_{,\underline{j}}S_{rs\underline{k}} + F_{,\underline{i}}F_{,\underline{k}}S_{rs\underline{j}} + F_{,\underline{j}}F_{,\underline{k}}S_{rs\underline{i}}\big)\\
&+ \tfrac{1}{4}(H^{,q}F_{,q})^{-4}F_{,\underline{i}}F_{,\underline{j}}F_{,\underline{k}}\,H^{,r}H^{,s}H^{,t}S_{rst} \qquad (67)
\end{aligned}$$

of equation (60). Equation (67) may be expressed more concisely in terms of normalized gradient (29),

$$\boxed{\begin{aligned}
\lambda_{\underline{i}\underline{j}\underline{k}} = (H^{,q}F_{,q})^{-1}\big[-S_{\underline{i}\underline{j}\underline{k}} &+ \tfrac{1}{2}\big(N_{\underline{i}}\,H^{,r}S_{r\underline{j}\underline{k}} + N_{\underline{j}}\,H^{,r}S_{r\underline{i}\underline{k}} + N_{\underline{k}}\,H^{,r}S_{r\underline{j}\underline{k}}\big)\\
&- \tfrac{1}{3}H^{,r}H^{,s}\big(N_{\underline{i}}N_{\underline{j}}S_{rs\underline{k}} + N_{\underline{i}}N_{\underline{k}}S_{rs\underline{j}} + N_{\underline{j}}N_{\underline{k}}S_{rs\underline{i}}\big)\\
&+ \tfrac{1}{4}N_{\underline{i}}N_{\underline{j}}N_{\underline{k}}\,H^{,r}H^{,s}H^{,t}S_{rst}\big] \quad , \qquad (68)
\end{aligned}}$$

where $S_{\underline{i}\underline{j}\underline{k}}$ is given by (61). Equation (57) with (58) and (68) represents the explicit equation for the transformation of the fourth–order derivatives of travel time at a general interface between two general media.

Note that equation (57) with inserted Lagrange multipliers (68) reads

$$\begin{aligned}
\tau_{,\underline{i}\underline{j}\underline{k}\underline{l}} = T_{\underline{i}\underline{j}\underline{k}\underline{l}} &- N_{\underline{i}}S_{\underline{j}\underline{k}\underline{l}} - N_{\underline{j}}S_{\underline{i}\underline{k}\underline{l}} - N_{\underline{k}}S_{\underline{i}\underline{j}\underline{l}} - N_{\underline{l}}S_{\underline{i}\underline{j}\underline{k}}\\
&+ N_{\underline{i}}N_{\underline{j}}H^{,r}S_{r\underline{k}\underline{l}} + N_{\underline{i}}N_{\underline{k}}H^{,r}S_{r\underline{j}\underline{l}} + N_{\underline{i}}N_{\underline{l}}H^{,r}S_{r\underline{j}\underline{k}}\\
&+ N_{\underline{j}}N_{\underline{k}}H^{,r}S_{r\underline{i}\underline{l}} + N_{\underline{j}}N_{\underline{l}}H^{,r}S_{r\underline{i}\underline{k}} + N_{\underline{k}}N_{\underline{l}}H^{,r}S_{r\underline{i}\underline{j}}\\
&- H^{,r}H^{,s}\big(N_{\underline{i}}N_{\underline{j}}N_{\underline{k}}S_{rs\underline{l}} + N_{\underline{i}}N_{\underline{j}}N_{\underline{l}}S_{rs\underline{k}} + N_{\underline{i}}N_{\underline{k}}N_{\underline{l}}S_{rs\underline{j}} + N_{\underline{j}}N_{\underline{k}}N_{\underline{l}}S_{rs\underline{i}}\big)\\
&+ N_{\underline{i}}N_{\underline{j}}N_{\underline{k}}N_{\underline{l}}\,H^{,r}H^{,s}H^{,t}S_{rst} \quad . \qquad (69)
\end{aligned}$$

