Point sources of acoustic waves at interfaces (reciprocity relations approach)

Vlastislav Červený

Charles University in Prague, Faculty of Mathematics and Physics,
Department of Geophysics, Ke Karlovu 3, 121 16 Praha 2, Czech Republic.
E-mail: vcerveny@seis.karlov.mff.cuni.cz

Summary

Point sources of acoustic waves situated on material interfaces in fluid inhomogeneous, non-moving, non-dissipative media are studied. Two types of point sources are considered, namely the volume injection point sources, and volume force point sources. The volume injection point sources correspond to the local violation of constitutive relation. The reciprocity relations by Fokkema and van den Berg are used to prove that the acoustic pressure wave field at any point of the medium does not change when the volume-injection point source crosses a material interface. This general property of volume injection point sources at material interfaces is verified on the analytical ray-theory expressions for pressure wave field in fluid containing interfaces. The results obtained for the acoustic wave fields in fluid media allow also to make certain conclusions for strain-glut and stress-glut point sources at material interfaces in elastic isotropic and anisotropic media.

Key words: Acoustic waves. Reciprocity relations. Point sources at interfaces. Stress glut. Strain glut.

1 Introduction

Consider an unbounded fluid inhomogeneous non-dissipative non-moving medium, specified by two material parameters $\rho(\mathbf{x})$ and $\kappa(\mathbf{x})$, where $\rho(\mathbf{x})$ is the mass density (kg m^{-3}) and $\kappa(\mathbf{x})$ the compressibility (Pa^{-1}, kg^{-1}m s^2). Instead of compressibility $\kappa(\mathbf{x})$, we can also use the incompressibility $k(\mathbf{x}) = 1/\kappa(\mathbf{x})$ $(\mathrm{Pa}, \mathrm{kg\,m^{-1}s^{-2}})$.

The most common field variables in acoustic wave fields propagating in fluids are the acoustic pressure $p\,(\mathrm{Pa}, \mathrm{kg\,m^{-1}s^{-2}})$ and particle velocity vector $\mathbf{v}$, with Cartesian components $v_i\,(\mathrm{m\,s^{-1}})$. In a smooth medium, these field variables are mutually connected by the constitutive relation

$$\kappa\dot{p} + v_{i,i} = 0\ . \tag{1}$$

Seismic Waves in Complex 3-D Structures, Report 23, Charles University in Prague, Faculty of Mathematics and Physics, Department of Geophysics, Praha 2013, pp. 237–251

Here the dot above the letter denotes the partial derivative with respect to time, and the index behind the comma in the subscript of a letter denotes the partial derivative with respect to relevant Cartesian coordinate (i.e., $v_{i,i} = \partial v_i/\partial x_i$). The Einstein summation convention is used (e.g., $\partial v_i/\partial x_i = \partial v_1/\partial x_1 + \partial v_2/\partial x_2 + \partial v_3/\partial x_3$).

The linearized acoustic wave equations for fluid inhomogeneous, non-dissipative and non-moving media have been expressed in various forms. Often, they are expressed in terms of a scalar partial differential equation of the second order for the acoustic pressure p. We shall use here an alternative form, consisting of two coupled linear partial differential equation of the first order. The first is a vectorial equation for particle velocity $\mathbf{v}$ (or the three scalar equations for v_i, $i = 1, 2, 3$, the Cartesian components of $\mathbf{v}$) and the second the scalar equation for acoustic pressure p. The equations read

$$\rho \dot{v}_i + p_{,i} = f_i \ , \qquad \kappa \dot{p} + v_{i,i} = q \ . \tag{2}$$

The quantities f_i represent the Cartesian components of a vectorial source term $\mathbf{f}$, which represents the volume force density (force per volume, $N/m^3 = \mathrm{kg\,m^{-2}s^{-2}}$). The quantity q represents a scalar source term, representing volume injection rate density. It is measured in second^{-1}. We shall call it in an abbreviated way: volume injection source. In exploration seismology, a well known example of this source is the air-gun source.

The physical meaning of both acoustic equations in (2) is as follows: The first equation of (2) represents an equation of motion corresponding to the second Newton law, in which the wave motion is caused by the volume source density $\mathbf{f}$ of volume force. The second equation of (2), called the deformation equation, has a different meaning. It describes the wave motion caused by the volume source density q of volume injection rate. If the constitutive relation (1) is violated in some region and the zero on the R.H.S. is replaced by a function q, we obtain the second acoustic wave equation in (2). Physically, this q corresponds to the volume injection rate density.

We are interested here mainly in acoustic wave fields generated by point sources. As we have two different source terms in acoustic equations (2), we have to consider two types of point sources:

a) The **volume force** point source, situated at point A, is specified by two relations

$$q(\mathbf{x}) = 0 \ , \qquad \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}^A)\delta(\mathbf{x} - \mathbf{x}^A) \ . \tag{3}$$

b) The **volume injection** point source situated at point A is specified by two relations:

$$q(\mathbf{x}) = q(\mathbf{x}^A)\delta(\mathbf{x} - \mathbf{x}^A) \ , \qquad \mathbf{f}(\mathbf{x}) = 0 \ . \tag{4}$$

The fluid media with different medium parameters may be in contact along interfaces. To compute the field variables across the interface, we have to use the boundary conditions (also called interface conditions). The boundary conditions at an interface between two non-moving non-dissipative fluid media are well known from literature. The boundary conditions require that the acoustic pressure p and the normal component to the interface of the particle velocity are continuous across the interface. Consequently, we require continuity of

$$p \ \text{ and } \ n_k v_k \quad \text{across the interface} \ , \tag{5}$$

where p is the acoustic pressure, $\mathbf{v}$ particle velocity and $\mathbf{n}$ the unit vector perpendicular to the interface.

