Green function as an integral superposition of Gaussian beams in inhomogeneous anisotropic layered structures in Cartesian coordinates

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Summary

Integral superposition of Gaussian beams is a useful generalization of the standard ray theory. It removes some of the deficiencies of the ray theory like its failure to describe properly behaviour of waves in caustic regions. It also leads to a more efficient computation of seismic wavefields since it does not require the time-consuming two-point ray tracing. We present the Gaussian beam integral superposition of Green function for inhomogeneous, isotropic or anisotropic, layered structures based on the dynamic ray tracing (DRT) in Cartesian coordinates. For the evaluation of the superposition formula, it is sufficient to solve the DRT in Cartesian coordinates just for the point-source initial conditions. Moreover, instead of seeking 3×3 paraxial matrices, it is sufficient to seek just 3×2 parts of these matrices. The presented formulae can be used for the computation of wavefields generated by various types of point sources (explosive, moment-tensor). Receivers may be situated at an arbitrary point of the medium, including the ray-theory shadow regions. Arbitrary direct, multiply reflected/transmitted, unconverted or converted elementary waves, propagating independently, can be considered.

Keywords: elastodynamic Green function, inhomogeneous anisotropic media, integral superposition of Gaussian beams

1 Introduction

The method of integral superposition of Gaussian beams, correctly called paraxial Gaussian beams in order to distinguish them from exact Gaussian beams, is a powerful extension of the ray method. It removes several problems, which complicate applications of

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the ray method. The most important is the regular behaviour of wavefields evaluated by the superposition at caustics and their vicinities, where the ray method fails. The method of integral superposition of Gaussian beams is not only regular at the caustics, but it yields at it and in its vicinity amplitudes close to correct. Although the superposition of Gaussian beams does not provide entirely correct values in shadow regions, it guarantees, in contrast to the ray method, which yields abrupt changes of the wavefield at the boundary of shadow region, a smooth transition from the illuminated to shadow region. Computations of seismic wavefields based on the superposition of Gaussian beams do not require time-consuming two-point ray tracing, which is a basic part of every ray tracing code. The integral superposition can be evaluated at any point of the medium, including points in shadow regions, without necessity to trace rays to these points.

The Gaussian beams are approximate solutions of the elastodynamic equation concentrated close to rays of seismic body waves. Thus, their travel times and amplitudes can be evaluated, approximately, not only along rays, as in the ray method, but also in their vicinities. Their frequency-dependent amplitudes decrease exponentially with the square of the distance from the ray in any plane intersecting the ray. This behaviour is a consequence of the use of the complex-valued matrix of spatial second-order derivatives of travel time in the Taylor expansion of travel time up-to quadratic terms.

In this paper, we pay attention to the integral superposition of Gaussian beams, not to individual Gaussian beams. There is an extensive literature devoted to individual Gaussian beams propagating in inhomogeneous acoustic, isotropic or anisotropic media. For references, see, for example, Červený and Pšenčík (1983), Hanyga (1986), Červený (2001), Bleistein (2007), Kravtsov and Berczynski(2007), Červený and Pšenčík (2010) or Protasov (2015). The references to integral superposition (summation) of Gaussian beams are mostly devoted to acoustic or elastic isotropic inhomogeneous media, see Babich and Popov (1981), Červený, Popov and Pšenčík (1982), Popov (1982), Klimeš (1984a), Červený (1985), George, Virieux and Madariaga (1987), Weber (1988), White, Norris, Bayliss and Burridge (1987), Červený, Klimeš and Pšenčík (2007), Vinje, Roberts and Taylor (2008), and others. Červený (2000) uses wavefront orthonormal coordinates to derive the integral superposition for inhomogeneous anisotropic media. Explicit references to integral superposition of Gaussian beams in inhomogeneous anisotropic media are difficult to find. Let us mention that Gaussian beams propagating in inhomogeneous anisotropic media have been applied in migration in seismic exploration, see Hill (1990, 2001), Alkhalifah (1995), Gray (2005), Zhu, Gray and Wang (2005, 2007), Popov, Semtchenok, Verdel and Popov (2007), Vinje, Roberts and Taylor (2008), Gray and Bleistein (2009), Bleistein and Gray (2010), Protasov and Tcheverda (2012), Protasov (2015).

We start the derivation of the integral superposition formula from the superposition formula of Červený (2001) in wavefront orthonormal coordinates, and Červený et al. (2007) in ray-centred coordinates, which have been derived originally by Klimeš (1994a). We exploit the fact that the integral superposition of Červený (2001) and Červený et al. (2007) can be used not only for laterally varying, layered isotropic, but also anisotropic media.

Basic procedure in the computation of the superposition formula is the dynamic ray tracing (DRT), see again Červený et al. (2007). The dynamic ray tracing system, and, consequently, the expressions for the integral superposition, can be expressed in various coordinate systems, most often in ray-centred, wavefront orthonormal, or Cartesian coordinate systems. The advantage of the use of the DRT in the ray-centred or wavefront orthonormal coordinates is the physical clarity of the concept and the low number of the DRT equations because the solution is sought in the form of 2×2 paraxial matrices. The disadvantage of such an approach is its relative complexity. The quantities in the ray-centred or wavefront orthonormal coordinates are computed in the coordinate system, which varies along the ray. In this respect, the straightforward use of the DRT in Cartesian coordinates is advantageous because the coordinate system, in which computations are performed, is fixed. Also, the structure of the right-hand sides of DRT equations in Cartesian coordinates is considerably simpler than in the ray-centred coordinates. The DRT in Cartesian coordinates is obtained by the differentiation of ray tracing equations with respect to ray parameters, see e.g., Červený (1972), Ralston (1983), Gajewski and Pšenčík (1990), Leung, Qian and Burridge (2007). Certain disadvantage of this approach is, however, a larger number of equations, from which the DRT consists. The solution of the DRT system in Cartesian coordinates has the form of 3×3 paraxial matrices. In this paper, we show that for the evaluation of the superposition formula, it is sufficient to seek the solution of the DRT in Cartesian coordinates in the form of only 3×2 paraxial matrices. We call such a DRT system the *reduced* DRT system. Let us note that the reduced DRT system (Pšenčík and Teles, 1996) is currently used in the ANRAY program package (Gajewski and Pšenčík, 1990). The package ANRAY is freely available on the web pages of the SW3D Consortium Seismic Waves in Complex 3D Structures (http://sw3d.cz).

The transformation of the superposition formula of Červený et al. (2007), which requires results of the DRT in ray-centred coordinates, into the formula requiring results of reduced the DRT in Cartesian coordinates represents an important contribution of this paper. We present superposition formula for a wavefield generated by a unit-force point source, i.e., for the Green-function wavefield. In principle, only spreading-free ray-theory amplitudes are sufficient for the evaluation of the superposition formula. The Green-function wavefield is considered intentionally. The Green-function expressions can be used for the construction of any type of point sources (explosive, moment-tensor), and, using the representation theorem (Aki and Richards, 1980), even for finite surface or finite volume sources.

We separate the study of wave propagation in anisotropic media based on the superposition of Gaussian beams in two parts. In this paper, we concentrate on study of elementary waves propagating in anisotropic media independently. This is always the case of P waves and it is also the case of split S waves. In the forthcoming paper, we intend to concentrate on wave propagation in weakly anisotropic media and in vicinities of shear-wave singularities. In such regions, the two S waves propagate coupled, and different tools must be used for their study. The study of P-wave propagation in weakly anisotropic media leads to simplified, and thus more efficient, formulae.

There are other summation method, which are regular at caustics. The Maslov asymptotic theory (Maslov, 1965) was used to derive suitable expressions for theoretical seismograms in inhomogeneous media by Chapman and Drummond (1982), Thomson and Chapman (1983), Chapman (2004), and others. Klimeš (1984b) studied the relation between integral superposition of Gaussian beams and Maslov asymptotic theory. Another method is the integral superposition of Gaussian packets. General expression for the integral superposition of Gaussian packets in heterogeneous anisotropic media was derived by Klimeš (2014a). Klimeš (2015) also showed how can the integral superposition of 3-D Gaussian packets be reduced to the superposition of Gaussian beams.

In the main text, we explain the basic concepts, on which the integral superposition of Gaussian beams is built. Auxiliary procedures like ray tracing and dynamic ray tracing are shifted to Appendices. Formulae given in this paper, with two exceptions mentioned below, represent a complete and uniformly described set of formulae necessary for the evaluation of the integral superposition. The reason for this way of presentation is that the required formulae are given in many papers often in different forms and under different assumptions. This applies even to the authors' papers. The only exceptions not presented here are formulae for normalized energy reflection/transmission coefficients, which are presented in Červený (2001; Secs 5.3 and 5.4.7), both for anisotropic and isotropic media, and formulae for the phase shift due to caustics, where we refer to the derivations by Garmany (2001), Klimeš (2010, 2014b).

