Calculation of the amplitudes of elastic waves in anisotropic media in Cartesian or ray-centred coordinates

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Summary

We derive and compare various expressions for the amplitude of the ray–theory approximation of elastic waves in heterogeneous anisotropic media. The amplitude of a wavefield with general initial conditions is expressed in terms of two paraxial vectors of geometrical spreading in Cartesian coordinates, and in terms of the 2×2 matrix of geometrical spreading in ray–centred coordinates. The amplitude of the Green tensor is expressed in six different ways: (a) in terms of the paraxial vectors corresponding to two ray parameters in Cartesian coordinates, (b) in terms of the 2×2 paraxial matrices corresponding to two ray parameters in ray–centred coordinates, (c) in terms of the 3×3 upper right submatrix of the 6×6 propagator matrix of geodesic deviation in Cartesian coordinates, (d) in terms of the 2×2 upper right submatrix of the 4×4 propagator matrix of geodesic deviation in ray–centred coordinates, (e) in terms of the 3×3 matrix of the mixed second–order spatial derivatives of the characteristic function with respect to source and receiver Cartesian coordinates, and (f) in terms of the 2×2 matrix of the mixed second–order spatial derivatives of the characteristic function with respect to source and receiver ray–centred coordinates.

Keywords

Amplitude, transport equation, elastic Green tensor, geodesic deviation, paraxial ray approximation, second–order derivatives of the characteristic function, anisotropy, heterogeneity.

1. Introduction

Solution of the transport equation for the wavefield amplitude is closely related to the equations of geodesic deviation. The *Hamiltonian equations of geodesic deviation*, also called *paraxial ray equations* or *dynamic ray tracing equations*, were proposed by Červený (1972) in order to calculate the wavefield amplitude. The Hamiltonian equations of geodesic deviation have a considerably simpler form than the equivalent Finslerian equations of geodesic deviation (Klimeš, 2013a). The Hamiltonian equations of geodesic deviation were first expressed in general Cartesian coordinates (Červený, 1972), but are also often expressed and solved in the *ray–centred coordinates* connected with a particular ray (Klimeš, 2006b). The solution of the Hamiltonian equations of geodesic deviation describes the coordinate deviations of paraxial rays (paraxial vectors of geometrical spreading) and slowness–vector deviations of paraxial rays.

For an orthonomic system of rays corresponding to a wavefield with general initial conditions, we may calculate two paraxial vectors of geometrical spreading corresponding to two ray parameters using the Hamiltonian equations of geodesic deviation. These paraxial vectors may be calculated in Cartesian coordinates or ray–centred coordinates. The amplitude of a general wavefield may then be expressed in terms of the paraxial vectors in Cartesian coordinates (Section 4.1), or in terms of the 2×2 matrix of geometrical spreading in ray–centred coordinates (Section 4.2).

Rays corresponding to the Green tensor are calculated from a point source, and we may again calculate two paraxial vectors of geometrical spreading corresponding to two arbitrary ray parameters at the point source. The amplitude of the Green tensor can then be expressed in terms of the paraxial vectors in Cartesian coordinates (Section 5.3), or in terms of the 2×2 paraxial matrices in ray–centred coordinates (Section 5.2).

We may also calculate the propagator matrix of geodesic deviation. The propagator matrix may be defined and calculated in Cartesian coordinates or in ray–centred coordinates. If we calculate the propagator matrix of geodesic deviation, we may obtain the paraxial vectors using the corresponding initial conditions and the propagator matrix. The paraxial vectors may then be used to determine the amplitude of a general wavefield or the amplitude of the Green tensor as mentioned above.

If we calculate the propagator matrix of geodesic deviation, we may also determine the amplitude of the Green tensor directly from the upper right submatrix of the propagator matrix of geodesic deviation, either from the 3×3 submatrix in Cartesian coordinates (Section 5.4) or from the 2×2 submatrix in ray–centred coordinates (Section 5.1).

We may also consider the mixed second–order spatial derivatives of the characteristic function with respect to source and receiver coordinates. In this case, we may determine the amplitude of the Green tensor using the 3×3 matrix of these derivatives in Cartesian coordinates (Section 5.6) or the 2×2 matrix of these derivatives in ray–centred coordinates (Section 5.5).

We do not present the expressions for the amplitude in terms of the surface–to–surface paraxial vectors of geometrical spreading corresponding to two ray parameters, in terms of the 2×2 surface–to–surface paraxial matrices nor in terms of the upper right 2×2 submatrix of the 4×4 surface–to–surface propagator matrix of geodesic deviation, refer to Moser & Červený (2007). We also do not demonstrate that the expression for the amplitude of the Green tensor in terms of the three different 2×2 matrices of the homogeneous second–order spatial derivatives of travel time and the characteristic function with respect to source and receiver ray–centred coordinates by Červený (2001, eq. 4.10.43) is applicable to anisotropic media.

The Einstein summation over repetitive lower–case Roman indices, corresponding to the 3 spatial coordinates, is used throughout the paper. Upper–case Roman indices correspond to the first two ray–centred coordinates or to two ray parameters.

2. Rays and geodesic deviation

2.1. Hamilton–Jacobi equation, travel time, Characteristic function, slowness vector, Hamilton's equations, ray coordinates

The Hamilton–Jacobi equation is a general partial differential equation of the first order. The Hamilton–Jacobi equation for *travel time* (*action*, *distance*) $\tau(x^m)$ reads

$$H\big(x^i, \tfrac{\partial\tau}{\partial x^j}(x^m)\big) = C \quad , \tag{1}$$

where the function $H(x^i, p_j)$ of coordinates x^i and of covariant vector p_j from the cotangent space at point x^i is referred to as the *Hamiltonian function.* The multivalued solution $\tau(x^m)$ of the Hamilton–Jacobi equation is determined by the initial conditions.

The *characteristic function* (*two–point travel time*, *point–to–point distance*) from point $\tilde{x}^n$ to point x^m, denoted by $\tau(x^m, \tilde{x}^n)$, is the solution of Hamilton–Jacobi equation (1) with initial conditions $\tau(\tilde{x}^m, \tilde{x}^n) = 0$.

Gradient

$$p_i = \frac{\partial\tau}{\partial x^i} \tag{2}$$

of travel time or characteristic function is referred to as the *slowness vector.*

When differentiating the Hamilton–Jacobi equation with respect to coordinates x^j, we find that the multivalued solution τ of the Hamilton–Jacobi equation can be calculated along *rays* (*geodesics*). *Hamilton's equations* (*equations of rays*, *ray tracing equations*, *equations of geodesics*) read

$$\frac{\mathrm{d}x^i}{\mathrm{d}\gamma^3} = \frac{\partial H}{\partial p_i}(x^m, p_n) \quad , \tag{3}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}\gamma^3} = -\frac{\partial H}{\partial x^i}(x^m, p_n) \quad , \tag{4}$$

where d in the derivatives denotes the differentiation along the ray. The meaning of independent parameter γ^3 along the rays depends on the particular form of the Hamiltonian function.