The field variables p and $\mathbf{v}$ in acoustic wave equations are functions of position, expressed in Cartesian coordinates x_i, and of time t, $p = p(\mathbf{x}, t)$ and $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$. We can also apply the Fourier, Laplace, or some other transform to eliminate time t, and work in the transformed domain.

2 Reciprocity relations

We now investigate the situation that the point source and the receiver interchange their position. In one situation, the point source is situated at a point A and the receiver at a point B. In the other situation, the position of a point source and receiver are interchanged. In this case, we can apply the well-known Rayleigh reciprocity principle (Rayleigh, 1894), which is however considerably more general than needed here. The reciprocity principles used here are taken from the book by Fokkema and van den Berg (1993), where the Rayleigh principle is simply and strictly derived and broadly discussed. See also DeHoop (1995).

Fokkema and van den Berg (1993) do not consider the acoustic wave equations in time domain, as are given in (2), but use the Laplace transform with the variable s, and work with Laplace transformed acoustic wave equations:

$$p_{,i} + s\rho v_i = f_i \ , \qquad v_{i,i} + s\kappa p = q \ . \tag{6}$$

For simplicity, we do not change the notation of variables in Laplace domain and time domain, we only distinguish them by arguments t and s. Consequently, in (6) we have $p = p(\mathbf{x}, s)$, $v_i = v_i(\mathbf{x}, s)$, $f_i = f_i(\mathbf{x}, s)$, $q = q(\mathbf{x}, s)$.

In the following, we shall use two important reciprocity relations:

a) Reciprocity relation for volume injection point source:

$$q(\mathbf{x}^B, s) p^A(\mathbf{x}^B, s) = q(\mathbf{x}^A, s) p^B(\mathbf{x}^A, s) \ , \tag{7}$$

see Fokkema and van den Berg (1993, eq. 6.9). Here $p^A(\mathbf{x}^B, s)$ denotes the acoustic pressure at point $\mathbf{x}^B$, due to the point source of the volume injection type $q(\mathbf{x}^A, s)$, acting at point A. The function $p^B(\mathbf{x}^A, s)$ has an analogous meaning. Equation (7) thus shows a full reciprocity if the point source of volume injection type and relevant acoustic pressure receiver interchange their positions.

b) Reciprocity relation for volume-force point source:

$$f_k(\mathbf{x}^B, s) v_k^A(\mathbf{x}^B, s) = f_k(\mathbf{x}^A, s) v_k^B(\mathbf{x}^A, s) \ , \tag{8}$$

see Fokkema and van den Berg (1993, eq. 6.13). Here $v_k^A(\mathbf{x}^B, s)$ denotes the k-th Cartesian component of the particle velocity at point $\mathbf{x}^B$ due to the volume force point source $\mathbf{f}(\mathbf{x}^A, s)$ acting at point A. The function $v_k^B(\mathbf{x}^A, s)$ has an analogous meaning. Note that the reciprocity relation (8), related to the volume-source point source $\mathbf{f}$, does not have such a general meaning in a fluid medium as in a solid medium. It applies only to scalar product of two quantities, not to quantities themselves.

The reciprocity relation (8) can be formally rewritten in several other forms. For example, we can express the volume-force point source as follows:

$$\mathbf{f}(\mathbf{x}^A, s) = f(\mathbf{x}^A, s)\mathbf{n}^f(\mathbf{x}^A) \ , \quad \mathbf{f}(\mathbf{x}^B, s) = f(\mathbf{x}^B, s)\mathbf{n}^f(\mathbf{x}^B) \ . \tag{9}$$

Here $\mathbf{n}^f(\mathbf{x}^A)$ and $\mathbf{n}^f(\mathbf{x}^B)$ are unit vectors specifying the directions of single-force point sources $\mathbf{f}(\mathbf{x}^A, s)$ and $\mathbf{f}(\mathbf{x}^B, s)$. Then we can express the reciprocity relation (8) in the following form

$$f(\mathbf{x}^B, s)n_k^f(\mathbf{x}^B)v_k^A(\mathbf{x}^B, s) = f(\mathbf{x}^A, s)n_k^f(\mathbf{x}^A)v_k^B(\mathbf{x}^A, s) \ . \tag{10}$$

Equation (10) formally represents a reciprocity relation between two scalar quantities, namely $f(\mathbf{x}^A, s)$ and $n_k^f(\mathbf{x}^B)v_k^A(\mathbf{x}^B, s)$. Relation (10) is very useful if points $\mathbf{x}^A$ and $\mathbf{x}^B$ are situated on interfaces and if the unit vectors $\mathbf{n}^f(\mathbf{x}^A)$ and $\mathbf{n}^f(\mathbf{x}^B)$ are chosen perpendicular to these interfaces.