In Section 2, we present and discuss quantities, which can be obtained by solving several forms of dynamic ray tracing for inhomogeneous anisotropic layered media. We also describe there the computation of ray-theory Green function amplitudes. Section 3 is devoted to the replacement of the quantities used in the formula for the integral superposition of Gaussian beams in inhomogeneous anisotropic media and obtained from the DRT in ray-centred coordinates by quantities obtained from the reduced DRT in Cartesian coordinates. This replacement makes possible to use only 3×2 parts of 3×3 paraxial matrices for the evaluation of the integral superposition of Gaussian beams. In section 4 some important concluding remarks are presented. Appendices A and B contain all the necessary equations for the ray tracing and dynamic ray tracing in Cartesian coordinates, respectively. The initial conditions at a point source and transformation relations for the ray tracing and dynamic ray tracing at structural interfaces are included.

A specific role in this study is played by the following three points. One is the initial point S, from which ray tracing and dynamic ray tracing calculations, whose solutions are necessary for the construction of Gaussian beams, start. Another point is the receiver R, at which the wavefield is evaluated by the summation of the contributions of nearby Gaussian beams. The contributions are extrapolated from the reference points P situated on rays Ω, along which Gaussian beam contributions are calculated. The points P should be chosen so that the receiver R is situated in their close, paraxial, vicinity. In this contribution, we choose points P as points of intersection of rays Ω with a smooth surface called a *target surface*. Target surface is a smooth surface, on which rays Ω terminate. The surface may, but need not, represent a structural surface (interface, free surface). In principle, for each receiver R one may choose a target surface for each computed elementary wave. On the other hand, one may use a common target surface for all considered receivers R and all computed elementary waves. The receiver R may be situated on the target surface or close to it. The points S and P situated on the rays Ω can be specified either by their Cartesian coordinates or by a variable parameter along the ray. Since we use the travel time τ along a ray as the variable parameter along the ray, we use either x_m^S or τ_0 for the specification of the point S, and x_m^P or τ for the points P. The receiver R is specified by its Cartesian coordinates x_m^R. The ray passing directly through the receiver R is not required.

To express the equations in the paper in a concise form, we use alternatively the component and matrix notation for vectors and matrices. In the component notation, the upper-case indices (I, J, K,...) take the values 1 or 2, and the lower-case indices (i, j, k,...) the values 1, 2, or 3. The Einstein summation convention is used throughout the paper. The matrices and vectors are denoted by bold upright symbols. The dynamic ray tracing is used here in two coordinate systems, namely in ray-centred coordinates q_i and in Cartesian coordinates x_i. To distinguish the matrices in ray-centred coordinates q_i from the analogous matrices in Cartesian coordinates x_i, we use superscripts (q) and (x) over them. Further, to distinguish between 2×2 and 3×3 matrices, we use the circumflex $(\hat{\ })$ above symbols for 3×3 matrices. The vectors are considered as column matrices. Whenever there may be reason for confusion, the dimensions of the matrices are explicitly indicated. The index following the comma in the subscript indicates a partial derivative with respect to the relevant Cartesian coordinate.

2 Ray theory for inhomogeneous anisotropic layered structures

In this section, we discuss results of several formulations of the dynamic ray tracing and describe the evaluation of the ray-theory complex-valued amplitudes of separate elementary body waves generated by a unit-force point source, and propagating in inhomogeneous anisotropic layered structures. We start with the presentation of expressions in ray-centred coordinates since they represent an important available basis for the derivations of formulae in Cartesian coordinates. We skip detailed derivation and discussion of basic formulae of the ray method. They can be found in Červený (2001).

In the following, we consider a ray Ω, specified by two ray parameters γ_1, γ_2. They may be, for example, the take-off angles of the ray Ω or two components of the slowness vector of the ray Ω at the point source. Ray tracing equations and the initial and boundary conditions can be found in Appendix A. The dynamic ray tracing equations in Cartesian coordinates with the corresponding initial and boundary conditions are given in Appendix B.

2.1 Dynamic ray tracing

The ray tracing can be used to compute necessary quantities only on the considered ray Ω, not in its vicinity. This is, however, not sufficient for the calculation of the ray-theory amplitudes and/or Gaussian beams concentrated to ray Ω. This is because the ray-theory amplitudes depend on geometrical spreading, which is related to the ray field, not to a single ray. In the case of Gaussian beams, we also need to compute complex-valued paraxial travel times (the complex-valued travel times in the vicinity of the ray Ω). For the computation of the quantities related to the ray field, it is necessary to compute system of rays around Ω or to supplement the ray tracing by an additional procedure called the *dynamic ray tracing.*

The dynamic ray tracing (DRT) is a basic procedure for the computation of geometrical spreading and for the computation of the second derivatives of the travel time field with respect to the used coordinates, along the ray. Geometrical spreading is a basic quantity in the computation of the ray-theory amplitudes along the ray. Therefore, we speak of dynamic ray tracing in order to distinguish it from the standard (kinematic) ray tracing, which provides kinematic quantities. Dynamic ray tracing consists in the solution of a system of linear ordinary differential equations of the first order along the ray Ω. The system may be solved together with ray tracing, or along an already known ray Ω.

The DRT system can be expressed in various coordinate systems (Cartesian x_i, ray-centred q_i, etc.). The DRT system in ray-centred coordinates designed for the computation of Gaussian beams was described in detail in Červený and Pšenčík (2010). In this paper we discuss three versions of the DRT. First we briefly discuss the DRT system in Cartesian coordinates x_i, and then in ray-centred coordinates q_i. The third version of the DRT is the reduced DRT in Cartesian coordinates, in which the 3×2 matrices instead of 3×3 matrices are considered.

2.1.1 Dynamic ray tracing system in Cartesian coordinates

The dynamic ray tracing system in Cartesian coordinates is commonly used to determine 3×3 paraxial matrices $\hat{\mathbf{Q}}^{(x)}(\tau)$, $\hat{\mathbf{P}}^{(x)}(\tau)$, with elements

$$Q_{ij}^{(x)}(\tau) = \partial x_i / \partial \gamma_j \ , \qquad P_{ij}^{(x)}(\tau) = \partial p_i / \partial \gamma_j \ . \tag{1}$$

Here p_i are Cartesian components of the slowness vector $\mathbf{p}$, and γ_j are the ray coordinates. The ray coordinates γ_j may be introduced in various ways. In this paper, we take γ_1 and γ_2 to be the ray parameters specifying the ray Ω, and $\gamma_3 = \tau$, where τ is a monotonic parameter along the ray Ω, which has the meaning of the travel time, see Appendix A. The 3×3 paraxial matrices $\hat{\mathbf{Q}}^{(x)}(\tau)$ and $\hat{\mathbf{P}}^{(x)}(\tau)$ describe off-ray paraxial changes of the position of the point $\mathbf{x}(\tau)$ on the ray and of the slowness vector $\mathbf{p}(\tau)$ caused by the change of ray coordinates γ_j. The DRT system for $\hat{\mathbf{Q}}^{(x)}(\tau)$ and $\hat{\mathbf{P}}^{(x)}(\tau)$ is given in (B-1).

The 3×3 matrices $\hat{\mathbf{Q}}^{(x)}(\tau)$ and $\hat{\mathbf{P}}^{(x)}(\tau)$ can be used to compute the 3×3 symmetric matrix $\hat{\mathbf{M}}^{(x)}(\tau)$ of the second-order partial derivatives of the travel time T with respect to spatial coordinates x_i. Its elements have the form:

$$M_{ij}^{(x)}(\tau) = \partial^2 T / \partial x_i \partial x_j \ . \tag{2}$$

The 3×3 symmetric matrix $\hat{\mathbf{M}}^{(x)}(\tau)$, with six independent elements $M_{ij}^{(x)}(\tau)$, can be simply expressed in terms of the 3×3 matrices $\hat{\mathbf{Q}}^{(x)}(\tau)$ and $\hat{\mathbf{P}}^{(x)}(\tau)$, obtained from the DRT. As

$$M_{ij}^{(x)}(\tau) Q_{jk}^{(x)}(\tau) = \frac{\partial^2 T}{\partial x_i \partial x_j} \frac{\partial x_j}{\partial \gamma_k} = \frac{\partial p_i}{\partial \gamma_k} = P_{ik}^{(x)}(\tau) \ , \tag{3}$$

we obtain

$$\hat{\mathbf{M}}^{(x)}(\tau) = \hat{\mathbf{P}}^{(x)}(\tau) (\hat{\mathbf{Q}}^{(x)}(\tau))^{-1} \ . \tag{4}$$

2.1.2 Dynamic ray tracing in ray-centred coordinates

In the ray-centred coordinates q_i, the selected ray Ω represents its coordinate axis q_3. The coordinate axes q_1 and q_2 are usually introduced as mutually perpendicular straight lines situated in the plane tangent to the wavefront, with the origin at the intersection of the wavefront with the ray Ω. The transformation from ray-centred to Cartesian coordinates and vice versa is controlled by the 3×3 transformation matrices $\hat{\mathbf{H}}(\tau)$ and $\hat{\overline{\mathbf{H}}}(\tau)$ with elements:

$$H_{im}(\tau) = \partial x_i/\partial q_m \ , \qquad \overline{H}_{im}(\tau) = \partial q_i/\partial x_m \ . \tag{5}$$

The transformation matrices have several useful properties. First of all, $\hat{\mathbf{H}} \cdot \hat{\overline{\mathbf{H}}} = \hat{\mathbf{I}}$, where $\hat{\mathbf{I}}$ is the 3×3 identity matrix. Further, $H_{i3} = \mathcal{U}_i$ and $H_{3i} = p_i$, where $\boldsymbol{\mathcal{U}}$ is the ray-velocity vector and $\mathbf{p}$ is the slowness vector, both known from ray tracing, see Appendix A. For more details and the description how to evaluate the transformation matrices $\hat{\mathbf{H}}$ and $\hat{\overline{\mathbf{H}}}$ see Červený and Pšenčík (2010; Sec. 2.3).