In addition to independent parameter γ^3 corresponding to the Hamiltonian function, we may also parametrize the rays by travel time τ, and define the *ray velocity vector*

$$V^i = \frac{\mathrm{d}x^i}{\mathrm{d}\tau} \tag{5}$$

as the derivative of coordinates x_i of the ray with respect to travel time τ along the ray.

In 3–D space, the orthonomic system of rays corresponding to the given initial conditions consists of a two–parametric system of rays which are parametrized by two *ray parameters* γ^1 and γ^2. The ray parameters together with the independent parameter along the rays form *ray coordinates* γ^1, γ^2, γ^3.

Since the solution of Hamilton–Jacobi equation (1) calculated using Hamilton's equations (3) and (4) is multivalued, it is parametrized by ray coordinates γ^a. Multivalued solution $\tau(x^m)$ or $\tau(x^m, \tilde{x}^n)$ is thus expressed in form $x^m = x^m(\gamma^a)$, $\tau = \tau(\gamma^a)$.

2.2. Ray–centred coordinates

Along a particular ray, we may define ray–centred coordinates q^a (Klimeš, 2006b).

We parametrize the points along the ray by an arbitrary monotonic variable q^3. At each point $x^i(q^3)$ of the ray, we choose two contravariant basis vectors $h_1^i(q^3)$ and $h_2^i(q^3)$ perpendicular to slowness vector p_i,

$$h_A^i(q^3)\, p_i = 0 \quad . \tag{6}$$

Contravariant basis vectors h_A^i should vary smoothly along the ray.

The transformation from the *ray–centred coordinates* q^a to Cartesian coordinates x^i is defined by relation

$$x^i = x^i(q^3) + h_A^i(q^3)\, q^A \quad . \tag{7}$$

Three contravariant basis vectors of the ray–centred coordinate system are

$$h_a^i = \frac{\partial x^i}{\partial q^a} \quad . \tag{8}$$

In the matrix notation, we shall denote the first two contravariant basis vectors as $\mathbf{h}_1$ and $\mathbf{h}_2$.

Three covariant basis vectors of the ray–centred coordinate system are

$$\hat{h}_i^a = \frac{\partial q^a}{\partial x^i} \quad . \tag{9}$$

The slowness vector in ray–centred coordinates at the central ray then reads

$$p_a^{(q)} = p_i\, h_a^i \quad . \tag{10}$$

In the matrix notation, we shall denote the first two covariant basis vectors as $\hat{\mathbf{h}}_1$ and $\hat{\mathbf{h}}_2$.

2.3. Paraxial ray matrices of an orthonomic system of rays

For an orthonomic system of rays corresponding to given initial conditions, we define the 3×3 paraxial matrix

$$X_a^i = \frac{\partial x^i}{\partial \gamma^a} \tag{11}$$

of geometrical spreading in Cartesian coordinates. We analogously define the 3×3 paraxial matrix

$$Y_{ia} = \frac{\partial p_i}{\partial \gamma^a} \tag{12}$$

describing the paraxial slowness vectors in Cartesian coordinates.

The third columns of the paraxial matrices can be obtained from the solution of Hamilton's equations (3) and (4). We shall refer to the first two columns X_1^i, X_2^i or Y_{i1}, Y_{i2} of the paraxial matrices as the paraxial vectors, and denote them as $\mathbf{X}_1$, $\mathbf{X}_2$ or $\mathbf{Y}_1$, $\mathbf{Y}_2$ in the matrix notation. The paraxial vectors can be calculated using the *Hamiltonian equations of geodesic deviation* (*paraxial ray equations*, *dynamic ray tracing equations*), which can be obtained by differentiating Hamilton's equations (3) and (4) with respect to ray coordinates γ^a (Červený, 1972).

We also define the analogous paraxial matrices

$$Q_a^i = \frac{\partial q^i}{\partial \gamma^a} \tag{13}$$

and

$$P_{ia} = \frac{\partial p_i^{(q)}}{\partial \gamma^a} \tag{14}$$

in ray–centred coordinates.

For fixed ray parameters γ^A, we have $q^I = 0$ and $p_I^{(q)} = 0$, see definition (7) and definition (10) with (6). Then

$$Q_3^I = 0 \tag{15}$$

and

$$P_{I3} = 0 \quad . \tag{16}$$

In ray–centred coordinates, we can thus calculate just the 2×2 paraxial matrices Q_A^I and P_{IA} using the Hamiltonian equations of geodesic deviation. In the matrix notation, we shall denote these 2×2 paraxial matrices as $\mathbf{Q}$ and $\mathbf{P}$.

We compare definitions (5) and (11), and obtain relation

$$X_3^k = V^k \frac{\mathrm{d}\tau}{\mathrm{d}\gamma^3} \quad . \tag{17}$$

Definition (13) yields identity

$$Q_3^k = \delta_3^k \frac{\mathrm{d}q^3}{\mathrm{d}\gamma^3} \quad . \tag{18}$$

2.4. Propagator matrices of geodesic deviation

The propagator matrices of geodesic deviation represent the solution of the Hamiltonian equations of geodesic deviation with identity initial conditions, both in Cartesian coordinates and in ray–centred coordinates (Kendall, Guest & Thomson, 1992; Klimeš, 1994).

The 6×6 propagator matrix of geodesic deviation in Cartesian coordinates may also be defined as the matrix of derivatives of x^i and p_i with respect to their initial conditions $\tilde{x}^j$ and $\tilde{p}_j$ for fixed γ^3. In this paper, we shall use just the 3×3 upper right submatrix

$$X_2^{ij} = \frac{\partial x^i}{\partial \tilde{p}_j} \tag{19}$$

of this propagator matrix. The partial derivatives are calculated for fixed γ^3.

The 6×6 propagator matrix of geodesic deviation in ray–centred coordinates may also be defined as the matrix of derivatives of q^i and $p_i^{(q)}$ with respect to their initial conditions $\tilde{q}^j$ and $\tilde{p}_j^{(q)}$ for fixed γ^3.

As the consequence of identities (15) and (16), we may also define the 4×4 propagator matrix of geodesic deviation in ray–centred coordinates as the matrix of derivatives of q^I and $p_I^{(q)}$ with respect to their initial conditions $\tilde{q}^J$ and $\tilde{p}_J^{(q)}$ for fixed γ^3. In this paper, we shall use just the upper right 2×2 submatrix

$$Q_2^{IJ} = \frac{\partial q^I}{\partial \tilde{p}_J^{(q)}} \tag{20}$$

of this propagator matrix. In the matrix notation, we shall denote this 2×2 matrix as $\mathbf{Q}_2$.