3 Behaviour of point sources at interfaces

The reciprocity relations remain valid even when the fluid media under consideration contain interfaces. Across interfaces, however, the accustic presure p and normal components of the particle velocity $n_k v_k$ must be continuous, see (5). We shall now study the behaviour of the field variables p and $\mathbf{v}$ when the point source crosses the interface. For this, we work here in Laplace domain and exploit the reciprocity relations (7) and (10).

a) Volume injection point source q at an interface

Consider first an volume injection point source situated at an arbitrary point S in the medium. Then we consider two points, R_1 and R_2, situated on any interface, formally at the same position, but on different sides of the interface. Consequently, the points R_1 and R_2 are situated in different media which are in contact along the interface under consideration. The medium parameters ρ and κ are different at R_1 and R_2, $\rho(\mathbf{x}^{R_1}) \neq \rho(\mathbf{x}^{R_2})$, $\kappa(\mathbf{x}^{R_1}) \neq \kappa(\mathbf{x}^{R_2})$. At points R_1 and R_2, we can compute the pressure $p(\mathbf{x}^{R_1})$ and $p(\mathbf{x}^{R_2})$ due to a volume injection point source situated at S. But $p(\mathbf{x}^{R_1})$ must be the same as $p(\mathbf{x}^{R_2})$, $p(\mathbf{x}^{R_1}) = p(\mathbf{x}^{R_2})$, see (5).

Let us first apply the reciprocity relation (7) to points S and R_1. Then we can interchange the point source from S to R_1 and the receiver from R_1 to S. We obtain $p(\mathbf{x}^{R_1}) \to p(\mathbf{x}^S)$, $q(\mathbf{x}^S) \to q(\mathbf{x}^{R_1})$. Similarly, if we consider points S and R_2, we obtain $p(\mathbf{x}^{R_2}) \to p(\mathbf{x}^S)$, $q(\mathbf{x}^S) \to q(\mathbf{x}^{R_2})$. Thus, the reciprocity relation (7) dictates that we obtain the same acoustic pressure $p(\mathbf{x}^S)$ at point S, and the same volume injection point sources at points R_1 and R_2.

The final conclusion is: When the volume injection point source is shifted across the interface from point R_1 to point R_2, the acoustic pressure $p(\mathbf{x}^S)$ at point S, situated arbitrarily in the medium, is not changed.

b) Volume-force point source $\mathbf{f}$ at an interface.

We again consider three points S, R_1, R_2 as above. At a point S, situated in a smooth medium, we choose a volume-force point source using (9), where $\mathbf{n}^f(S)$ is specified arbitrarily. For this source, we can compute particle-velocity at any point of medium, including points on interfaces. At points R_1 and R_2, we obtain $\mathbf{v}(R_1)$ and $\mathbf{v}(R_2)$, and determine $\mathbf{n}^f(R_1)\mathbf{v}(R_1)$ and $\mathbf{n}^f(R_2)\mathbf{v}(R_2)$, where $\mathbf{n}^f$ is a unit vector perpendicular to Σ^R. Both these quantities, however, are the same, see (5). Using the reciprocity relation (10), we then obtain, similarly as sub a), the following conclusion: When the volume-force point source, perpendicular to the interface, is shifted across the interface, the component of the particle velocity vector oriented in the direction of the relevant volume-force point source is not changed at any point of the medium.

4 Ray-theory amplitudes of acoustic waves for a point source situated on an interface

Equations given in previous sections are quite general, but they do not give expressions for the amplitude of the wave generated by a given point source. Using the zero-order ray theory, we can simply find analytical expressions for amplitudes of individual waves, and verify the general conclusions for point sources of various types crossing a material interface. The receiver may be also situated at an interface. Actually, we can take many results from Červený (2001). References of equations consisting of three numbers separated by dots refer to the book Červený (2001). For example, the reference (4.10.11) refer to the equation (4.10.11) in the book Červený (2001). Similarly, the references of sections consisting of three members refer to that book.

Only zero-order ray amplitudes are considered. Point source is situated at the point S, the receiver at point R. The points S and R are connected by the ray $\Omega(R, S)$. In the acoustic (fluid) model, the interfaces between S and R may exist. Equations which will be given here are valid even for the source and receiver situated at interfaces.

First we give the expression for the **auxiliary function** $A(R, S)$, which will be broadly used later.

$$A(R,S) = \exp[iT^c(R,S)]\mathcal{R}^C(R,S)/\mathcal{L}(R,S) \ . \tag{11}$$

Here $\mathcal{L}(R, S)$ is the *relative geometrical spreading*, see (4.10.11). The relative geometrical spreading is reciprocal, $\mathcal{L}(R, S) = \mathcal{L}(S, R)$. For homogeneous acoustic medium, it is given by the relation $\mathcal{L}(R, S) = cl(R, S)$, where c is acoustic velocity $\sqrt{k/\rho} = 1/\sqrt{\kappa\rho}$ and $l(R, S)$ is the distance between S and R, see (4.10.18).

Next, $T^c(R, S)$ is the *phase shift due to caustics*. The phase shift due to caustics is reciprocal, $T^c(R, S) = T^c(S, R)$. For homogeneous medium, $T^c(R, S) = 0$. See Section 4.14.13.

Next, $\mathcal{R}^C(R,S)$ is the *complete pressure reflection/transmission coefficient* along the ray $\Omega(R,S)$, see (5.1.35). It represents the product of normalized R/T coefficients at all points of incidence on structural interfaces *between* S and R. It is the same for pressure and particle velocity R/T coefficients. The complete pressure R/T coefficient is reciprocal, $\mathcal{R}^C(R,S) = \mathcal{R}^C(S,R)$. For smooth media without interfaces between S and R, $\mathcal{R}^C(R,S) = 1$.