The dynamic ray tracing in ray-centred coordinates is used to determine two 2×2 matrices $\mathbf{Q}^{(q)}(\tau)$ and $\mathbf{P}^{(q)}(\tau)$, with elements

$$Q^{(q)}_{IJ}(\tau) = \partial q_I/\partial \gamma_J \ , \qquad P^{(q)}_{IJ}(\tau) = \partial p^{(q)}_I/\partial \gamma_J \ . \tag{6}$$

The parameters γ_J may be again specified as the ray parameters.

From the 2×2 paraxial matrices $\mathbf{Q}^{(q)}(\tau)$ and $\mathbf{P}^{(q)}(\tau)$, we can obtain the important 2×2 matrix $\mathbf{M}^{(q)}(\tau)$ of the second derivatives of travel-time field T with respect to ray-centred coordinates q_1 and q_2, whose elements have the form:

$$M^{(q)}_{IJ}(\tau) = \partial^2 T/\partial q_I \partial q_J \ . \tag{7}$$

The 2×2 symmetric matrix $\mathbf{M}^{(q)}(\tau)$, with three independent elements $M^{(q)}_{IJ}(\tau)$, can be expressed in terms of $\mathbf{Q}^{(q)}(\tau)$ and $\mathbf{P}^{(q)}(\tau)$. As

$$M^{(q)}_{IJ}(\tau) Q^{(q)}_{JK}(\tau) = \frac{\partial^2 T}{\partial q_I \partial q_J} \frac{\partial q_J}{\partial \gamma_K} = \frac{\partial p^{(q)}_I}{\partial q_J} \frac{\partial q_J}{\partial \gamma_K} = \frac{\partial p^{(q)}_I}{\partial \gamma_K} = P^{(q)}_{IK}(\tau) \ , \tag{8}$$

we obtain

$$\mathbf{M}^{(q)}(\tau) = \mathbf{P}^{(q)}(\tau)(\mathbf{Q}^{(q)}(\tau))^{-1} \ . \tag{9}$$

2.1.3 Reduced DRT in Cartesian coordinates

The disadvantage of the DRT system in the ray-centred coordinates with respect to the DRT in Cartesian coordinates is that the DRT system in ray-centred coordinates is more complicated than in Cartesian coordinates. It is because we have to rotate appropriately the whole system at any step of the computation. The disadvantage of the DRT system in Cartesian coordinates is a larger number of its equations. We can, however, reduce the number of equations of the DRT in Cartesian coordinates, and compute only the first two columns of 3×3 paraxial matrices $\hat{\mathbf{Q}}^{(x)}(\tau)$ and $\hat{\mathbf{P}}^{(x)}(\tau)$, namely the elements $Q^{(x)}_{iJ}(\tau)$ and

$P_{iJ}^{(x)}(\tau)$. These four columns are sufficient for the computation of Gaussian beams at any point of the ray Ω. The procedure is as follows:

a) At the initial point S of the ray, we specify the initial conditions (B-6) and (B-8) with (B-9) for $Q_{iN}^{(x)}(S)$ and $P_{iN}^{(x)}(S)$.

b) We perform the reduced DRT in Cartesian coordinates (B-1), only for elements $Q_{iN}^{(x)}(P)$ and $P_{iN}^{(x)}(P)$ of the paraxial matrices at the point P.

Let us note that this procedure is used in the program package ANRAY (Gajewski and Pšenčík, 1990) designed for the computation of seismic wavefields in 3D laterally varying layered structures composed of isotropic or anisotropic layers.

2.1.4 Comments on dynamic ray tracing

The application of the DRT is much broader than discussed in the previous sections. As the travel time $T = \tau$ along the ray Ω and its first derivatives $p_i = \partial T/\partial x_i$ are known from ray tracing, and as its second derivatives $M_{ij}^{(x)} = \partial^2 T/\partial x_i \partial x_j$ can be determined using the DRT, we can make a spatial quadratic expansion of the travel time at an arbitrary point of the ray. Consequently, we can determine approximately the travel time T even at points situated in the vicinity of the ray Ω. We speak of paraxial travel times and of paraxial (quadratic) vicinity of the ray Ω. We can also construct linear expansion of the slowness vectors and compute paraxial rays in the paraxial vicinity of the ray Ω. Actually, the DRT system itself represents an approximate ray tracing system for paraxial rays in a vicinity of the ray Ω. For this reason, the dynamic ray tracing is sometimes called paraxial ray tracing. Here, we are not interested in paraxial rays and, therefore, we speak of dynamic ray tracing. For more details on the applications of DRT, see Červený (2001).

2.2 Ray-theory complex-valued amplitudes

Let us again consider a harmonic high-frequency seismic body wave propagating in a laterally varying, anisotropic layered structure and the ray Ω corresponding to this wave. We consider that the wavefield is generated by a unit-force point source located at the point S, i.e., we consider the Green-function wavefield. For the references, see, for example Kendall, Guest and Thomson (1992), Pšenčík and Teles (1996), Červený (2001), Červený et al. (2007), Klimeš (2012). At a point P, arbitrarily chosen on the ray Ω, the wavefield is represented by the zero-order approximation of the Green-function displacement tensor with Cartesian components $u_{ij}(P)$:

$$u_{ij}(P) = U_{ij}(P)\exp[-\mathrm{i}\omega(t - T(P))] = A(P)g_i(P)g_j(S)\exp[-\mathrm{i}\omega(t - T(P))] \; . \tag{10}$$

The symbol $U_{ij}(P)$ denotes Cartesian components of the tensorial zero-order ray-theory amplitude, $A(P)$ is the corresponding scalar zero-order ray-theory amplitude. The amplitude $A(P)$ can be, in general, complex-valued. The symbols $g_i(P)$ and $g_j(S)$ denote

Cartesian components of the real-valued polarization vectors $\mathbf{g}$ specified at points P and S, respectively. $T(P)$ is the travel time at the point P of the ray Ω. Due to the homogeneity of the considered Hamiltonian (A-4), $T(P) = \tau(P)$, where τ is the parameter along the ray Ω with the meaning of time. The travel time $T(P)$ and the polarization vectors $\mathbf{g}(S)$ and $\mathbf{g}(P)$ are determined during the ray tracing. It remains to discuss the computation of the scalar ray-theory amplitude $A(P)$. For this, results of the DRT are required.

The scalar ray-theory amplitude $A(P)$ of the Green function can be expressed as:

$$A(P) = \frac{\mathcal{R}^C(P,S)\exp[\mathrm{i}T^G(P,S)]}{4\pi[\rho(P)\rho(S)\mathcal{C}(P)\mathcal{C}(S)]^{1/2}\mathcal{L}(P,S)} , \qquad (11)$$

see Červený et al. (2007, eq.78). In equation (11), ρ is the density, $\mathcal{C}$ the phase velocity (known from ray tracing), $\mathcal{R}^C(P,S)$ is the product of all normalized energy reflection/transmission coefficients along ray Ω from S to P. For its detailed description, see Červený (2001, Sec. 5.4.7). The function $T^G(P,S)$ in (11) is the complete phase shift due to caustics along the ray from the source S to the point P. It is related to the so-called KMAH index. The possible phase shift directly at the source is included in $T^G(P,S)$. For details and computation of the phase shift due to caustics in anisotropic media, see Klimeš (1997, 2010, 2014b), Bakker (1998). The function $\mathcal{L}(P,S)$ is the relative geometrical spreading defined as:

$$\mathcal{L}(P,S) = |\det \mathbf{Q}^{(q)}_{\mathcal{L}}(P,S)|^{1/2} , \qquad (12)$$

where the 2×2 matrix $\mathbf{Q}^{(q)}_{\mathcal{L}}(P,S)$ is the solution $\mathbf{Q}^{(q)}(P)$ of the DRT in ray-centred coordinates with the initial conditions $\mathbf{Q}^{(q)}(S) = \mathbf{0}$ and $\mathbf{P}^{(q)}(S) = \mathbf{I}$, see Červený (2001, eq. (4.10.11)). The symbols $\mathbf{0}$ and $\mathbf{I}$ denote the 2×2 null and identity matrices, respectively. The subscript $\mathcal{L}$ of the matrix $\mathbf{Q}^{(q)}_{\mathcal{L}}(P,S)$ indicates the special initial conditions used for its generation. Equation (12) can be simply generalized for the initial conditions $\mathbf{Q}^{(q)}(S) = \mathbf{0}$, $\mathbf{P}^{(q)}(S) = \mathbf{P}^{(q)}_0$, where $\mathbf{P}^{(q)}_0$ is an arbitrary finite and non-zero 2×2 matrix. Since the DRT is linear, the DRT solution for the new initial conditions, obtained by the multiplication of the original ones by a constant matrix, is the original solution multiplied by this constant matrix. Thus, the multiplication of the initial conditions used for the generation of $\mathbf{Q}^{(q)}_{\mathcal{L}}(P,S)$ by $\mathbf{P}^{(q)}_0$ yields $\mathbf{Q}^{(q)}(P) = \mathbf{Q}^{(q)}_{\mathcal{L}}(P,S)\mathbf{P}^{(q)}_0$. It is then easy to generalize equation (12) to:

$$\mathcal{L}(P,S) = \left[\frac{|\det \mathbf{Q}^{(q)}(P)|}{|\det \mathbf{P}^{(q)}(S)|}\right]^{1/2} . \qquad (13)$$