7. Transformation of higher–order derivatives of travel time

For higher–order derivatives, we obtain transformation equation

$$\boxed{\tau_{,\underline{i}\underline{j}\dots\underline{m}\underline{n}} = T_{,\underline{i}\underline{j}\dots\underline{m}\underline{n}} + \sum_{\substack{\{\underline{a}\}\subset\{\underline{i}\underline{j}\dots\underline{m}\underline{n}\}\\ \{\underline{b}\dots\underline{h}\}=\{\underline{i}\underline{j}\dots\underline{m}\underline{n}\}\setminus\{\underline{a}\}}} F_{,\underline{a}}\,\lambda_{\underline{b}\dots\underline{h}}} \tag{70}$$

analogous to equations (18), (38) and (57). Here $\{\underline{i}\underline{j}...\underline{m}\underline{n}\}$ represents the set of indices $\underline{i}$, $\underline{j}$, ..., $\underline{m}$, $\underline{n}$, and $\sum_{\{\underline{a}\}\subset\{\underline{i}\underline{j}\dots\underline{m}\underline{n}\}}$ implies the summation over all single–element subsets of set $\{\underline{i}\underline{j}...\underline{m}\underline{n}\}$, i.e., over the subscripts of $\tau_{,\underline{i}\underline{j}\dots\underline{m}\underline{n}}$. Notation $\{\underline{b}...\underline{h}\} = \{\underline{i}\underline{j}...\underline{m}\underline{n}\}\setminus\{\underline{a}\}$ means that $\underline{b}...\underline{h}$ represents all indices other than $\underline{a}$.

In equation (70), we have put

$$\begin{aligned}
T_{,\underline{i}\underline{j}\dots\underline{m}\underline{n}} = \widetilde{\tau}_{,\underline{i}\underline{j}\dots\underline{m}\underline{n}} &+ F_{,\underline{i}\underline{j}\dots\underline{m}\underline{n}}\,\lambda \\
&+ \sum_{\substack{\{\underline{a}\}\subset\{\underline{i}\underline{j}\dots\underline{m}\underline{n}\}\\ \{\underline{b}\dots\underline{h}\}=\{\underline{i}\underline{j}\dots\underline{m}\underline{n}\}\setminus\{\underline{a}\}}} F_{,\underline{b}\dots\underline{h}}\,\lambda_{\underline{a}} \\
&+ \sum_{\substack{\{\underline{a}\underline{b}\}\subset\{\underline{i}\underline{j}\dots\underline{m}\underline{n}\}\\ \{\underline{c}\dots\underline{h}\}=\{\underline{i}\underline{j}\dots\underline{m}\underline{n}\}\setminus\{\underline{a}\underline{b}\}}} F_{,\underline{c}\dots\underline{h}}\,\lambda_{\underline{a}\underline{b}} \\
&+ \sum_{\substack{\{\underline{a}\underline{b}\underline{c}\}\subset\{\underline{i}\underline{j}\dots\underline{m}\underline{n}\}\\ \{\underline{d}\dots\underline{h}\}=\{\underline{i}\underline{j}\dots\underline{m}\underline{n}\}\setminus\{\underline{a}\underline{b}\underline{c}\}}} F_{,\underline{d}\dots\underline{h}}\,\lambda_{\underline{a}\underline{b}\underline{c}} \\
&\ \vdots \\
&+ \sum_{\substack{\{\underline{g}\underline{h}\}\subset\{\underline{i}\underline{j}\dots\underline{m}\underline{n}\}\\ \{\underline{a}\dots\underline{f}\}=\{\underline{i}\underline{j}\dots\underline{m}\underline{n}\}\setminus\{\underline{g}\underline{h}\}}} F_{,\underline{g}\underline{h}}\,\lambda_{\underline{a}\dots\underline{f}}
\end{aligned} \tag{71}$$

analogously to expressions (19), (39) and (58). Here $\sum_{\{\underline{a}\underline{b}\}\subset\{\underline{i}\underline{j}\dots\underline{m}\underline{n}\}}$ implies the summation over all two–element subsets of set $\{\underline{i}\underline{j}...\underline{m}\underline{n}\}$, and $\{\underline{c}...\underline{h}\} = \{\underline{i}\underline{j}...\underline{m}\underline{n}\}\setminus\{\underline{a}\underline{b}\}$ means that $\underline{c}...\underline{h}$ represents all indices other than $\underline{a}$ and $\underline{b}$. The values of all Lagrange multipliers in equation (71) are known from the transformations of the derivatives of orders lower than $\tau_{,\underline{i}\underline{j}\dots\underline{m}\underline{n}}$.