3. Amplitude

3.1. Transport equation

Multivalued zero–order ray–theory amplitude $A = A(x_m)$ of a general elastic wavefield satisfies *transport equation*

$$\frac{\partial}{\partial x_i}\big(A^2 \rho\, V^i\big) = 0 \tag{21}$$

(Klimeš, 2006a, eq. 10), where ray velocity vector V^i is given by definition (5).

Function $\rho = \rho(x_m, f_\kappa)$ is a function parametrizing the transport equation. If A is the amplitude of the displacement of an elastic wavefield, ρ is the density. From the point of view of differential geometry, amplitude A is a scalar if ρ is the scalar density of weight -1 (scalar per volume).

The solution of transport equation (21) can be calculated separately along each ray (Babich, 1961; Klimeš, 2006a).

3.2. Phase shift due to caustics

Transport equation (21) is a partial differential equation for the square A^2 of the amplitude, not for the amplitude itself. Even if the solution A^2 of transport equation (21) is real–valued, amplitude A becomes complex–valued if its square A^2 becomes negative. Amplitude A is thus complex–valued. Since the complex–valued square root has two branches, it is difficult to determine amplitude A from its square A^2. We must determine which branch of the amplitude calculated along the ray is correct.

We thus separate square root $A = \sqrt{A^2}$ into complex modulus $|A| = \sqrt{|A^2|}$ and complex argument $\exp(\mathrm{i}\varphi)$,

$$A = |A| \exp(\mathrm{i}\varphi) \quad . \tag{22}$$

Quantity φ in expression (22) is the *phase shift due to caustics*.

If the rays are real–valued, there is a real–valued solution of transport equation (21) for the square A^2 of the amplitude. Then the phase shift due to caustics is constant in the regions of equal sign of A^2, and changes by an integer multiple of $\pi/2$ at the boundaries between the regions.

The calculation of amplitude A along the ray is thus composed of the calculation of its complex modulus $|A|$ along the ray and of the determination of the phase shift φ due to caustics. This paper is devoted to the the calculation of the complex modulus $|A|$ of the amplitude.

The rules for the determination of the phase shift due to caustics along the ray were proposed by Lewis (1965), Orlov (1981), Kravtsov & Orlov (1993, 1999), Bakker (1988), Garmany (2001) and Klimeš (2010, 2014).

For an orthonomic system of rays corresponding to a wavefield with general initial conditions, the phase shift due to caustics may be determined using the 3×3 paraxial matrices in Cartesian coordinates with their derivatives (Klimeš, 2014, Secs. 2.2 and 3.2), or using the 2×2 paraxial matrices in ray–centred coordinates with their derivatives (Klimeš, 2014, Secs. 2.1 and 3.1).

The phase shift of the Green tensor due to caustics may be determined using the 3×3 right–hand submatrices of the 6×6 propagator matrix of geodesic deviation in Cartesian coordinates with their derivatives (Klimeš, 2010, Secs. 2.2 and 3.2), or using the 2×2 right–hand submatrices of the 4×4 propagator matrix of geodesic deviation in ray–centred coordinates with their derivatives (Klimeš, 2010, Secs. 2.1 and 3.1).

4. Amplitude of a general wavefield

4.1. Amplitude in terms of the matrix of geometrical spreading in Cartesian coordinates

The solution of transport equation (21) may be expressed in various forms. The square of the complex–valued amplitude may be expressed as

$$A^2 = C^2 \, \frac{1}{\varrho} \frac{\mathrm{d}\tau}{\mathrm{d}\gamma_3} \, \frac{1}{\det(X^i_a)} \tag{23}$$

(Klimeš, 2006a, eqs. 12, 34), where the 3×3 paraxial matrix X^i_a of geometrical spreading in Cartesian coordinates is given by definition (11).

This equation represents the generalization of the analogous equation by Babich (1961, eq. 3.7) from a homogeneous Hamiltonian function to a general Hamiltonian function.

The complex–valued amplitude then reads

$$A = C \, \sqrt{\frac{1}{\varrho} \frac{\mathrm{d}\tau}{\mathrm{d}\gamma_3} \, \frac{1}{|\det(X^i_a)|}} \, \exp(\mathrm{i}\varphi) \quad , \tag{24}$$

where φ is the phase shift due to caustics.

Complex–valued factor $C = C(\gamma^1, \gamma^2)$ is constant along the ray in a smooth medium and is determined by the initial conditions. It is often referred to as the *reduced amplitude* (Červený, Klimeš & Pšenčík, 1988, eq. 5.19). Reduced amplitude C depends on the selection of ray parameters γ^1 and γ^2.

We insert relation (17) into expression (24) and arrive at expression

$$\boxed{A = \frac{C}{\sqrt{\varrho \, |\varepsilon_{ijk} X^i_1 X^j_2 V^k|}} \, \exp(\mathrm{i}\varphi)} \tag{25}$$

(Gajewski & Pšenčík, 1990, eq. 7; Kendall, Guest & Thomson, 1992, eqs. 3–4), where paraxial vectors X^i_1 and X^i_2 represent the first two columns of the 3×3 paraxial matrix (11) of geometrical spreading.

For the special case of a *homogeneous* Hamiltonian function, paraxial vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ are tangent to the wavefront, and their cross product is thus normal to the wavefront,

$$\varepsilon_{ijk} X^i_1 X^j_2 = \pm|\mathbf{X}_1 \times \mathbf{X}_2| \, v \, p_k \quad . \tag{26}$$

In this case,

$$A = \frac{C}{\sqrt{\varrho \, v \, |\mathbf{X}_1 \times \mathbf{X}_2|}} \, \exp(\mathrm{i}\varphi) \tag{27}$$

(Červený, 1972, eq. 29b; Kendall & Thomson, 1989, eqs. 29–30). This equation is not applicable to a general Hamiltonian function.

4.2. Amplitude in terms of the matrix of geometrical spreading in ray–centred coordinates

Definition (11) yields relation

$$X^k_a = \frac{\partial x^k}{\partial q^i}\,\frac{\partial q^i}{\partial \gamma^a} \tag{28}$$

for the transformation from ray–centred coordinates to Cartesian coordinates. Considering definitions (8) and (13), relation (28) reads

$$X^k_a = h^k_i Q^i_a \quad , \tag{29}$$

which implies relation

$$|\det(X^g_a)| = |\det(h^g_i)|\,|\det(Q^i_a)| \tag{30}$$

for the determinants. The determinant of the transformation matrix (8) from ray–centred to Cartesian coordinates is

$$|\det(h^i_a)| = |\varepsilon_{ijk} h^i_1 h^j_2 h^k_3| \quad . \tag{31}$$

Since contravariant basis vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ of the ray–centred coordinate system are tangent to the wavefront, their cross product is normal to the wavefront, and

$$\varepsilon_{ijk} h^i_1 h^j_2 = \pm|\mathbf{h}_1 \times \mathbf{h}_2|\, v\, p_k \quad . \tag{32}$$