Auxiliary function $A(R,S)$ is reciprocal,

$$A(R,S) = A(S,R) \tag{12}$$

For homogeneous medium, the auxiliary function $A(R,S)$ is given by the relation

$$A(R,S) = \frac{1}{\mathcal{L}(R,S)} = \frac{1}{cl(R,S)} \ . \tag{13}$$

4.1 Acoustic pressure amplitude $P(R)$ for a volume injection point source S and a receiver R situated on interfaces

It is given by the expression, see (5.1.61):

$$P(R) = \left[\frac{\rho(R)c(R)}{\rho(S)c(S)}\right]^{1/2} A(R,S)\mathcal{D}(R)\mathcal{G}(S;\gamma_1,\gamma_2) \ , \tag{14}$$

where $\rho(S)$ and $\rho(R)$ are densities, $c(S)$ and $c(R)$ are acoustic velocities, $c = \sqrt{k/\rho} = 1/\sqrt{\kappa\rho}$, $\mathcal{D}(R)$ is conversion coefficient at the receiver R, and $\mathcal{G}(S;\gamma_1,\gamma_2)$ is radiation function, at the point S. It may be a function of ray parameters γ_1, γ_2. For omni-directional point source $\mathcal{G}(S;\gamma_1,\gamma_2) = \mathcal{G}(S)$.

a) **A note to the conversion coefficient $\mathcal{D}(R)$ at the receiver R.**
For point R situated in a smooth medium, $\mathcal{D}(R) = 1$. For point R situated on an interface Σ^R, we understand that the point R is situated on Σ^R from the side of incident wave (ray $\Omega(R,S)$). Then we also introduce the point R^+, situated on the opposite side of Σ^R from R. Then, see (5.1.49):

$$\mathcal{D}(R) = 1 + R^r(R) \ , \quad \mathcal{D}(R^+) = R^t(R) \ , \tag{15}$$

where R^r is the pressure reflection coefficient at R, see (5.1.47),

$$R^r(R) = \frac{\rho(R^+)c(R^+)W(R) - \rho(R)c(R)W(R^+)}{\rho(R^+)c(R^+)W(R) + \rho(R)c(R)W(R^+)} \ , \tag{16}$$

and $R^t(R)$ is the pressure transmission coefficient, see (5.1.45),

$$R^t(R) = \frac{2\rho(R^+)c(R^+)W(R)}{\rho(R^+)c(R^+)W(R) + \rho(R)c(R)W(R^+)} \ . \tag{17}$$

Here $W(R)$ is the cosine of the angle of incidence of the ray $\Omega(R,S)$ at R, and $W(R^+)$ is the relevant cosine of the angle of transmission at R^+,

$$W(R) = \cos i(R) \ , \quad W(R^+) = \cos i(R^+) \ . \tag{18}$$

Using the Snell's law, $\sin i(R^+) = [c(R^+)/c(R)] \sin i(R)$, we can determine $W(R^+)$ from $W(R)$:

$$W(R^+) = [1 - \frac{c^2(R^+)}{c^2(R)}(1 - W^2(R))]^{1/2} \ . \tag{19}$$

We also introduce the critical angle of incidence $i^*(R)$ by the relation

$$\sin i^*(R) = c(R)/c(R^+) \ . \tag{20}$$

For subcritical angles of incidence, $i(R) < i^*(R)$, $W(R^+)$ is real-valued, but for overcritical angles of incidence, $i(R) > i^*(R)$, $W(R^+)$ is complex-valued.

Comparing the expressions for $R^r(R)$ and $R^t(R)$, we get finally:

$$\mathcal{D}(R) = \mathcal{D}(R^+) = \frac{2\rho(R^+)c(R^+)W(R)}{\rho(R^+)c(R^+)W(R) + \rho(R)c(R)W(R^+)} \ . \tag{21}$$

Conclusion to conversion coefficients: Conversion coefficients at the receiver point situated at the interface are the same at the points R and R^+. Thus, it is not necessary to distinguish between points R and R^+ in the computation of pressure amplitudes $P(R)$ and $P(R^+)$; they are the same.

b) **A note to the radiation function of pressure waves** $\mathcal{G}(S;\gamma_1,\gamma_2)$

We shall consider only the radiation function of the elementary ray-theory pressure Green function. It corresponds to the volume injection rate density $q(x,s)$ given by (4), taken at point S. In a smooth medium, it is given by the relation

$$\mathcal{G}(S;\gamma_1,\gamma_2) = \frac{1}{4\pi}\rho(S)c(S) \ , \tag{22}$$

see (5.1.60). When the source is situated on an interface Σ^S we understand that it is situated on the interface Σ^S from the side of the ray $\Omega(R,S)$ (from the side of the medium, in which the wave from S to R propagates). We also introduce point S^+ on the opposite side of Σ^S from S. Then we obtain:

$$\mathcal{G}(S,\gamma_1,\gamma_2) = \mathcal{G}(S^+;\gamma_1,\gamma_2) = \frac{\rho(S)c(S)}{4\pi}\mathcal{D}(S) \ , \tag{23}$$

see (5.1.60). The conversion coefficients $\mathcal{D}(S)$ and $\mathcal{D}(S^+)$ at the source are again given by analogous equation as at the receiver:

$$\mathcal{D}(S) = \mathcal{D}(S^+) = \frac{2\rho(S^+)c(S^+)W(S)}{\rho(S^+)c(S^+)W(S) + \rho(S)c(S)W(S^+)} \ . \tag{24}$$