We then differentiate the Hamilton–Jacobi equation (3) with respect to $x^{\underline{i}}$, $x^{\underline{j}}$, ..., $x^{\underline{m}}$, insert (70) for $\tau_{,\underline{i}\underline{j}\dots\underline{m}r}$, and obtain equation

$$H^{,r}F_{,r}\,\lambda_{\underline{i}\underline{j}\dots\underline{m}} + \sum_{\substack{\{\underline{a}\}\subset\{\underline{i}\underline{j}\dots\underline{m}\}\\ \{\underline{b}\dots\underline{h}\}=\{\underline{i}\underline{j}\dots\underline{m}\}\setminus\{\underline{a}\}}} F_{,\underline{a}}\,\lambda_{\underline{b}\dots\underline{h}r}H^{,r} = -S_{\underline{i}\underline{j}\dots\underline{m}} \tag{72}$$

for Lagrange multipliers $\lambda_{\underline{i}\underline{j}\dots\underline{m}}$, where

$$\boxed{\begin{aligned} S_{\underline{i}\underline{j}\dots\underline{m}} = &\left(\frac{\partial}{\partial x^{\underline{i}}}+\tau_{,\underline{i}r}\frac{\partial}{\partial p_r}\right)\left(\frac{\partial}{\partial x^{\underline{j}}}+\tau_{,\underline{j}s}\frac{\partial}{\partial p_s}\right)\dots\left(\frac{\partial}{\partial x^{\underline{m}}}+\tau_{,\underline{m}z}\frac{\partial}{\partial p_z}\right)H \\ &- \tau_{,\underline{i}\underline{j}\dots\underline{m}r}H^{,r} + T_{,\underline{i}\underline{j}\dots\underline{m}r}H^{,r} \end{aligned}} \tag{73}$$

represents the generalization of expressions (24), (43) and (61) to higher–order derivatives. Equation (73) contains the derivatives of orders lower than $\tau_{,\underline{i}\underline{j}...\underline{m}\underline{n}}$ only.

The Lagrange multipliers in equation (70) for the transformation of the higher–order derivatives of travel time at a general interface between two general media then read

$$\boxed{\begin{aligned}\lambda_{\underline{i}\underline{j}...\underline{m}} = (H^{,q}F_{,q})^{-1}\Big[& - S_{\underline{i}\underline{j}...\underline{m}} \\ & + \tfrac{1}{2} \sum_{\substack{\{\underline{a}\}\subset\{\underline{i}\underline{j}...\underline{m}\} \\ \{\underline{b}...\underline{h}\}=\{\underline{i}\underline{j}...\underline{m}\}\setminus\{\underline{a}\}}} N_{\underline{a}}\, H^{,r} S_{r\underline{b}...\underline{h}} \\ & - \tfrac{1}{3} \sum_{\substack{\{\underline{a}\underline{b}\}\subset\{\underline{i}\underline{j}...\underline{m}\} \\ \{\underline{c}...\underline{h}\}=\{\underline{i}\underline{j}...\underline{m}\}\setminus\{\underline{a}\underline{b}\}}} N_{\underline{a}} N_{\underline{b}}\, H^{,r} H^{,s} S_{rs\underline{c}...\underline{h}} \\ & + \tfrac{1}{4} \sum_{\substack{\{\underline{a}\underline{b}\underline{c}\}\subset\{\underline{i}\underline{j}...\underline{m}\} \\ \{\underline{d}...\underline{h}\}=\{\underline{i}\underline{j}...\underline{m}\}\setminus\{\underline{a}\underline{b}\underline{c}\}}} N_{\underline{a}} N_{\underline{b}} N_{\underline{c}}\, H^{,r} H^{,s} H^{,t} S_{rst\underline{d}...\underline{h}} \\ & \vdots \\ & + \tfrac{(-1)^K}{K} N_{\underline{i}} N_{\underline{j}} ... N_{\underline{m}}\, H^{,r} H^{,s} ... H^{,z} S_{rs...z} \Big] \quad ,\end{aligned}} \qquad (74)$$

where K is the order of derivative $\tau_{,\underline{i}\underline{j}...\underline{m}\underline{n}}$. We may easily check that Lagrange multipliers (74) satisfy equation (72). Equation (74) represents the generalization of equations (30), (50) and (68) to the higher–order derivatives.