We insert relation

$$h^k_3 = \frac{\partial x^k}{\partial q^3} \quad , \tag{33}$$

following from definitions (7) and (8), into relation (31) with (32) and obtain relation

$$|\det(h^g_i)| = |\mathbf{h}_1 \times \mathbf{h}_2|\, v\, \frac{\mathrm{d}\tau}{\mathrm{d}q^3} \quad . \tag{34}$$

Relation (30) with relation (34) reads

$$|\det(X^i_a)| = |\mathbf{h}_1 \times \mathbf{h}_2|\,|\det(Q^i_a)|\, v\, \frac{\mathrm{d}\tau}{\mathrm{d}q^3} \quad . \tag{35}$$

Due to identities (15) and (18), we have relation

$$\det(Q^i_a) = \det(Q^I_A)\,\frac{\mathrm{d}q^3}{\mathrm{d}\gamma^3} \quad . \tag{36}$$

We insert relation (35) with relation (36) into expression (24) and arrive at expression

$$\boxed{A(\mathbf{x},\tilde{x}) = \frac{C}{\sqrt{\varrho\, v\, |\mathbf{h}_1 \times \mathbf{h}_2|\,|\det(Q^I_A)|}}\,\exp(\mathrm{i}\varphi) \qquad (37)}$$

(Klimeš, 2012, eqs. 7, 9) for the amplitude of a general wavefield.

5. Amplitude of the Green tensor

5.1. Amplitude in terms of the propagator matrix of geodesic deviation in ray–centred coordinates

Using the representation theorem for elastic waves and relation (37) in ray–centred coordinates, we can derive expression

$$\boxed{A^{\mathrm{G}}(\mathbf{x},\tilde{\mathbf{x}}) = \frac{1}{4\pi}\,\frac{1}{\sqrt{\varrho(\mathbf{x})\,v(\mathbf{x})\,\varrho(\tilde{\mathbf{x}})\,v(\tilde{\mathbf{x}})}\,L(\mathbf{x},\tilde{\mathbf{x}})}\,\exp[\mathrm{i}\varphi(\mathbf{x},\tilde{\mathbf{x}})] \qquad (38)}$$

(Klimeš, 2012, eq. 55) for the amplitude of the Green tensor from point $\tilde{\mathbf{x}}$ to point $\mathbf{x}$ in the frequency domain. Here

$$\boxed{L(\mathbf{x},\tilde{\mathbf{x}}) = \sqrt{|\mathbf{h}_1(\mathbf{x})\times\mathbf{h}_2(\mathbf{x})|\,|\det[\mathbf{Q}_2(\mathbf{x},\tilde{\mathbf{x}})]|\,|\mathbf{h}_1(\tilde{\mathbf{x}})\times\mathbf{h}_2(\tilde{\mathbf{x}})|} \qquad (39)}$$

(Klimeš, 2012, eq. 13) is the *relative geometrical spreading* defined by Červený (2001, eq. 4.14.45). Amplitude (38) corresponds to the Fourier transform

$$G_{ik}(\mathbf{x},\tilde{\mathbf{x}},\omega) = \int \mathrm{d}t\; G_{ik}(\mathbf{x},\tilde{\mathbf{x}},t)\,\exp(\mathrm{i}\omega t) \qquad (40)$$

of the Green tensor from time t to circular frequency ω. If the right–hand side of the Fourier transform included a multiplicative factor of $(2\pi)^{-\frac{1}{2}}$ or $(2\pi)^{-1}$, the right–hand side of expression (38) should be multiplied by the same factor.

The determinant of the transformation matrix (9) from Cartesian to ray–centred coordinates is

$$|\det(\hat{h}^i_a)| = |\varepsilon^{abc}\hat{h}^1_a\hat{h}^2_b\hat{h}^3_c| \quad . \qquad (41)$$

Since the covariant basis vectors $\hat{\mathbf{h}}^1$ and $\hat{\mathbf{h}}^2$ of the ray–centred coordinate system are perpendicular to the ray, their cross product is tangent to the ray, and we have relation

$$\varepsilon^{abc}\hat{h}^1_a\hat{h}^2_b = \pm|\hat{\mathbf{h}}^1\times\hat{\mathbf{h}}^2|\,V^{-1}V^c \qquad (42)$$

analogous to relation (32) for the contravariant basis vectors. Here V is the ray velocity and V^c is the ray–velocity vector. We insert definition (5) and part

$$\hat{h}^3_i = \frac{\partial q^3}{\partial x^i} \qquad (43)$$

of definition (9) into relation (41) with (42) and obtain relation

$$\hat{h}^3_c\,V^c = \frac{\mathrm{d}q^3}{\mathrm{d}\tau} \quad . \qquad (44)$$

Inserting (42) into (41) and considering (44), we arrive at

$$|\det(\hat{h}^i_a)| = |\hat{\mathbf{h}}^1\times\hat{\mathbf{h}}^2|\,V^{-1}\frac{\mathrm{d}q^3}{\mathrm{d}\tau} \quad . \qquad (45)$$

Since

$$|\det(h^a_i)|\,|\det(\hat{h}^j_b)| = 1 \quad , \qquad (46)$$

relations (34) and (45) yield identity

$$|\mathbf{h}_1\times\mathbf{h}_2| = |\hat{\mathbf{h}}^1\times\hat{\mathbf{h}}^2|^{-1}\,V\,v^{-1} \quad , \qquad (47)$$

which can be inserted into expression (39), both at point $\tilde{\mathbf{x}}$ or point $\mathbf{x}$, e.g.,

$$L(\mathbf{x},\tilde{\mathbf{x}}) = \sqrt{|\mathbf{h}_1(\mathbf{x})\times\mathbf{h}_2(\mathbf{x})|\,|\det[\mathbf{Q}_2(\mathbf{x},\tilde{\mathbf{x}})]|\,|\hat{\mathbf{h}}^1(\tilde{\mathbf{x}})\times\hat{\mathbf{h}}^2(\tilde{\mathbf{x}})|^{-1}V(\tilde{\mathbf{x}})\,[v(\tilde{\mathbf{x}})]^{-1}} \quad , \tag{48}$$

or

$$L(\mathbf{x},\tilde{\mathbf{x}}) = \sqrt{\frac{|\det[\mathbf{Q}_2(\mathbf{x},\tilde{\mathbf{x}})]|}{|\hat{\mathbf{h}}^1(\mathbf{x})\times\hat{\mathbf{h}}^2(\mathbf{x})|\,|\hat{\mathbf{h}}^1(\tilde{\mathbf{x}})\times\hat{\mathbf{h}}^2(\tilde{\mathbf{x}})|}\,\frac{V(\mathbf{x})}{v(\mathbf{x})}\,\frac{V(\tilde{\mathbf{x}})}{v(\tilde{\mathbf{x}})}} \quad . \tag{49}$$