The expressions for the scalar ray-theory amplitude $A(P)$ can be now written as:

$$A(P) = \overline{A}(P)\left[\frac{|\det \mathbf{P}^{(q)}(S)|}{|\det \mathbf{Q}^{(q)}(P)|}\right]^{1/2} = \frac{\overline{A}(P)}{\mathcal{L}(P,S)} . \qquad (14)$$

Here $\overline{A}(P)$ is the spreading-free scalar ray-theory amplitude, which is given by the relation:

$$\overline{A}(P) = \frac{\mathcal{R}^C(P,S)\exp[\mathrm{i}T^G(P,S)]}{4\pi[\rho(S)\rho(P)\mathcal{C}(S)\mathcal{C}(P)]^{1/2}} . \qquad (15)$$

Let us emphasize that $A(P)$ is singular at caustic points, where $|\det \mathbf{Q}^{(q)}(P)|^{1/2} = 0$, and thus $\mathcal{L}(P, S) = 0$.

There is an important aspect related to the use of equation (11) for the scalar zero-order ray-theory amplitude $A(P)$ of the Green function. The equation is valid along any ray situated in a smoothly varying inhomogeneous medium with smooth structural interfaces. With increasing strength of variations of the medium properties, i.e., with decreasing smoothness of the medium properties and/or interfaces, one must, however, expect decrease of the accuracy of the scalar ray-theory amplitude $A(P)$.

3 Time-harmonic Gaussian beam integral superposition of Green function

In Section 3.1, we introduce the formula for the time-harmonic Gaussian beam integral superposition of Green function for inhomogeneous anisotropic layered structures. The formula depends on quantities whose evaluation requires DRT in ray-centred coordinates. In Section 3.2, we show how to calculate these DRT quantities using the reduced DRT in Cartesian coordinates. Finally, in Section 3.3, we summarize the evaluation of the integral superposition with the help of the reduced DRT in Cartesian coordinates.

3.1 Gaussian beam integral superposition of Green function in ray-centred coordinates

The expression for Gaussian beam integral superposition of Green function in 3D inhomogeneous isotropic media was first derived by Klimeš (1984a, equation (77)). Here we use it in the form given in equation (169) of Červený et al. (2007). The expression, whose important elements are specified in the ray-centred coordinates, can be used not only in inhomogeneous isotropic, but also in inhomogeneous anisotropic media. In the notation adopted in this paper, the expression for the integral supperposition of Gaussian beams $u^B_{ij}(R, \omega)$ reads:

$$\begin{aligned} u^B_{ij}(R,\omega) &= (\omega/2\pi) \iint\limits_{\mathcal{D}} \mathrm{d}\gamma_1 \mathrm{d}\gamma_2 U_{ij}(P) |\det \mathbf{Q}^{(q)}(P)| \\ &\times \{-\det[\mathbf{M}^{(q)}(P) - \mathbf{M}(P)]\}^{1/2} \exp[\mathrm{i}\omega\theta(R,P)] \ , \end{aligned} \tag{16}$$

where the symbol $U_{ij}(P)$ represents an element of the tensorial ray-theory amplitude (10). $\mathbf{M}^{(q)}(P)$ is a 2×2 real-valued matrix (9) of the second derivatives of travel time with respect to ray-centred coordinates q_I. It can be obtained from the DRT in ray-centred coordinates. The function $\theta(R, P)$ is given by the relation:

$$\theta(R,P) = \tau(P) + (x^R_k - x^P_k)p_k(P) + \tfrac{1}{2}(x^R_k - x^P_k)\mathcal{M}_{kl}(P)(x^R_l - x^P_l) \ . \tag{17}$$

Here P denotes a point on the ray Ω with coordinates x^P_m, close to the receiver R. We assume that the point R is situated in the paraxial vicinity of the ray Ω. The quantity

$\tau(P)$ is the real-valued travel time at P, measured along Ω from the initial point S, and $p_k(P)$ is the k-th Cartesian component of the real-valued slowness vector $\mathbf{p}$ at the point P. The 3×3 complex-valued matrix $\hat{\boldsymbol{\mathcal{M}}}(P)$ is related to the 2×2 complex-valued matrix $\mathbf{M}(P)$ of parameters of Gaussian beams, which controls the shape of Gaussian beam along the ray Ω. The matrix $\mathbf{M}(P)$ satisfies the existence conditions of Gaussian beams, i.e., it is finite, symmetric and its imaginary part is positively definite. Otherwise, it can be chosen arbitrarily. It can be recalculated from one point of the ray Ω to another using an appropriate DRT. If $\mathbf{M}$ satisfies the existence conditions at one point of the ray Ω, then it satisfies them at any point of the ray Ω, see Sec.4 of Červený and Pšenčík (2010).

The relation of the 3×3 complex-valued matrix $\hat{\boldsymbol{\mathcal{M}}}(P)$ to the 2×2 complex-valued matrix $\mathbf{M}(P)$ of the parameters of Gaussian beams is given by the relation derived by Červený and Klimeš (2010):

$$\mathcal{M}_{ij} = \bar{H}_{Ji} M_{JK} \bar{H}_{Kj} + p_i \eta_j + \eta_i p_j - p_i p_j \mathcal{U}_k p_k \ . \tag{18}$$

The real-valued quantities p_i, η_i and $\mathcal{U}_i$ are components of the slowness, eta and ray-velocity vectors, which are known from ray tracing, respectively.

In equation (16), R is again the receiver point with coordinates x_m^R, P is the reference point with coordinates x_m^P, situated on the ray Ω, specified by ray parameters γ_1, γ_2. The reference point P should be chosen so that the receiver R is situated in its paraxial vicinity. In this paper, we specify the reference point P as the point of intersection of the ray Ω with the *target surface* Σ^T. The target surface is a smooth surface passing through the receiver R or close to it. The superposition works best when the rays Ω are roughly perpendicular to Σ^T.

Note that the factor $|\det \mathbf{Q}^{(q)}(P)|$ in the integral superposition (16) cancels the factor $|\det \mathbf{Q}^{(q)}(P)|^{1/2}$ appearing in the denominator of the scalar ray-theory amplitude $A(P)$ in equation (14), and thus removes the singularity caused by this factor when it vanishes. Consequently the integrand of (16) is always regular, even at caustics. This is a significant difference of the integral superposition (16) of Gaussian beams with respect to the standard ray-theory, in which amplitudes rise to infinity when $|\det \mathbf{Q}^{(q)}(P)|^{1/2}$ approaches zero.

For each ray specified by ray parameters γ_1, γ_2 in the integral superposition (16), the relevant Gaussian beam is concentrated to this ray, and vanishes at some distance from it. The region of the integration $\mathcal{D} = \mathcal{D}(\gamma_1, \gamma_2)$ in (16) represents the region in the ray-parameter domain, for which the value of the integrand on the target surface Σ^T is greater than a prescribed treshold.

Let us rewrite the integral (16) into the form convenient for further transformations.. We denote

$$|\det \mathbf{Q}^{(q)}(P)| \{-\det[\mathbf{M}^{(q)}(P) - \mathbf{M}(P)]\}^{1/2} = |\det \mathbf{Q}^{(q)}(P)|^{1/2} [-\det \boldsymbol{\mathcal{N}}(P)]^{1/2} \ , \tag{19}$$

where the 2×2 complex-valued matrix $\boldsymbol{\mathcal{N}}(P)$ is given by the relation

$$\boldsymbol{\mathcal{N}}(P) = \mathbf{P}^{(q)}(P) - \mathbf{M}(P)\mathbf{Q}^{(q)}(P) \ . \tag{20}$$

The argument of $[-\det \boldsymbol{\mathcal{N}}(P)]^{1/2}$ is chosen so that it satisfies the relation

$$\mathrm{Re}[-\det \boldsymbol{\mathcal{N}}(P)]^{1/2} > 0 \qquad \text{for} \ \ \mathrm{Im}\boldsymbol{\mathcal{N}}(P) \neq 0 \ . \tag{21}$$

The term $|\det \mathbf{Q}^{(q)}(P)|^{1/2}$ on the right-hand side of (19) cancels in the integrand of (16) with the corresponding term in (14) so that it is possible to write the integral superposition of Gaussian beams for the Green-function wavefield in ray-centred coordinates in the following form:

$$\begin{aligned} u_{ij}^{B}(R,\omega) &= (\omega/2\pi) \iint\limits_{\mathcal{D}} \mathrm{d}\gamma_1 \mathrm{d}\gamma_2 \overline{A}(P) g_i(P) g_j(S) |\det \mathbf{P}^{(q)}(S)|^{1/2} [-\det \boldsymbol{\mathcal{N}}(P)]^{1/2} \\ &\times \exp[\mathrm{i}\omega\theta(R,P)] \ . \end{aligned} \tag{22}$$

Here $\overline{A}(P)$ is given in (15), $\boldsymbol{\mathcal{N}}(P)$ in (20) and $\theta(R,P)$ in (17). For $\mathbf{P}^{(q)}(S)$, see the text before equation (13). The reference points P with Cartesian coordinates x_m^P are situated at intersections of rays, specified by the ray parameters γ_1, γ_2, with the target surface Σ^T.