Note that equation (70) with inserted Lagrange multipliers (74) reads

$$\begin{aligned}\tau_{,\underline{i}\underline{j}...\underline{m}\underline{n}} = T_{\underline{i}\underline{j}...\underline{m}\underline{n}} & - \sum_{\substack{\{\underline{a}\}\subset\{\underline{i}\underline{j}...\underline{m}\underline{n}\} \\ \{\underline{b}...\underline{h}\}=\{\underline{i}\underline{j}...\underline{m}\underline{n}\}\setminus\{\underline{a}\}}} N_{\underline{a}}\, S_{\underline{b}...\underline{h}} \\ & + \sum_{\substack{\{\underline{a}\underline{b}\}\subset\{\underline{i}\underline{j}...\underline{m}\underline{n}\} \\ \{\underline{c}...\underline{h}\}=\{\underline{i}\underline{j}...\underline{m}\underline{n}\}\setminus\{\underline{a}\underline{b}\}}} N_{\underline{a}} N_{\underline{b}}\, H^{,r} S_{r\underline{c}...\underline{h}} \\ & - \sum_{\substack{\{\underline{a}\underline{b}\underline{c}\}\subset\{\underline{i}\underline{j}...\underline{m}\underline{n}\} \\ \{\underline{d}...\underline{h}\}=\{\underline{i}\underline{j}...\underline{m}\underline{n}\}\setminus\{\underline{a}\underline{b}\underline{c}\}}} N_{\underline{a}} N_{\underline{b}} N_{\underline{c}}\, H^{,r} H^{,s} S_{rs\underline{d}...\underline{h}} \\ & + \sum_{\substack{\{\underline{a}\underline{b}\underline{c}\underline{d}\}\subset\{\underline{i}\underline{j}...\underline{m}\underline{n}\} \\ \{\underline{e}...\underline{h}\}=\{\underline{i}\underline{j}...\underline{m}\underline{n}\}\setminus\{\underline{a}\underline{b}\underline{c}\underline{d}\}}} N_{\underline{a}} N_{\underline{b}} N_{\underline{c}} N_{\underline{d}}\, H^{,r} H^{,s} H^{,t} S_{rst\underline{e}...\underline{h}} \\ & \vdots \\ & + (-1)^K N_{\underline{i}} N_{\underline{j}} ... N_{\underline{m}} N_{\underline{n}}\, H^{,r} H^{,s} ... H^{,z} S_{rs...z} \quad ,\end{aligned} \qquad (75)$$

where K is the order of derivative $\tau_{,\underline{i}\underline{j}...\underline{m}\underline{n}}$. Equation (75) represents the generalization of equations (32), (51) and (69) to the higher–order derivatives.

Acknowledgements

The research has been supported by the Grant Agency of the Czech Republic under contracts 205/07/0032 and P210/10/0736, by the Ministry of Education of the Czech Republic within research project MSM0021620860, and by the members of the consortium "Seismic Waves in Complex 3–D Structures" (see "http://sw3d.cz").

References

Červený, V. (1972): Seismic rays and ray intensities in inhomogeneous anisotropic media. *Geophys. J. R. astr. Soc.*, **29**, 1–13, online at "http://sw3d.cz".

Červený, V. (2001): *Seismic Ray Theory.* Cambridge Univ. Press, Cambridge.

Hamilton, W.R. (1837): Third supplement to an essay on the theory of systems of rays. *Trans. Roy. Irish Acad.*, **17**, 1–144, read January 23, 1832, and October 22, 1832.

Klimeš, L. (2002): Second–order and higher–order perturbations of travel time in isotropic and anisotropic media. *Stud. geophys. geod.*, **46**, 213–248, online at "http://sw3d.cz".