Here $W(S)$ and $W(S^+)$ have an analogous meaning as $W(R)$ and $W(R^+)$, but are related to the radiation angle at S. $W(S)$ is the cosine of the radiation angle (of the ray $\Omega(R,S)$) at S, $W(S) = \cos i(S)$, and $W(S^+) = \cos i(S^+)$. Similarly as the critical angle of incidence at R, we also introduce the critical radiation angle $\sin i^*(S) = c(S)/c(S^+)$. For subcritical radiation angles $i(S) < i^*(S)$, $W(S^+)$ is real-valued, and for overcritical radiation angles $i(S) > i^*(S)$, $W(S^+)$ $\mathcal{D}(S)$ and $\mathcal{D}(S^+)$ are complex-valued.

c) **Final equation**

Finally, for the amplitude of the ray-theory pressure Green function, corresponding to the volume injection point source at S, we obtain, see (5.1.62):

$$P(R) = P(R^+) = \frac{1}{4\pi}[\rho(S)\rho(R)c(S)c(R)]^{1/2}A(R,S)\mathcal{D}(R)\mathcal{D}(S) \ . \qquad (25)$$

4.2 Discussion of final equation for acoustic pressure amplitudes $P(R)$ at a point R for a volume injection point source at S

.

1) Source S at an interface: The pressure amplitude $P(R)$ at any point of the medium R is the same if the volume injection point source is situated at S or S^+. This is consistent with general results derived in Section 3a.

2) Receiver at an interface: The pressure amplitude $P(R)$ at the receiver is the same, if the receiver is situated at R or R^+.

3) Source S in a smooth medium: In this case, we can put $\mathcal{D}(S) = 1$, i.e. $\rho(S) = \rho(S^+)$, $c(S) = c(S^+)$, $W(S) = W(S^+)$. Thus, the equation (25) is valid both for the source situated in a smooth medium and on an interface.

4) Receiver R in a smooth medium: In this case, we can put $\mathcal{D}(R) = 1$, i.e. $\rho(R) = \rho(R^+)$, $c(R) = c(R^+)$, $W(R) = W(R^+)$. Thus, the equation (25) is valid both for the receiver situated in a smooth medium and on an interface.

5) Source-receiver reciprocity: The presure amplitude does not change if the volume injection point source S and receiver R interchange their positions. The points S and S^+ in $\mathcal{D}(S)$ and the points R and R^+ in $\mathcal{D}(R)$ can be used alternatively.

6) Homogeneous medium between S and R: If the medium along the ray $\Omega(R,S)$ is homogeneous, the pressure amplitudes at the receiver for a volume injection point source at S are given by the relation

$$P(R) = P(R^+) = \frac{\rho\mathcal{D}(R)\mathcal{D}(S)}{4\pi l(R,S)} \ . \qquad (26)$$

7) Complex-valued pressure amplitudes. In general, the equations for ray-theory pressure amplitudes are real-valued if i) the ray $\Omega(R,S)$ is real-valued (non-dissipative medium, no overcritical transmission, no caustics along $\Omega(R,S)$), ii) the complete pressure R/T coefficient $\mathcal{R}^C(R,S)$ is real-valued (no overcritical reflections), iii) the cosines $W(R^+)$ and/or $W(S^+)$ are real valued (no overcritical incidence at R, no overcritical radiation at S).

8) The ray-theory pressure amplitudes $P(R)$ are frequency independent.

4.3 Acoustic particle velocity amplitudes at the receiver R for volume-force point source

Expression for the particle velocity amplitudes in acoustic (fluid) media may be simply obtained from analogous expressions for the particle displacement amplitudes of unconverted P waves in solid media, given in (5.2.89). We only take the rigidity μ in the medium zero. The ray $\Omega(R,S)$ from the point source S to the receiver R is the same in both cases. The i-th Cartesian component $u_i^{(x)}(R)$ of the particle displacement vector is given by the relation (5.2.89):

$$u_i^{(x)}(R) = \frac{1}{4\pi[\rho(S)\rho(R)\alpha(S)\alpha(R)]^{1/2}} A(R,S)\mathcal{D}_{i3}(R)\mathcal{D}_{n3}(S)f_n^{(x)}(S) \ . \qquad (27)$$

The local Cartesian components are used at S and R, with the third axis perpendicular to the interface, at which S or R are situated. If they are situated in a smooth medium, the Cartesian coordinate system may be chosen arbitrarily. The auxiliary function $A(R,S)$ is the same as given in the beginning of Section 4, see equation (11). The source function $f_n^{(x)}(S)$ represent the n-th local Cartesian component of the volume force point source (force per volume). Functions $\mathcal{D}_{i3}(R)$ are the local Cartesian components of the 3×3 conversion matrix at the point R of the receiver. Their general expressions for $D_{i3}(R)$ and $D_{i3}(R^+)$ at the points R and R^+ are different, see (5.2.93) and (5.2.94). Similarly, functions $\mathcal{D}_{n3}(S)$ and $\mathcal{D}_{n3}(S^+)$ are the local Cartesian components of the 3×3 conversion matrix at the point source. They are analogous to $\mathcal{D}_{i3}(R)$ and $\mathcal{D}_{i3}(R^+)$.