5.2. Amplitude in terms of the paraxial matrices in ray–centred coordinates

Paraxial matrices $\mathbf{Q}(\mathbf{x})$ and $\mathbf{P}(\tilde{\mathbf{x}})$ corresponding to arbitrarily parametrized rays from a point source at $\tilde{\mathbf{x}}$ are related by equation

$$\mathbf{Q}(\mathbf{x}) = \mathbf{Q}_2(\mathbf{x},\tilde{\mathbf{x}})\mathbf{P}(\tilde{\mathbf{x}}) \quad . \tag{50}$$

Inserting relation (50) into expression (39), we obtain expression

$$\boxed{L(\mathbf{x},\tilde{\mathbf{x}}) = \sqrt{|\mathbf{h}_1(\mathbf{x})\times\mathbf{h}_2(\mathbf{x})|\,|\det[\mathbf{Q}(\mathbf{x})]|\,|\det[\mathbf{P}(\tilde{\mathbf{x}})]|^{-1}\,|\mathbf{h}_1(\tilde{\mathbf{x}})\times\mathbf{h}_2(\tilde{\mathbf{x}})|} \qquad (51)}$$

for the relative geometrical spreading.

We may insert identity (47) into expression (51), both at point $\tilde{\mathbf{x}}$ or point $\mathbf{x}$, e.g.,

$$L(\mathbf{x},\tilde{\mathbf{x}}) = \sqrt{\frac{|\mathbf{h}_1(\mathbf{x})\times\mathbf{h}_2(\mathbf{x})|\,|\det[\mathbf{Q}(\mathbf{x})]|}{|\det[\mathbf{P}(\tilde{\mathbf{x}})]|\,|\hat{\mathbf{h}}^1(\tilde{\mathbf{x}})\times\hat{\mathbf{h}}^2(\tilde{\mathbf{x}})|}\,\frac{V(\tilde{\mathbf{x}})}{v(\tilde{\mathbf{x}})}} \quad , \tag{52}$$

or

$$L(\mathbf{x},\tilde{\mathbf{x}}) = \sqrt{\frac{|\det[\mathbf{Q}(\mathbf{x})]|}{|\hat{\mathbf{h}}^1(\mathbf{x})\times\hat{\mathbf{h}}^2(\mathbf{x})|\,|\det[\mathbf{P}(\tilde{\mathbf{x}})]|\,|\hat{\mathbf{h}}^1(\tilde{\mathbf{x}})\times\hat{\mathbf{h}}^2(\tilde{\mathbf{x}})|}\,\frac{V(\mathbf{x})}{v(\mathbf{x})}\,\frac{V(\tilde{\mathbf{x}})}{v(\tilde{\mathbf{x}})}} \quad . \tag{53}$$

5.3. Amplitude in terms of the paraxial vectors in Cartesian coordinates

We supplement paraxial vectors Y_{i2} and Y_{i1} with slowness vector p_i to create a 3×3 matrix. The transformation of this matrix from ray–centred coordinates to Cartesian coordinates reads

$$\begin{pmatrix} Y_{i1} & Y_{i2} & p_i \end{pmatrix} = \hat{h}^i_k \begin{pmatrix} P_{k1} & P_{k2} & \delta_{k3}\,\frac{\mathrm{d}\tau}{\mathrm{d}q^3} \end{pmatrix} \quad . \tag{54}$$

It follows that the corresponding determinants are transformed as

$$|\varepsilon^{ijk}Y_{i1}Y_{j2}p_k| = |\det(\hat{h}^i_k)|\,|\varepsilon^{mn3}P_{m1}P_{n2}|\,\frac{\mathrm{d}\tau}{\mathrm{d}q^3} \quad . \tag{55}$$

We insert relation (45) into relation (55) and obtain

$$|\varepsilon^{ijk}Y_{i1}Y_{j2}p_k| = |\hat{\mathbf{h}}^1\times\hat{\mathbf{h}}^2|\,|\det(P_{IA})|\,V^{-1} \quad . \tag{56}$$

Relations (17), (35) and (36) yield relation

$$|\varepsilon_{ijk}X^i_1X^j_2V^k| = |\mathbf{h}_1\times\mathbf{h}_2|\,|\det(Q^I_A)|\,v \quad . \tag{57}$$

We insert relations (56) and (57) into expression (52) and obtain expression

$$L(\mathbf{x},\tilde{\mathbf{x}}) = \sqrt{\frac{|\varepsilon_{ijk}X^i_1(\mathbf{x})X^j_2(\mathbf{x})V^k(\mathbf{x})|}{v(\mathbf{x})\,|\varepsilon^{lmn}Y_{l1}(\tilde{\mathbf{x}})Y_{m2}(\tilde{\mathbf{x}})p_n(\tilde{\mathbf{x}})|\,v(\tilde{\mathbf{x}})}} \tag{58}$$

for the relative geometrical spreading. Since vectors Y_{i1} and Y_{i2} are tangent to the slowness surface at point $\tilde{\mathbf{x}}$, their cross product is normal to the slowness surface, and

$$\varepsilon^{ijk} Y_{i1}(\tilde{\mathbf{x}}) Y_{j2}(\tilde{\mathbf{x}}) = \pm |\mathbf{Y}_1(\tilde{\mathbf{x}}) \times \mathbf{Y}_2(\tilde{\mathbf{x}})| \, V^k(\tilde{\mathbf{x}}) \, V^{-1}(\tilde{\mathbf{x}}) \quad . \tag{59}$$

We multiply equation (59) by the slowness vector and obtain

$$|\varepsilon^{ijk} Y_{i1}(\tilde{\mathbf{x}}) Y_{j2}(\tilde{\mathbf{x}}) p_k(\tilde{\mathbf{x}})| = |\mathbf{Y}_1(\tilde{\mathbf{x}}) \times \mathbf{Y}_2(\tilde{\mathbf{x}})| \, V^{-1}(\tilde{\mathbf{x}}) \quad . \tag{60}$$

We now insert relation (60) into expression (58) and arrive at expression

$$\boxed{L(\mathbf{x}, \tilde{\mathbf{x}}) = \sqrt{\frac{|\varepsilon_{ijk} X_1^i(\mathbf{x}) X_2^j(\mathbf{x}) V^k(\mathbf{x})| \, V(\tilde{\mathbf{x}})}{v(\mathbf{x}) \, |\mathbf{Y}_1(\tilde{\mathbf{x}}) \times \mathbf{Y}_2(\tilde{\mathbf{x}})| \, v(\tilde{\mathbf{x}})}} \tag{61}}$$

(Chapman, 2004, eq. 5.4.19) for the relative geometrical spreading.