3.2 Evaluation of the Gaussian beam integral superposition of Green function from the results of the reduced DRT

In this section, we derive formulae for the evaluation of quantities appearing in the integral superposition of Gaussian beams from the results of the reduced dynamic ray tracing in Cartesian coordinates, see Section 2.1.3 and equations (B-1) with (B-5) and (B-7) with (B-8). The reduced dynamic ray tracing provides elements $Q_{iN}^{(x)} = \partial x_i/\partial\gamma_N$ and $P_{iN}^{(x)} = \partial p_i/\partial\gamma_N$ of paraxial matrices $\hat{\mathbf{Q}}^{(x)}$ and $\hat{\mathbf{P}}^{(x)}$ along the ray specified by ray parameters γ_1, γ_2. The 3×2 matrices with elements $Q_{iN}^{(x)}$ and $P_{iN}^{(x)}$ are sufficient for our purpose; the complete DRT system (B-1) for 3×3 matrices $\hat{\mathbf{Q}}$ and $\hat{\mathbf{P}}$ with elements $Q_{ij}^{(x)} = \partial x_i/\partial\gamma_j$ and $P_{ij}^{(x)} = \partial p_i/\partial\gamma_j$, respectively, is not required. The transformation of the reduced DRT at a structural interface is given by equations (B-12)-(B-14).

If we wish to evaluate the integral superposition (22) using the solution of the reduced dynamic ray tracing in Cartesian coordinates, we need to express the matrices $\mathbf{Q}^{(q)}$, $\mathbf{P}^{(q)}$ and $\boldsymbol{\mathcal{N}}$ in terms of the solution of the reduced DRT. In the following, we use equations derived by Klimeš (1994). In contrast to Klimeš (1994) we use only elements $Q_{iJ}^{(x)}$ and $P_{iJ}^{(x)}$ of 3×3 matrices $\hat{\mathbf{Q}}^{(x)}$ and $\hat{\mathbf{P}}^{(x)}$. From $Q_{iK}^{(q)} = \partial q_i/\partial\gamma_K$ we simply get

$$Q_{iK}^{(q)} = \overline{H}_{ij} Q_{jK}^{(x)} \ , \tag{23}$$

where $\overline{H}_{ij}$ are elements of the transformation matrix $\hat{\overline{\mathbf{H}}}$, given in equation (5). The derivation of transformation relation for $P_{IK}^{(q)} = \partial p_I^{(q)}/\partial\gamma_K$ is slightly more complicated, and yields:

$$P_{IK}^{(q)} = H_{mI} P_{mK}^{(x)} + F_{Ij} \overline{H}_{jK} \ . \tag{24}$$

Here H_{iJ} are elements of the transformation matrix $\hat{\mathbf{H}}$, given in equation (5). The elements F_{mn} of the matrix $\hat{\mathbf{F}}$ have the form:

$$F_{mn} = p_k \frac{\partial^2 x_k}{\partial q_m \partial q_n} \ . \tag{25}$$

Klimeš (1994, eq. 36) proved that

$$F_{MN} = 0 \ , \qquad F_{I3} = -H_{iI}\eta_i \ . \tag{26}$$

Inserting (26) into (24) yields the final transformation relation between $P^{(q)}_{IJ}$ obtained from the DRT in ray-centred coordinates and $P^{(x)}_{iJ}$ and $Q^{(x)}_{iJ}$ obtained from the reduced DRT in Cartesian coordinates:

$$P^{(q)}_{IK} = H_{mI}(P^{(x)}_{mK} - \eta_m p_j Q^{(x)}_{jK}) \ . \tag{27}$$

Use of equations (23) and (27) in equation (20), makes possible to express $\mathcal{N}_{IJ}$ in terms of quantities obtained from the reduced DRT in Cartesian coordinates:

$$\mathcal{N}_{IJ} = H_{mI}(P^{(x)}_{mJ} - \eta_m p_j Q^{(x)}_{jJ}) - M_{IL}\overline{H}_{Lj}Q^{(x)}_{jJ} \ . \tag{28}$$

It is important to emphasize that the 2×2 matrix $\mathbf{M}$ of parameters of Gaussian beams in equation (28) is considered to be specified at the same point of the ray as the other quantities in equation (28), see the discussion below.

3.3 Gaussian beam integral superposition of Green function in Cartesian coordinates

Results of the preceding section can be used in the transformation of the integral superposition formula (22) to Cartesian coordinates. We take into account that for the 2×2 matrix $\mathbf{P}^{(q)}(S)$ required in equation (22), equation (27) yields:

$$P^{(q)}_{IK}(S) = H_{mI}(S)P^{(x)}_{mK}(S) \tag{29}$$

because $Q^{(x)}_{iJ}(S) = 0$ at a point source. Inserting equations (28) and (29) into (22), we arrive at the final expression for the Gaussian beam integral superposition of the Green function in Cartesian coordinates:

$$\begin{aligned} u^B_{ij}(R,\omega) \ &= \ \omega/2\pi \iint_{\mathcal{D}} \mathrm{d}\gamma_1 \mathrm{d}\gamma_2 \overline{A}(P) g_i(P) g_j(S) |\det[H_{mI}(S)P^{(x)}_{mK}(S)]|^{1/2} \\ &\times \ [-\det[H_{mI}(P)\Big(P^{(x)}_{mJ}(P) - \eta_m(P)p_j(P)Q^{(x)}_{jJ}(P)\Big) - M_{IL}(P)\bar{H}_{Lj}(P)Q^{(x)}_{jJ}(P)]^{1/2} \\ &\times \ \exp[\mathrm{i}\omega\theta(R,P)] \ . \end{aligned} \tag{30}$$

All the quantities in (30) can be obtained from ray tracing and reduced dynamic ray tracing in Cartesian coordinates. $\overline{A}(P)$ is given in (15), $\theta(R,P)$ in (17). The Gaussian beam integral superposition of Green function (30) is regular everywhere, including caustics.

Let us briefly discuss the 2×2 matrix of Gaussian beam parameters $\mathbf{M}$, which appears in equation (30) and also enters it through the matrix $\hat{\boldsymbol{\mathcal{M}}}(P)$, see equation (18). The matrix $\mathbf{M}$ is specified at the reference points P. In principle, it could be specified at any point of the ray Ω. A typical option is, for example, to specify it at the initial point S of the ray Ω. Then, however, an additional solution of the reduced DRT in Cartesian coordinates, which allows transformation of $\mathbf{M}(S)$ into $\mathbf{M}(P)$ appearing in the integral superposition, is necessary.

4 Concluding remarks

Here we present several additional remarks and comments to the presented equations for the Gaussian beam integral superposition of Green function in inhomogeneous anisotropic layered structures in Cartesian coordinates.

Integral superposition proposed in this paper has the following advantages:

1) All computations, including dynamic ray tracing, are performed in Cartesian coordinates. Similarly, the position of the initial point S of the ray (a point source), of the receiver point R and of the points P of the intersection of rays with the target surface Σ^T are specified in Cartesian coordinates.

2) It is sufficient to use only the reduced version of the dynamic ray tracing described in Section 2.1.3, and compute only 3×2 columns $Q^{(x)}_{iJ}(\tau)$ and $P^{(x)}_{iJ}(\tau)$ of the 3×3 matrices $\hat{\mathbf{Q}}^{(x)}(\tau)$ and $\hat{\mathbf{P}}^{(x)}(\tau)$. Consequently, it is sufficient to solve only 12 DRT equations instead of 18 DRT equations. Elements $Q^{(x)}_{i3}(\tau)$ and $P^{(x)}_{i3}(\tau)$ of paraxial matrices $\hat{\mathbf{Q}}^{(x)}(\tau)$ and $\hat{\mathbf{P}}^{(x)}(\tau)$ are not needed at all. Moreover, due to the choice of the travel time τ as a variable parameter along rays, the elements $Q^{(x)}_{i3}(\tau)$ and $P^{(x)}_{i3}(\tau)$ can be determined directly from the ray tracing.

3) There is no need for the two point ray tracing. Only an orthonomic system of rays which intersect the target surface Σ^T should be computed.

4) Equation (30) is applicable to models with structural interfaces, with which the wavefield interacts on the way from the source to the target surface Σ^T. It is only necessary to take into account the transformation relations for ray tracing and dynamic ray tracing at interfaces and consider the complete energy reflection/transmission coefficients when evaluating amplitudes.

5) Equation (30) is applicable to any direct, reflected, multiply reflected, converted, or multiply converted elementary wave propagating in inhomogeneous anisotropic or isotropic media.