Instead of the particle displacement vector, we can use particle velocity vector, and the differences may be compensated by taking $\dot{f}_n$ instead of f_n. Then, for $v_3(R) = \mathbf{v}(R)\cdot\mathbf{n}(R)$, we obtain

$$v_3(R) = \frac{1}{4\pi[\rho(S)\rho(R)c(S)c(R)]^{1/2}} A(R,S)\mathcal{D}_{33}(R)\mathcal{D}_{33}(S)\dot{f}_3(S) \ . \qquad (28)$$

Here the conversion coefficients $\mathcal{D}_{33}(R)$ and $\mathcal{D}_{33}(S)$ are given by the relations

$$\begin{aligned} \mathcal{D}_{33}(R) = \mathcal{D}_{33}(R^+) &= \frac{2\rho(R)c(R)W(R)W(R^+)}{\rho(R^+)c(R^+)W(R) + \rho(R)c(R)W(R^+)} \ , \\ \mathcal{D}_{33}(S) = \mathcal{D}_{33}(S^+) &= \frac{2\rho(S)c(S)W(S)W(S^+)}{\rho(S^+)c(S^+)W(S) + \rho(S)c(S)W(S^+)} \ , \end{aligned} \qquad (29)$$

$\dot{f}_3(S) = \dot{\mathbf{f}}(S)\cdot\mathbf{n}(S)$, $\mathbf{n}(R)$ and $\mathbf{n}(S)$ are unit vectors perpendicular to Σ^R at R and to Σ^S at S.

Discussion of final equations for acoustic particle velocity amplitude $v_3(R)$ for a volume-force point source at S.

We consider here the component $v_3(R) = \mathbf{v}(R)\cdot\mathbf{n}(R)$ of the particle-velocity vector $\mathbf{v}(R)$, where $\mathbf{n}(R)$ is the unit vector perpendicular to Σ^R. Similarly, we consider the component $f_3(S) = \mathbf{f}(S)\cdot\mathbf{n}(S)$ of the volume-force point source $\mathbf{f}(S)$, where $\mathbf{n}(S)$ is the unit vector perpendicular to Σ^S. If points S and/or R are situated in a smooth medium, $\mathbf{n}(S)$ and/or $\mathbf{n}(R)$ may be taken arbitrarily.

In this case, the conclusions related to $v_3(R)$ and $f_3(S)$ are the same as the conclusion related to acoustic pressure amplitude $P(R)$ for an volume injection point source at S, given in Section 4.2. Only the expressions (26) for the homogeneous medium are slightly changed:

$$v_3(R) = v_3(R^+) = \frac{1}{4\pi\rho c^2 l(R,S)} \mathcal{D}_{33}(R)\mathcal{D}_{33}(S)\dot{f}_3(S) \ . \tag{30}$$

5 High-frequency acoustic waves, generated by point sources situated at interfaces

We consider the high-frequency acoustic waves, generated by point sources of volume injection type or by the volume-force type, situated on a planar interface between two homogeneous fluid medium. When the acoustic velocity $c(S^+)$ is smaller then the acoustic velocity $c(S)$, the radiation angle $i(S)$ is subcritical and the zero-order ray-theory expressions may be used with a good accuracy. If, however, acoustic velocity $c(S^+)$ is greater than $c(S)$, the situation is more complicated. The situation is different for subcritical and overcritical radiation angles, where the critical radiation angle $i^*(S)$ is given by the relation:

$$\sin i^*(S) = c(S)/c(S^+) \ . \tag{31}$$

a) For subcritical radiation angles $i(S) < i^*(S)$, only one regular ray-theory wave exists at both sides of interface Σ^S. The computation of acoustic amplitude of this wave can be performed as shown in Section 4, without any change. See also Jílek and Červený (1996).

b) For overcritical radiation angles $i(S) > i^*(S)$, the radiation conversion coefficients are complex-valued. Two high-frequency waves are generated at both sides of interface Σ^S by the point source. The zero-order ray theory is not sufficient to describe properly the wave field generated by a point source in this case. A great role is played by the grazing ray, which is parallel to the interface Σ^S in the medium of higher velocity (containing the point S^+). The grazing ray corresponds to the angle $i(S^+) = \frac{1}{2}\pi$ connected with the critical radiation angle $i^*(S)$.

The wave which propagates along the grazing ray generates the head wave in the medium of lower velocity. The head wave is not a regular zero-order ray-theory wave. Its ray is situated partly in the medium of higher velocity (grazing ray), partly in the medium of lower velocity (critical ray). It cannot be expressed in terms of reflection/transmission coefficients. The amplitude of head wave depends on frequency ω; it is proportional to $(i\omega)^{-1}$. Moreover, the regular wave which propagates along the interface in the medium of lower velocity generates inhomogeneous waves propagating in the medium of higher velocity, the amplitudes of which decrease exponentially with frequency. This inhomogeneous waves cannot be computed by the standard ray methods. Both the head wave in the medium of lower velocity and the inhomogeneous wave in the medium of higher velocity depend on medium parameters of both media.