For the special case of a *homogeneous* Hamiltonian function, we may insert relation (26) into expression (61). In this case,

$$L(\mathbf{x}, \tilde{\mathbf{x}}) = \sqrt{\frac{|\mathbf{X}_1(\mathbf{x}) \times \mathbf{X}_2(\mathbf{x})|}{|\mathbf{Y}_1(\tilde{\mathbf{x}}) \times \mathbf{Y}_2(\tilde{\mathbf{x}})|} \, \frac{V(\tilde{\mathbf{x}})}{v(\tilde{\mathbf{x}})}} \quad . \tag{62}$$

This equation is not applicable to a general Hamiltonian function. Equation (62) with special initial conditions $|\mathbf{Y}_1(\tilde{\mathbf{x}}) \times \mathbf{Y}_2(\tilde{\mathbf{x}})| = V(\tilde{\mathbf{x}})/v(\tilde{\mathbf{x}})$ was used by Pšenčík & Teles (1996, eqs. A.1, A.4), Farra & Pšenčík (2008, eqs. 25, 28) and Červený & Pšenčík (2014, eqs. 20, 22).

5.4. Amplitude in terms of the propagator matrix of geodesic deviation in Cartesian coordinates

Paraxial vectors $X_A^i(\mathbf{x})$ and $Y_{mA}(\tilde{\mathbf{x}})$ corresponding to arbitrarily parametrized rays from a point source at $\tilde{\mathbf{x}}$ are related by equation

$$X_A^i(\mathbf{x}) = X_2^{im}(\mathbf{x}, \tilde{\mathbf{x}}) Y_{mA}(\tilde{\mathbf{x}}) \quad , \tag{63}$$

where $X_2^{im}(\mathbf{x}, \tilde{\mathbf{x}})$ is the 3×3 upper right submatrix (19) of the 6×6 propagator matrix of geodesic deviation in Cartesian coordinates. Then

$$\varepsilon_{ijk} X_1^i(\mathbf{x}) X_2^j(\mathbf{x}) V^k(\mathbf{x}) = \varepsilon_{ijk} X_2^{im}(\mathbf{x}, \tilde{\mathbf{x}}) X_2^{jn}(\mathbf{x}, \tilde{\mathbf{x}}) V^k(\mathbf{x}) Y_{m1}(\tilde{\mathbf{x}}) Y_{n2}(\tilde{\mathbf{x}}) \quad . \tag{64}$$

We consider the skewness of Levi–Civita symbol ε_{ijk}, and rewrite relation (64) in form

$$\begin{aligned} &\varepsilon_{ijk} X_1^i(\mathbf{x}) X_2^j(\mathbf{x}) V^k(\mathbf{x}) \\ &\quad = \tfrac{1}{2} \varepsilon_{ijk} X_2^{im}(\mathbf{x}, \tilde{\mathbf{x}}) X_2^{jn}(\mathbf{x}, \tilde{\mathbf{x}}) V^k(\mathbf{x}) \big[Y_{m1}(\tilde{\mathbf{x}}) Y_{n2}(\tilde{\mathbf{x}}) - Y_{n1}(\tilde{\mathbf{x}}) Y_{m2}(\tilde{\mathbf{x}}) \big] \quad . \end{aligned} \tag{65}$$

We now insert identity

$$\varepsilon_{mnl} \varepsilon^{lrs} = \delta_{mr} \delta_{ns} - \delta_{ms} \delta_{nr} \tag{66}$$

into relation (65), and arrive at

$$\varepsilon_{ijk} X_1^i(\mathbf{x}) X_2^j(\mathbf{x}) V^k(\mathbf{x}) = \tfrac{1}{2} \varepsilon_{ijk} X_2^{im}(\mathbf{x}, \tilde{\mathbf{x}}) X_2^{jn}(\mathbf{x}, \tilde{\mathbf{x}}) V^k(\mathbf{x}) \varepsilon_{mnl} \varepsilon^{lrs} Y_{r1}(\tilde{\mathbf{x}}) Y_{s2}(\tilde{\mathbf{x}}) \quad . \tag{67}$$

We insert relation (59) into relation (67), and obtain relation

$$\begin{aligned} &\varepsilon_{ijk} X_1^i(\mathbf{x}) X_2^j(\mathbf{x}) V^k(\mathbf{x}) \\ &\quad = \pm \tfrac{1}{2} \varepsilon_{ijk} X_2^{im}(\mathbf{x}, \tilde{\mathbf{x}}) X_2^{jn}(\mathbf{x}, \tilde{\mathbf{x}}) V^k(\mathbf{x}) \varepsilon_{mnl} |\mathbf{Y}_1(\tilde{\mathbf{x}}) \times \mathbf{Y}_2(\tilde{\mathbf{x}})| \, V^l(\tilde{\mathbf{x}}) \, [V(\tilde{\mathbf{x}})]^{-1} \quad . \end{aligned} \tag{68}$$

We define the matrix

$$\boxed{C_{kl}(\mathbf{x},\tilde{\mathbf{x}}) = \tfrac{1}{2}\,\varepsilon_{kij}\,\varepsilon_{lmn}\,X_2^{im}(\mathbf{x},\tilde{\mathbf{x}})\,X_2^{jn}(\mathbf{x},\tilde{\mathbf{x}}) \qquad (69)}$$

of the cofactors of matrix $X_2^{im}(\mathbf{x},\tilde{\mathbf{x}})$, insert relation (68) with definition (69) into expression (61), and obtain relation

$$\boxed{L(\mathbf{x},\tilde{\mathbf{x}}) = \sqrt{\frac{|V^k(\mathbf{x})\,C_{kl}(\mathbf{x},\tilde{\mathbf{x}})V^l(\tilde{\mathbf{x}})|}{v(\mathbf{x})\,v(\tilde{\mathbf{x}})}} \qquad (70)}$$

(Kendall, Guest & Thomson, 1992, eq. 17b; Chapman, 2004, eq. 5.4.23) for the relative geometrical spreading.

5.5. Amplitude in terms of the second–order derivatives of the characteristic function in ray–centred coordinates

Klimeš (2013a, eq. 28) derived relation

$$\left(\frac{\partial^2\tau}{\partial\tilde{x}^i\partial x^j} + \frac{1}{\Gamma}\frac{\partial\gamma}{\partial\tilde{x}^i}\frac{\partial\gamma}{\partial x^j}\right)(\mathbf{x},\tilde{\mathbf{x}})\,X_2^{jk}(\mathbf{x},\tilde{\mathbf{x}}) = -\delta_i^k \qquad (71)$$

between the mixed second–order spatial derivatives of characteristic function $\tau(\mathbf{x},\mathbf{x}')$ and the 3×3 upper right submatrix of the 6×6 propagator matrix of geodesic deviation in general coordinates including Cartesian coordinates. The meaning of functions $\gamma(\mathbf{x},\tilde{\mathbf{x}})$ and $\Gamma(\mathbf{x},\tilde{\mathbf{x}})$ is not significant here. Interested readers may refer to Klimeš (2013a; 2013b).