6) Equation (30) could be modified to be applicable for the integral superposition of coupled shear waves. The concept of coupled shear waves removes the problems with the collapse of standard ray computations in the vicinities of shear-wave singularities. This application will be described in a forthcoming paper.

7) The target surface Σ^T used in the computation may be chosen in different ways. It may be chosen as a convenient geometrical surface, but it may also be taken along the Earth's surface or any structural interface. It may differ for each receiver and for each elementary wave. It is, however, also possible to compute the complete wave field, consisting of several elementary waves and recorded at several receivers, using the same target surface Σ^T.

8) The receiver R, at which we wish to use equation (30) to evaluate the wavefield, may be situated either at an arbitrary point of the target surface Σ^T or close to it.

9) The integral superposition for the Green function (30) represents a basis for comput-

ing the wavefield generated by various types of point sources, e.g. explosive, single-force or moment-tensor point sources.

10) As emphasized by Červený (2001; Sec. 5.8.4), when using the Gaussian beam integral superposition of Green function, one must take into account that the quality of the superposition in the singular regions of the ray method depends on the choice of the matrix $\mathbf{M}$ of Gaussian beam parameters. The accuracy of the formula in caustic or critical regions can be enhanced by the use of broad beams. On the contrary, for the calculation of edge diffractions, narrow Gaussian beams are required.

The computation of all quantities included in the integral superposition of Gaussian beams is described in the paper in detail. The exceptions are the formulae for the normalized energy reflection/transmission coefficients, which can be found in Červený (2001; Sec. 5.4.7) and for the phase shift due to caustics, which can be found in Klimeš (2014b).

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Appendix A

Ray-tracing in inhomogeneous anisotropic media with interfaces

Let us introduce the 3×3 *generalized Christoffel matrix* $\hat{\mathbf{\Gamma}}(x_m, p_n)$ with components Γ_{ik} given by the relation:

$$\Gamma_{ik} = a_{ijkl} p_j p_l = \rho^{-1} c_{ijkl} p_j p_l \ . \qquad (A-1)$$

Here $c_{ijkl} = c_{ijkl}(x_n)$ are real-valued elastic moduli (components of the fourth-order stiffness tensor), $\rho(x_m)$ is the density. The symbol a_{ijkl} denotes the density-normalized elastic moduli, $a_{ijkl} = c_{ijkl}/\rho$. The quantities $p_i = \partial T / \partial x_i$ are the components of the slowness vector, where $T(x_n)$ is the travel time. The generalized Christoffel matrix is symmetric and positive definite. It has three positive eigenvalues $G(x_i, p_j)$, which are in this paper considered to be mutually different, and three corresponding eigenvectors $\mathbf{g}(x_i, p_j)$. They correspond to the three elementary waves, P, S1 and S2, propagating in inhomogeneous anisotropic media. The eigenvalues $G(x_i, p_j)$ and the eigenvectors $\mathbf{g}(x_i, p_j)$ are the solutions of the Christoffel equation

$$(\Gamma_{ik} - G\delta_{ik}) g_k = 0 \qquad (A-2)$$

and of its characteristic equation (condition of solvability of (A-2)):

$$\det(\Gamma_{ik} - G\delta_{ik}) = 0 \ . \qquad (A-3)$$

The eigenvalues $G(x_i, p_j)$ of the generalized Christoffel matrix $\hat{\mathbf{\Gamma}}$ represent a basic element in the *eikonal equations* $G(x_i, p_j) = 1$ controlling kinematics of each of the three waves. In the *Hamiltonian form* the eikonal equation reads:

$$\mathcal{H}(x_i, p_j) = \tfrac{1}{2} G(x_i, p_j) = \tfrac{1}{2} \ . \qquad (A-4)$$

The *Hamiltonian* $\mathcal{H}(x_i, p_j)$ in (A-4) represents a homogeneous function of the second degree in p_i. The eikonal equation (A-4) is a nonlinear partial differential equation of the first order for the travel time $T = T(x_i)$. It can be solved in terms of characteristics. The equations of characteristics, *ray-tracing equations*, have the form:

$$\frac{\mathrm{d}x_i}{\mathrm{d}u} = \frac{\partial \mathcal{H}}{\partial p_i} \ , \quad \frac{\mathrm{d}p_i}{\mathrm{d}u} = -\frac{\partial \mathcal{H}}{\partial x_i} \ , \quad \frac{\mathrm{d}T}{\mathrm{d}u} = p_i \frac{\partial \mathcal{H}}{\partial p_i} \ . \qquad (A-5)$$

Here u is a parameter along the characteristics (ray). The parameter u can be chosen in various ways as, for example, travel time, arclength, etc. The choice of the Hamiltonian function $\mathcal{H}(x_i, p_j)$ in the form (A-4), i.e., as a homogeneous function of the second degree in p_i, and use of the the Euler's theorem lead to:

$$p_i \frac{\partial \mathcal{H}}{\partial p_i} = 2\mathcal{H} = 1 \ . \qquad (A-6)$$

As a consequence of (A-6), the third equation in (A-5) yields $dT/du = 1$, i.e., the parameter u along the ray is travel time. We denote this parameter τ. We emphasize that τ represents the travel time T only along the ray Ω, not in its vicinity.

A1) **Ray tracing system**

As shown above, it is not necessary to calculate T by solving the last equation in (A - 5). The travel time $T = \tau$ is automatically obtained by solving the ray-tracing system, which consists of six nonlinear ordinary differential equations of the first order:

$$\frac{\mathrm{d}x_i}{\mathrm{d}\tau} = \frac{\partial \mathcal{H}}{\partial p_i} \ , \quad \frac{\mathrm{d}p_i}{\mathrm{d}\tau} = -\frac{\partial \mathcal{H}}{\partial x_i} \ . \qquad (A-7)$$

The partial derivatives of the Hamiltonian in (A-7) can be expressed in alternative forms. The most straightforward is

$$\frac{\partial \mathcal{H}}{\partial p_i} = \mathcal{U}_i = a_{ijkl} p_l g_j g_k \ , \quad -\frac{\partial \mathcal{H}}{\partial x_i} = \eta_i = -\frac{1}{2} \frac{\partial a_{jkln}}{\partial x_i} p_k p_n g_j g_l \ . \qquad (A-8)$$

The symbols $\mathcal{U}_i$ and η_i denote Cartesian components of the ray-velocity and eta vectors, respectively. With τ chosen as a parameter along a ray, the components $\mathcal{U}_i$ and η_i constitute third columns of paraxial matrices $\hat{\mathbf{Q}}^{(x)}$ and $\hat{\mathbf{P}}^{(x)}$, see Appendix B.

An alternative form of the partial derivatives of the Hamiltonian does not contain the eigenvectors $\mathbf{g}$ explicitly. It reads:

$$\frac{\partial \mathcal{H}}{\partial p_i} = \mathcal{U}_i = a_{ijkl} p_l D_{jk}/D_{mm} \ , \quad -\frac{\partial \mathcal{H}}{\partial x_i} = \eta_i = -\frac{1}{2} \frac{\partial a_{jkln}}{\partial x_i} p_k p_n D_{jl}/D_{mm} \ . \qquad (A-9)$$

Here

$$D_{ij} = \tfrac{1}{2} \epsilon_{ikl} \epsilon_{jrs} (\Gamma_{kr} - \delta_{kr})(\Gamma_{ls} - \delta_{ls}) \ . \qquad (A-10)$$

A2) **Point-source initial conditions for rays**

To solve the ray-tracing system (A-7), the initial values $\mathbf{x}(\tau_0)$, $\mathbf{p}(\tau_0)$ for the initial travel time $\tau = \tau_0$ must be specified. We denote the initial point with coordinates $\mathbf{x}(\tau_0)$ by S, i.e., $\mathbf{x}(S) = \mathbf{x}(\tau_0)$. It remains to specify $\mathbf{p}(S)$. It is common to specify $\mathbf{p}(S)$ by the ray parameters γ_1, γ_2, i.e., by parameters determining one specific ray. At the point S, the ray parameters γ_1, γ_2 are usually taken as the take-off angles φ_0, δ_0. The initial conditions for $p_i(S)$ then read:

$$p_i(S) = \mathcal{C}^{-1}(S; \varphi_0, \delta_0) N_i(S) \; . \qquad (A-11)$$

Here $\mathbf{N}(S)$ is a unit vector at the point S specified by take-off angles φ_0 and δ_0 in the following way:

$$\mathbf{N}(S) \equiv (\cos\varphi_0 \cos\delta_0,\ \sin\varphi_0 \cos\delta_0,\ \sin\delta_0) \; . \qquad (A-12)$$

The symbol $\mathcal{C}(S; \varphi_0, \delta_0)$ denotes the phase velocity corresponding to the direction $\mathbf{N}(S)$,

$$\mathcal{C}(S; \varphi_0, \delta_0) = [G(x_i(S), N_j(S))]^{1/2} \; , \qquad (A-13)$$

where $G(x_i, N_j)$ is one of the three eigenvalues of the Christoffel matrix $\hat{\mathbf{\Gamma}}(x_i, p_j)$ defined in (A-1), with p_j replaced by N_j. It is important to emphasize that the take-off angles φ_0, δ_0 do not specify the initial direction of the ray, but the initial direction of the slowness vector $\mathbf{p}(S)$. The initial direction of the ray is given by the ray-velocity vector $\boldsymbol{\mathcal{U}}(S)$. The ray-velocity vector $\boldsymbol{\mathcal{U}}(S)$ may be simply calculated from $\mathbf{p}(S)$ using the first set of equations in (A-8) or (A-9). Determination of $\mathbf{p}(S)$ from $\boldsymbol{\mathcal{U}}(S)$ is, however, a more complicated task, for which no exact explicit formulae exist.