A similar discussion can be performed even at the point of incidence R on the interface

Σ^R. When $c(R^+) < c(R)$, the angle of incidence $i(R)$ is always subcritical, and the zero-order ray-theory expressions given in Section 4 can be used with a good accuracy. For $c(R^+) > c(R)$, these expressions can be used only for subcritical angles of incidence $i(R) < i^*(R)$, where the critical angle of incidence $i^*(R)$ is given by the relation $\sin i^*(R) = c(R)/c(R^+)$. For overcritical angles of incidence $i(R) > i^*(R)$, also the head waves on the side of Σ^R of incident wave and the inhomogeneous wave on the other side of Σ^R should be considered. Both these waves are frequency dependent and vanish for $\omega \to \infty$.

If the source S is not situated directly on the interface, but very close to it in a medium of higher velocity, also the pseudospherical wave must be considered.

Analytical expressions for amplitudes of non-ray waves (head waves, inhomogeneous waves, pseudospherical waves) are known for locally homogeneous media in the vicinity of points S and R. All these waves are frequency dependent and their amplitudes vanish for $\omega \to \infty$. The detailed discussion of resulting equations, however, is outside the frame of this article and is not given here.

6 Alternative deformation equation

In previous sections, we considered only volume-injection point source q, and the reciprocity relation (7) has been valid. This reciprocity principle implied the important conclusion related to the volume injection point source situated on a material interface, derived in Section 3: When the volume injection point source q is situated on a material interface, the relevant pressure amplitudes at any point of the medium do not depend on which side of the interface the source is situated. We also proved that this conclusion remains valid even for ray-theory pressure amplitudes, see conclusion 1 in Section 4.2.

We can, however, write the alternative form of the deformation equation. The constitutive relation (1) can be also written in the following way:

$$\dot{p} + kv_{i,i} = 0 \ . \tag{32}$$

In region, where the constitutive relation is violated, we can replace it by the equation:

$$\dot{p} + kv_{i,i} = Q \ , \tag{33}$$

where the source function Q has a similar meaning as q. Deformation equation is fully alternative to (33) if we put

$$Q = kq \ . \tag{34}$$

The point sources q and Q have certain important differences. The acoustic pressure wave field generated by a volume injection point source q does not change when the point source crosses a interface. This was proved in Section 3. The acoustic pressure wave field generated by a point source Q, however, changes when the point source crosses an interface. This is immediately obvious from equation (34). This also implies that the conclusions 1) and 5) of Section 4.2 are not valid in this case.

7 Elastodynamic equations

Elastodynamic equations for an anisotropic inhomogeneous non-dissipative elastic medium are usually expressed in terms of the partial differential equations of the second order for the Cartesian components u_i of the displacement vector $\mathbf{u}$:

$$\rho \ddot{u}_i = \sigma_{ij,j} + f_i \ . \tag{35}$$

Here σ_{ij} is the stress tensor and f_i are the Cartesian components of a body force, representing a source. In the source-free medium, the stress is connected with the strain tensor $\varepsilon_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k})$ by the Hooke's constitutive law

$$\sigma_{ij} - c_{ijkl}\varepsilon_{kl} = 0 \ . \tag{36}$$

Assume now that the Hooke's law is violated in some region and that it can be expressed in the form

$$\sigma_{ij} - c_{ijkl}\varepsilon_{kl} = \sigma^*_{ij} \ . \tag{37}$$

Here σ^*_{ij} is called the *stress glut* (see, e.g., Ampuero and Dahlen, 2005). It is simple to show that the region in which $\sigma^*_{ij} \neq 0$ acts as a source region. Consider $f_i = 0$ and insert (37) into (35). We obtain

$$\rho \ddot{u}_i = c_{ijkl}\varepsilon_{kl} + f^*_i \ , \tag{38}$$

where

$$f^*_i = -\sigma^*_{ij,j} \ . \tag{39}$$

Here f^*_i represents the equivalent body force, called also the *stress-glut source.*

Similarly as the stress glut, we can also introduce the *strain glut* ε^*_{kl}, using the relation

$$\sigma_{ij} = c_{ijkl}(\varepsilon_{kl} - \varepsilon^*_{kl}) \ . \tag{40}$$

The relation between the stress-glut and strain glut is immediately obtained from (37) and (40)

$$\sigma^*_{ij} = c_{ijkl}\varepsilon^*_{kl} \ . \tag{41}$$

The equivalent body force for the strain glut source reads

$$f^*_i = -(c_{ijkl}\varepsilon^*_{kl})_{,j} \ . \tag{42}$$

For more details on stress glut and strain glut, on their importance in seismology, and on many other references, see the review article by Ampuero and Dahlen (2005).

An alternative form of elastodynamic equations for anisotropic inhomogeneous non-dissipative elastic medium, called velocity–stress method, is expressed in terms of twelve partial differential equations of the first order; three for Cartesian components of the particle velocity vector $v_i = \dot{u}_i$, and nine for the components of the stress tensor σ_{ij} (Virieux, 1988; Carcione, 2007). These differential equations read:

$$\rho \dot{v}_i = \sigma_{ij,j} + f_i \ , \qquad \dot{\sigma}_{ij} = c_{ijkl}(v_{k,l} - \dot{\varepsilon}^*_{kl}) \ . \tag{43}$$

The second equation corresponds to the time-derivative of equation (40). This system of equations correspond to the strain glut source, expressed in terms of the time derivative of the strain glut $\dot{\varepsilon}^*_{kl}$.