If we define matrix $[X_2^{-1}]_{ji}(\mathbf{x},\tilde{\mathbf{x}})$ inverse to matrix $X_2^{ij}(\mathbf{x},\tilde{\mathbf{x}})$, we may express relation (71) as

$$\left(\frac{\partial^2\tau}{\partial\tilde{x}^i\partial x^j} + \frac{1}{\Gamma}\frac{\partial\gamma}{\partial\tilde{x}^i}\frac{\partial\gamma}{\partial x^j}\right)(\mathbf{x},\tilde{\mathbf{x}}) = -[X_2^{-1}]_{ij}(\mathbf{x},\tilde{\mathbf{x}}) \quad . \qquad (72)$$

We now transform relation (72) into the ray–centred coordinates.

We transform the submatrix $X_2^{ij}(\mathbf{x},\tilde{\mathbf{x}})$ of the propagator matrix of geodesic deviation from Cartesian to ray–centred coordinates,

$$Q_2^{ab}(\mathbf{x},\tilde{\mathbf{x}}) = \hat{h}_i^a(\mathbf{x})\,X_2^{ij}(\mathbf{x},\tilde{\mathbf{x}})\,\hat{h}_j^b(\tilde{\mathbf{x}}) \quad . \qquad (73)$$

Transformation of the inverse matrix then reads

$$[Q_2^{-1}]_{ba}(\mathbf{x},\tilde{\mathbf{x}}) = h_b^j(\tilde{\mathbf{x}})\,[X_2^{-1}]_{ji}(\mathbf{x},\tilde{\mathbf{x}})\,h_a^i(\mathbf{x}) \quad . \qquad (74)$$

We transform the mixed second–order derivatives of the characteristic function from Cartesian to ray–centred coordinates,

$$\frac{\partial^2\tau}{\partial q^a\partial\tilde{q}^b}(\mathbf{x},\tilde{\mathbf{x}}) = \frac{\partial x^k}{\partial q^a}(\mathbf{x})\,\frac{\partial^2\tau}{\partial x^k\partial\tilde{x}^l}(\mathbf{x},\tilde{\mathbf{x}})\,\frac{\partial\tilde{x}^l}{\partial\tilde{q}^b}(\tilde{\mathbf{x}}) \quad , \qquad (75)$$

which can be expressed in terms of the contravariant basis vectors (8) of the ray–centred coordinate system as

$$\frac{\partial^2\tau}{\partial q^a\partial\tilde{q}^b}(\mathbf{x},\tilde{\mathbf{x}}) = h_a^k(\mathbf{x})\,\frac{\partial^2\tau}{\partial x^k\partial\tilde{x}^l}(\mathbf{x},\tilde{\mathbf{x}})\,h_b^l(\tilde{\mathbf{x}}) \quad . \qquad (76)$$

Since

$$\frac{\partial H}{\partial p_k}(\mathbf{x})\,\frac{\partial^2\tau}{\partial x^k\partial\tilde{x}^l}(\mathbf{x},\tilde{\mathbf{x}}) = 0 \tag{77}$$

(Hamilton, 1837, eqs. U, I), and

$$\frac{\partial^2\tau}{\partial x^k\partial\tilde{x}^l}(\mathbf{x},\tilde{\mathbf{x}})\,\frac{\partial H}{\partial p_l}(\tilde{\mathbf{x}}) = 0 \tag{78}$$

(Hamilton, 1837, eqs. Y, I), we have identities

$$\frac{\partial^2\tau}{\partial q^A\partial\tilde{q}^3}(\mathbf{x},\tilde{\mathbf{x}}) = 0 \quad , \qquad \frac{\partial^2\tau}{\partial q^3\partial\tilde{q}^B}(\mathbf{x},\tilde{\mathbf{x}}) = 0 \quad . \tag{79}$$

Relation (72) in the ray–centred coordinates then reads

$$[Q_2^{-1}]_{ba}(\mathbf{x},\tilde{\mathbf{x}}) = -\begin{pmatrix} \frac{\partial^2\tau}{\partial\tilde{q}^B\partial q^A} + \frac{\partial\gamma}{\partial\tilde{q}^B}\frac{1}{\Gamma}\frac{\partial\gamma}{\partial q^A} & \frac{\partial\gamma}{\partial\tilde{q}^B}\frac{1}{\Gamma}\frac{\partial\gamma}{\partial q^3} \\ \frac{\partial\gamma}{\partial\tilde{q}^3}\frac{1}{\Gamma}\frac{\partial\gamma}{\partial q^A} & \frac{\partial\gamma}{\partial\tilde{q}^3}\frac{1}{\Gamma}\frac{\partial\gamma}{\partial q^3} \end{pmatrix}(\mathbf{x},\tilde{\mathbf{x}}) \quad . \tag{80}$$

We define 2×2 matrix

$$Q_2^{BA}(\mathbf{x},\tilde{\mathbf{x}}) = -\left(\frac{\partial^2\tau}{\partial\tilde{q}^A\partial q^B}\right)^{-1}(\mathbf{x},\tilde{\mathbf{x}}) \quad , \tag{81}$$

invert 3×3 matrices in relation (80), and arrive at

$$Q_2^{ab}(\mathbf{x},\tilde{\mathbf{x}}) = \begin{pmatrix} Q_2^{AB} & -Q_2^{AD}\frac{\partial\gamma}{\partial\tilde{q}^D}\left(\frac{\partial\gamma}{\partial\tilde{q}^3}\right)^{-1} \\ -\frac{\partial\gamma}{\partial q^C}Q_2^{CB}\left(\frac{\partial\gamma}{\partial q^3}\right)^{-1} & \left(\Gamma + \frac{\partial\gamma}{\partial q^C}Q_2^{CD}\frac{\partial\gamma}{\partial\tilde{q}^D}\right)\left(\frac{\partial\gamma}{\partial q^3}\frac{\partial\gamma}{\partial\tilde{q}^3}\right)^{-1} \end{pmatrix}(\mathbf{x},\tilde{\mathbf{x}}) \quad . \tag{82}$$

We see that 2×2 matrix $Q_2^{BA}(\mathbf{x},\tilde{\mathbf{x}})$ is a submatrix of 3×3 matrix (73). Definition (81) may then be understood as the relation between the 2×2 submatrix $Q_2^{BA}(\mathbf{x},\tilde{\mathbf{x}})$ of 3×3 matrix (73) and the mixed second–order spatial derivatives of characteristic function $\tau(\mathbf{x},\mathbf{x}')$ in the ray–centred coordinates. This relation represents a generalization of the relation for the mixed second–order spatial derivatives of characteristic function $\tau(\mathbf{x},\mathbf{x}')$ by Červený, Klimeš & Pšenčík (1988, eq. 22) to anisotropic media.