A3) **Rays across structural interfaces**

In the framework of the zero-order approximation of the ray method, the problem of reflection/transmission of a wave at a curved interface $\Sigma^{Int}(x_m) = 0$ separating two inhomogeneous media reduces to the problem of incidence of a plane wave at a plane interface separating two homogeneous media. The wavefronts of the incident and generated waves are approximated by plane wavefronts tangent to the actual wavefronts at the point of incidence. The interface Σ^{Int} is approximated by a plane tangent to Σ^{Int} at the point of incidence. Finally, the media on both sides of the plane interface are assumed to be homogeneous with properties corresponding to the properties at the point of incidence.

Let us denote the normal to the interface Σ^{Int} at the point of incidence by $\mathbf{n}$, $n_i = \pm|\Sigma^{Int}_{,i}|/(\Sigma^{Int}_{,k}\Sigma^{Int}_{,k})^{1/2}$. We choose the sign, for which the normal $\mathbf{n}$ points to the medium, in which the incident wave propagates. The vectorial component $\mathbf{p}^t$ of the slowness vector of the incident wave tangent to the interface then reads:

$$\mathbf{p}^t = \mathbf{p} - (\mathbf{p} \cdot \mathbf{n})\mathbf{n} \; . \qquad (A-14)$$

The vectorial component $\mathbf{p}^t$ is the same for the incident and all generated waves (this is an expression of the Snell's law). It is necessary to find the vectorial components of slowness vectors to the normal to the interface for all generated waves. This can be done by expressing the slowness vector of any of the generated waves as

$$\mathbf{p}^g = \sigma\mathbf{n} + \mathbf{p}^t \; , \qquad (A-15)$$

where σ is a scalar factor to be determined. The factor σ can be determined by solving the condition of solvability (A-3) of the Christoffel equation (A-2):

$$\det[a_{ijkl}(\sigma n_j + p_j^t)(\sigma n_l + p_l^t) - \delta_{ik}] = 0 \ . \qquad (A-16)$$

Equation (A-16) is an algebraic equation of the sixth degree, and thus, it has six roots. These roots must be sought in each halfspace. Criterion for the selection of the real-valued roots is the request that the ray-velocity vector $\boldsymbol{\mathcal{U}}$ points to the medium, in which is the generated wave supposed to propagate. From complex-valued roots, only those are selected, which guarantee the existence of an inhomogeneous wave whose amplitude decays with the distance from the interface Σ^{Int}. For more details, see Gajewski and Pšenčík (1987) or Červený (2001, Sec. 2.3.3).

Appendix B
Dynamic ray tracing in inhomogeneous anisotropic media

Dynamic ray tracing consists in the solution of a system of linear ordinary differential equations of the first order along the ray Ω. The dynamic ray tracing system in Cartesian coordinates was studied by several authors. For a detailed derivations and for references see Červený (2001; Chap. 4).

B1) **DRT system in Cartesian coordinates** x_i.

The DRT in Cartesian coordinates consists of 18 linear differential equations for $Q_{ik}^{(x)} = \partial x_i/\partial\gamma_k$ and $P_{ik}^{(x)} = \partial p_i/\partial\gamma_k$, where γ_k are ray coordinates. The DRT system reads:

$$\mathrm{d}Q_{ik}^{(x)}/\mathrm{d}\tau = A_{ij}^{(x)}Q_{jk}^{(x)} + B_{ij}^{(x)}P_{jk}^{(x)} \ , \qquad \mathrm{d}P_{ik}^{(x)}/\mathrm{d}\tau = -C_{ij}^{(x)}Q_{jk}^{(x)} - D_{ij}^{(x)}P_{jk}^{(x)} \ , \qquad (B-1)$$

where

$$A_{ij}^{(x)} = \partial^2\mathcal{H}/\partial p_i\partial x_j \ , \qquad B_{ij}^{(x)} = \partial^2\mathcal{H}/\partial p_i\partial p_j \ ,$$
$$C_{ij}^{(x)} = \partial^2\mathcal{H}/\partial x_i\partial x_j \ , \qquad D_{ij}^{(x)} = \partial^2\mathcal{H}/\partial x_i\partial p_j \ , \qquad (B-2)$$

and where the Hamiltonian $\mathcal{H}(x_i, p_j)$ is given by (A-4). The elements of 3×3 matrices $A_{ij}^{(x)}$, $B_{ij}^{(x)}$, $C_{ij}^{(x)}$ and $D_{ij}^{(x)}$ satisfy three symmetry relations:

$$B_{ij}^{(x)} = B_{ji}^{(x)} \ , \qquad C_{ij}^{(x)} = C_{ji}^{(x)} \ , \qquad D_{ij}^{(x)} = A_{ji}^{(x)} \ . \qquad (B-3)$$

The 3×3 matrices $\hat{\mathbf{Q}}^{(x)}$ and $\hat{\mathbf{P}}^{(x)}$ in (B-1) satisfy the following constraint relation:

$$\mathcal{U}_i P_{ik}^{(x)} - \eta_i Q_{ik}^{(x)} = 0 \qquad (B-4)$$

along the ray Ω. The symbols $\mathcal{U}_i$ and η_i denote components of the ray-velocity vector $\boldsymbol{\mathcal{U}}$ and eta vector $\boldsymbol{\eta}$, respectively.

As shown in the main text, for the evaluation of the integral superposition (22) it is not necessary to solve the complete DRT system (B-1). It is sufficient to solve the reduced DRT system for just six elements of the matrices $\mathbf{Q}^{(x)}$ and $\mathbf{P}^{(x)}$:

$$Q_{iN}^{(x)} = \partial x_i/\partial\gamma_N \ , \qquad P_{iN}^{(x)} = \partial p_i/\partial\gamma_N \ . \qquad (B-5)$$

Here γ_1 and γ_2 are the ray parameters. For the choice of τ as the monotonous parameter along the ray, the elements $Q^{(x)}_{i3}$ and $P^{(x)}_{i3}$ can be obtained from the ray tracing. The reduced DRT consists of only 12 equations.

B2) **Point-source initial conditions for the DRT**. Let us consider the ray parameters γ_1 and γ_2 to be the take-off angles $\gamma_1 = \varphi_0$ and $\gamma_2 = \delta_0$, for which the unit vector $\mathbf{N}(S)$, specifying the direction of the slowness vector at the point S, has the form (A-12). As before, we choose $\gamma_3 = \tau$.

The initial conditions for $\hat{\mathbf{Q}}^{(x)}$ are straightforward:

$$Q^{(x)}_{iJ}(S) = 0 \qquad (B-6)$$

for the first two columns, and

$$Q^{(x)}_{i3}(S) = \mathcal{U}_i(S) \qquad (B-7)$$

for the last one.

For the determination of the initial conditions for $P^{(x)}_{ij}$, we have to differentiate the slowness vector at the point S with respect to γ_1 and γ_2. For the first two columns we get

$$P^{(x)}_{iJ}(S) = [Z_{iJ}(S) - p_i(S)\mathcal{U}_k(S)Z_{kJ}(S)]/\mathcal{C}(S;\varphi_0,\delta_0) \ , \qquad (B-8)$$

where $Z_{iJ}(S) = \partial N_i(S)/\partial\gamma_J$. For Z_{iJ} we get:

$$\begin{aligned} &Z_{11}(S) = -\sin\varphi_0\cos\delta_0 \ , \quad Z_{21}(S) = \cos\varphi_0\cos\delta_0 \ , \qquad Z_{31}(S) = 0 \ , \\ &Z_{12}(S) = -\cos\varphi_0\sin\delta_0 \ , \quad Z_{22}(S) = -\sin\varphi_0\sin\delta_0 \ , \quad Z_{32}(S) = \cos\delta_0 \ . \end{aligned} \qquad (B-9)$$

See Pšenčík and Teles (1996, eq.A-4) for the derivation and more details. For $P^{(x)}_{i3} = \partial p_i/\partial\tau$, we have

$$P^{(x)}_{i3}(S) = \eta_i(S) \ . \qquad (B-10)$$

With the initial conditions (B-6)-(B-10), the elements of paraxial matrices $\hat{\mathbf{Q}}^{(x)}$ and $\hat{\mathbf{P}}^{(x)}$ along the ray Ω have the meaning of partial derivatives of spatial coordinates and slowness vectors with respect to γ_i. Let us note that for $\gamma_3 = \tau$, the quantities Q_{i3} and P_{i3} are available from the ray tracing equations (A-7). Thus, it is not necessary to seek them as the solution of (B-1). The quantities Q_{i3} and P_{i3} are not needed at all in the reduced DRT. For the reduced DRT, only equations (B-6) and (B-8) with (B-9) are necessary.