Elastodynamic equations (43) can be used for anisotropic and isotropic media, but also for fluid media. For fluid media, we must insert $\sigma_{ij} = -p\delta_{ij}$, where p is the acoustic pressure, and proper c_{ijkl}; corresponding to $\mu = 0$. The second equation in (43) then read

$$-\dot{p} = k(v_{k,k} - \dot{\varepsilon}^*_{kk}) \ . \tag{44}$$

Multiplying (44) by $-\kappa$, equation (44) reads

$$\kappa\dot{p} + v_{i,i} = \dot{\varepsilon}^*_{kk} \ , \tag{45}$$

as $\kappa k = 1$. As we can see, equations (43) can be then expressed in the form

$$\rho\dot{v}_i + p_{,i} = f_i \ , \qquad \kappa\dot{p} + v_{i,i} = q \ , \tag{46}$$

where

$$q = \dot{\varepsilon}^*_{kk} \ . \tag{47}$$

Thus, equations (43) reduce in fluid media strictly to acoustic wave equations (2), where the volume injection rate density q is represented by the time derivative of strain glut.

An alternative form of equation (43) uses the stress glut σ^*_{ij} instead of the strain glut ε^*_{ij}. From (43) and (37), we obtain

$$\rho\dot{v}_i = \sigma_{ij,j} + f_i \ , \qquad \dot{\sigma}_{ij} = c_{ijkl}\, v_{k,l} + \dot{\sigma}^*_{ij} \ . \tag{48}$$

Here $\dot{\sigma}^*_{ij}$ is the time derivative of the stress glut.

For fluid isotropic media, we proceed in the same way as previously. The second deformation equation in (48) reads

$$-\dot{p} = kv_{i,i} + \dot{\sigma}^*_{ii} \ . \tag{49}$$

This can be written in the following way:

$$\dot{p} + kv_{i,i} = Q \ , \tag{50}$$

where

$$Q = -\dot{\sigma}^*_{ii} \ . \tag{51}$$

But this is exactly the same as equation (33), where the function Q has a physical meaning of a negative time derivative of the stress glut.

8 Concluding remarks

In studies of acoustic waves propagating in fluid inhomogeneous non-dissipative and non-moving media, mostly the acoustic wave equations (2), corresponding to volume injection source are considered. Using the reciprocity relations by Fokkema and van den Berg (1993), it was proved that the amplitudes of acoustic pressure waves at any point of the medium do not change when the volume injection point source crosses a material interface. In the terminology of elastic media, the volume-injection point source is a special case of strain-glut point source.

In seismology, however, mostly elastodynamic wave equations with a stress-glut point source are used. The reason is that the stress-glut point source is a good representation of a moment tensor point source, which plays an important role in the description of natural earthquakes. For fluid media, the stress-glut point source reduces to a source described by equation (34). It was shown in Section 6 that for this point source the acoustic pressure waves at any point of the medium jump when the source crosses an interface. In general, this must remain valid for all stress-glut point sources in elastic media.

This fact has provoked a broad discussion in seismology, whether or not the moment tensor is ambiguous if the earthquake is situated close to the structural interface (Ampuero and Dahlen, 2005, and many other). Recently, Vavryčuk (2013) shows that the moment tensor must be determined from averaged elastic parameters known from the theory of effective media. He further shows that the moment tensor is not ambiguous in this case, even when situated close or at the structural interface.

Acknowledgements

The research has been supperted by the Grant Agency of the Czech Republic under Contracts P210/10/0736 and P210/11/0117, by the Ministry of Education of the Czech Republic within Research Project MSM0021620800 and by the Consortium Project "Seismic Waves in Complex 3-D Structures".

9 References

Aki, K., and Richards, P.G., 1980. Quantitative Seismology. Theory and Methods. Freeman, San Francisco.

Ampuero, J.-P., and Dahlen, P.A., 2005. Ambiguity of the moment tensor. *Bull. Seismol. Soc. Am.*, **95**, 390–400.

Carcione, J.M., 2007. Wave fields in real media. Wave propagation in anisotropic anelastic, porous and electromagnetic media. Handbook of geophysical exploration, **38**. Elsevier.

Červený, V., 2001. Seismic ray theory. Cambridge University Press, Cambridge.

DeHoop, A.T., 1995. Handbook of Radiation and Scattering of Waves. Academic Press, London. (Electronic reproduction 2008; http: //www.atdehoop.com).

Fokkema, J.T., and van den Berg, P.M., 1993. Seismic application of acoustic reciprocity. Elsevier, Amsterodam.

Jílek, P., and Červený, V., 1996. Radiation patterns of point sources situated close to structural interfaces and to the Earth's surface. *PAGEOPH*, **148**, 175–225.

Lord Rayleight (J.W. Strutt), 1894. Theory of sound. Macmillan, London.

Vavryčuk, V., 2013. Is the seismic moment tensor ambiguous at a material interface? *Geophys. J. Int.*, doi: 10.1093/gji/ggt084.

Virieux, J., 1988. P-SV-wave-propagation in heterogeneous media - velocity-stress finite-difference method. *Geophysics*, **51**, 889–901.

Wu, Z.L., and Chen, Y.T., 2003. Definition of seismic moment at a discontinuity interface. *Bull. Seismol. Soc. Am.*, **93**, 1832–1834.