Relation (81) yields relation

$$\det\left(\frac{\partial^2\tau}{\partial q^A\partial\tilde{q}^B}(\mathbf{x},\tilde{\mathbf{x}})\right) = -1/\det[\mathbf{Q}_2(\mathbf{x},\tilde{\mathbf{x}})] \quad . \tag{83}$$

We insert relation (83) into expression (39) and obtain expression

$$\boxed{L(\mathbf{x},\tilde{\mathbf{x}}) = \sqrt{|\mathbf{h}_1(\mathbf{x})\times\mathbf{h}_2(\mathbf{x})|\left|\det\left(\frac{\partial^2\tau}{\partial q^A\partial\tilde{q}^B}(\mathbf{x},\tilde{\mathbf{x}})\right)\right|^{-1}|\mathbf{h}_1(\tilde{\mathbf{x}})\times\mathbf{h}_2(\tilde{\mathbf{x}})|} \qquad (84)}$$

for the relative geometrical spreading in terms of the mixed second–order spatial derivatives of the characteristic function in ray–centred coordinates. The special case of expression (84), corresponding to orthonormal contravariant basis vectors $\mathbf{h}_1$ and $\mathbf{h}_2$, was presented by Schleicher et al. (2001, eqs. 5–6).

Expression (84) for the relative geometrical spreading may also be modified by inserting identity (47), both at point $\tilde{\mathbf{x}}$ or point $\mathbf{x}$, e.g.,

$$L(\mathbf{x},\tilde{\mathbf{x}}) = \sqrt{\frac{V(\mathbf{x})}{v(\mathbf{x})}\,\frac{V(\tilde{\mathbf{x}})}{v(\tilde{\mathbf{x}})}}\bigg/\sqrt{|\hat{\mathbf{h}}^1(\mathbf{x})\times\hat{\mathbf{h}}^2(\mathbf{x})|\left|\det\left(\frac{\partial^2\tau}{\partial q^A\partial\tilde{q}^B}(\mathbf{x},\tilde{\mathbf{x}})\right)\right||\hat{\mathbf{h}}^1(\tilde{\mathbf{x}})\times\hat{\mathbf{h}}^2(\tilde{\mathbf{x}})|} \quad . \tag{85}$$

5.6. Amplitude in terms of the second–order derivatives of the characteristic function in Cartesian coordinates

Transformation of the mixed second–order derivatives of the characteristic function from ray–centred to Cartesian coordinates reads

$$\frac{\partial^2\tau}{\partial x^k \partial\tilde{x}^l}(\mathbf{x},\tilde{\mathbf{x}}) = \frac{\partial q^a}{\partial x^k}(\mathbf{x})\,\frac{\partial^2\tau}{\partial q^a \partial\tilde{q}^b}(\mathbf{x},\tilde{\mathbf{x}})\,\frac{\partial\tilde{q}^b}{\partial\tilde{x}^l}(\tilde{\mathbf{x}}) \quad . \tag{86}$$

Considering identities (79), we express relation (86) in terms of the covariant basis vectors (9) of the ray–centred coordinate system as

$$\frac{\partial^2\tau}{\partial x^j \partial\tilde{x}^m}(\mathbf{x},\tilde{\mathbf{x}}) = \hat{h}^A_j(\mathbf{x})\frac{\partial^2\tau}{\partial q^A \partial\tilde{q}^B}(\mathbf{x},\tilde{\mathbf{x}})\hat{h}^B_m(\tilde{\mathbf{x}}) \quad . \tag{87}$$

We define the matrix

$$\boxed{W^{il}(\mathbf{x},\tilde{\mathbf{x}}) = \tfrac{1}{2}\varepsilon^{ijk}\varepsilon^{lmn}\frac{\partial^2\tau}{\partial x^j \partial\tilde{x}^m}(\mathbf{x},\tilde{\mathbf{x}})\frac{\partial^2\tau}{\partial x^k \partial\tilde{x}^n}(\mathbf{x},\tilde{\mathbf{x}}) \qquad (88)}$$

of the cofactors of matrix (86). We insert relation (87) into definition (88) and arrive at relation

$$\begin{aligned} W^{il}(\mathbf{x},\tilde{\mathbf{x}}) = {} & \varepsilon^{ijk}\,\hat{h}^1_j(\mathbf{x})\,\hat{h}^2_k(\mathbf{x})\,\varepsilon^{lmn}\,\hat{h}^1_m(\tilde{\mathbf{x}})\,\hat{h}^2_n(\tilde{\mathbf{x}}) \\ & \times \left[\frac{\partial^2\tau}{\partial q^1 \partial\tilde{q}^1}(\mathbf{x},\tilde{\mathbf{x}})\,\frac{\partial^2\tau}{\partial q^2 \partial\tilde{q}^2}(\mathbf{x},\tilde{\mathbf{x}}) - \frac{\partial^2\tau}{\partial q^1 \partial\tilde{q}^2}(\mathbf{x},\tilde{\mathbf{x}})\,\frac{\partial^2\tau}{\partial q^2 \partial\tilde{q}^1}(\mathbf{x},\tilde{\mathbf{x}})\right] \quad . \end{aligned} \tag{89}$$

We insert relation (42) into relation (89) and obtain relation

$$W^{il}(\mathbf{x},\tilde{\mathbf{x}}) = \pm|\hat{\mathbf{h}}^1(\mathbf{x})\times\hat{\mathbf{h}}^2(\mathbf{x})|\,\frac{V^i(\mathbf{x})}{V(\mathbf{x})}\,|\hat{\mathbf{h}}^1(\tilde{\mathbf{x}})\times\hat{\mathbf{h}}^2(\tilde{\mathbf{x}})|\,\frac{V^l(\tilde{\mathbf{x}})}{V(\tilde{\mathbf{x}})}\,\det\!\left(\frac{\partial^2\tau}{\partial q^A \partial\tilde{q}^B}(\mathbf{x},\tilde{\mathbf{x}})\right) \quad . \tag{90}$$

We multiply relation (90) by $p_i(\mathbf{x})$ and $p_l(\tilde{\mathbf{x}})$, insert the product into expression (85), and obtain expression

$$L(\mathbf{x},\tilde{\mathbf{x}}) = 1\Big/\sqrt{|p_i(\mathbf{x})\,W^{il}(\mathbf{x},\tilde{\mathbf{x}})\,p_l(\tilde{\mathbf{x}})|\,v(\mathbf{x})\,v(\tilde{\mathbf{x}})} \quad . \tag{91}$$

for the relative geometrical spreading in terms of the mixed second–order derivatives of the characteristic function in Cartesian coordinates.

We insert expression (91) for the relative geometrical spreading into expression (38) and obtain new expression

$$\boxed{A^{\mathrm{G}}(\mathbf{x},\tilde{\mathbf{x}}) = \frac{1}{4\pi}\sqrt{\frac{|p_i(\mathbf{x})\,W^{il}(\mathbf{x},\tilde{\mathbf{x}})\,p_l(\tilde{\mathbf{x}})|}{\varrho(\mathbf{x})\,\varrho(\tilde{\mathbf{x}})}}\;\exp[\mathrm{i}\,\varphi(\mathbf{x},\tilde{\mathbf{x}})] \qquad (92)}$$

for the amplitude of the Green tensor.

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