If we choose the ray parameters γ_1 and γ_2 as follows: $\gamma_1 = p_1(S)$, $\gamma_2 = p_2(S)$, where $p_I(S)$ are components of the slowness vector at the point S, and $\gamma_3 = \tau$, the initial conditions for $Q^{(x)}_{iJ}$ are again as in (B-5). For $P^{(x)}_{iJ}$, we get:

$$P^{(x)}_{IJ}(S) = \delta_{IJ}, \qquad P^{(x)}_{3J}(S) = -\mathcal{U}_J(S)/\mathcal{U}_3(S) \ . \qquad (B-11)$$

The initial conditions for the elements $P^{(x)}_{i3}(S)$ remain the same as in (B-10). For the reduced DRT, only equations (B-6) and (B-11) are needed. The expressions (B-11) fail, however, for the rays whose initial directions are situated in the plane (x_1, x_2), because $\mathcal{U}_3(S) = 0$ in this case. It is then necessary to rotate the Cartesian coordinate system correspondingly.

B3) **Transformation of DRT in Cartesian coordinates across an interface**. We consider a smooth structural interface $\Sigma^{Int}(x_i) = 0$ described in Appendix A3. We use the symbols $Q^{(x)}_{kJ}$ and $P^{(x)}_{kJ}$ for the elements of the paraxial matrices of the incident wave, and denote $Q^{(x)G}_{kJ}$ and $P^{(x)G}_{kJ}$ the elements of the paraxial matrices of any generated (reflected, transmitted) wave. The relations between the elements of generated and incident waves are given by the following relations:

$$Q^{(x)G}_{iJ} = W_{ik} Q^{(x)}_{kJ} \ , \qquad P^{(x)G}_{iJ} = R_{ik} Q^{(x)}_{kJ} + S_{ik} P^{(x)}_{kJ} \ . \qquad (B-12)$$

Here the 3×3 matrices $\hat{\mathbf{W}}, \hat{\mathbf{R}}, \hat{\mathbf{S}}$ are given by relations:

$$W_{ij} = \delta_{ij} + \eta^{-1}(Q^{(x)G}_{i3} - Q^{(x)}_{i3}) n_j \ , \qquad S_{ij} = \delta_{ij} - (\eta^G)^{-1}(Q^{(x)G}_{j3} - Q^{(x)}_{j3}) n_i \ ,$$

$$R_{ij} = \eta^{-1}(P^{(x)G}_{i3} - P^{(x)}_{i3}) n_j + (\eta^G)^{-1}(P^{(x)G}_{j3} - P^{(x)}_{j3}) n_i$$
$$+ (\eta\eta^G)^{-1}(Q^{(x)G}_{k3} P^{(x)}_{k3} - Q_{k3} P^{(x)G}_{k3}) n_i n_j$$
$$+ (\Sigma^{Int}_{,m} \Sigma^{Int}_{,m})^{-1/2} (\zeta^G - \zeta) \Sigma^{Int}_{,lk} [\delta_{il} - (\eta^G)^{-1} Q^{(x)G}_{l3} n_i][\delta_{kj} - \eta^{-1} Q^{(x)}_{k3} n_j] \ . \qquad (B-13)$$

In (B-13), $\mathbf{n}$ is the unit normal to the interface at the point of incidence, $\Sigma^{Int}_{,i}$ and $\Sigma^{Int}_{,ij}$ are spatial derivatives of $\Sigma^{Int} = \Sigma^{Int}(x_m)$, and

$$\zeta^G = p^G_m n_m \ , \quad \zeta = p_m n_m \ , \quad \eta^G = Q^{(x)G}_{m3} n_m \ , \quad \eta = Q^{(x)}_{m3} n_m \ . \qquad (B-14)$$

For the derivation and more details see Farra and LeBégat (1995) or Pšenčík and Farra (2014).

The values of the elements $Q^{(x)G}_{i3}$ and $P^{(x)G}_{i3}$ are available from the ray-tracing equations (A-7) specified for the corresponding generated wave.

References

Aki, K., and P.G. Richards, 1980. *Quantitative seismology. Theory and methods.* W.H. Freeman: S. Francisco.

Alkhalifah, T., 1995. Gaussian beam depth migration for anisotropic media. *Geophysics*, **60**, 1474 - 1484.

Babich, V,M,, and Popov, M.M., 1981. Propagation of concentrated acoustical beams in three-dimensional inhomogeneous media (in Russian). *Akust. Zh.*, **27**, 828 - 835.

Bakker, P.M., 1998. Phase shift at caustics along rays in anisotropic media *Geophys. J. Int.*, **134**, 515-518.

Bleistein, N., 2007. Mathematical modeling, migration and inversion with Gaussian beams. Lecture Notes. Center for Wave Phenomena, Colorado School of Mines, Golden, 118 p.

Bleistein, N., and Gray, S.H., 2010. Amplitude calculation for 3-D Gaussian beam migration using complex-valued traveltimes. *Inverse Problems*, **26**, Article Number 085017.

Červený, V., 1972. Seismic rays and ray intensities in inhomogeneous anisotropic media. *Geophys. J. R. astr. Soc.*, **29**, 1-13.

Červený, V., 1985. Gaussian beam synthetic seismograms. *J. Geophys.*, **58**, 44 - 72.

Červený, V., 2000. Summation of paraxial Gaussian beams and of paraxial ray approximations in inhomogeneous anisotropic layered structures. In: *Seismic waves in Complex 3-D Structures*, Report 10, 121–159. Charles Univ. in Prague, Faculty of Mathematics and Physics, Department of Geophysics Praha, online at "http: //sw3d.cz".

Červený, V., 2001. *Seismic ray theory.* Cambridge: Cambridge Univ. Press.

Červený, V., and Klimeš, L., 2010. Transformation relations for second order derivatives of travel time in anisotropic media. *Stud. Geophys. Geod.*, **54**, 257–267.

Červený, V., and Pšenčík, I., 1983. Gaussian beams and paraxial ray approximation in three-dimensional elastic inhomogeneous media. *J. Geophys.*, **53**, 1–15.

Červený, V., and Pšenčík, I., 2010. Gaussian beams in inhomogeneous anisotropic layered structures. *Geophys. J. Int.*, **180**, 798–812.

Červený, V., Klimeš, L., and Pšenčík, I., 2007. Seismic ray method: Recent developments. *Advances in Geophysics*, **48**, 1–126.

Červený, V., Popov, M.M., and Pšenčík, I., 1982. Computation of wave fields in inhomogeneous media. Gaussian beam approach. *Geophys. J.R. astr. Soc.*, **88**, 43–79.

Chapman, C. H., 2004. *Fundamentals of seismic wave propagation.* Cambridge: Cambridge Univ. Press.

Chapman, C. H., and Drummond, R., 1982. Body-wave seismograms in inhomogeneous media using Maslov asymptotic theory. *Bull. seism. Soc. Am.*, **72**, S277–S317.

Farra, V., and LeBégat, S., 1995. Sensitivity of qP-wave traveltimes and polarisation vectors to heterogeneity, anisotropy and interfaces. *Geophys. J. Int.*, **121**, 371–384.

Gajewski, D., and Pšenčík, I., 1987. Computation of high-frequency seismic wavefields in 3-D laterally inhomogeneous anisotropic media. *Geophys. J. R. astr. Soc.*, **91**, 383 - 411.

Gajewski, D., and Pšenčík, I., 1990. Vertical seismic profile synthetics by dynamic ray tracing in laterally varying layered anisotropic structures. *J. Geophys. Res.*, **95**, 11301 - 11315.

Garmany, J., 2001. Phase shifts at caustics in anisotropic media. In: *Anisotropy 2000: Fractures, Converted Waves and Case Studies*, eds. L. Ikelle and A. Gangi, pp. 419–425, Soc. Explor. Geophysicists, Tulsa.

George, T., Virieux,J., and Madariaga, R., 1987. Seismic wave synthesis by Gaussian beam summation: A comparison with finite differences. *Geophysics*, **52**, 1065 - 1073.

Gray, S., 2005. Gaussian beam migration of common shot records. *Geophysics*, **70**, S71-S77.

Gray, S.H., and Bleistein, N., 2009. True-amplitude Gaussian-beam migration. *Geophysics*, **74**, S11-S23.

Hanyga, A., 1986. Gaussian beams in anisotropic elastic media. *Geophys. J. R. astr. Soc.*, **85**, 473 - 503.

Hill, N., 1990. Gaussian beam migration. *Geophysics*, **55**, 1416 - 1428.

Hill, N.R., 2001. Prestack Gaussian beam depth migration. *Geophysics*, **66**, 1240 - 1250.

Kendall, J.-M., Guest, W., and Thomson, C., 1995. Ray theory Green's function reciprocity and ray-centered coordinates in anisotropic media. *Geophys. J. Int.*, **108**, 364–371.

Klimeš, L., 1984a. Expansion of high-frequency time harmonic wavefield given on an initial surface into Gaussian beams. *Geophys. J.R. astr. Soc.*, **79**, 105 - 118.

Klimeš, L., 1984b. The relation between Gaussian beams and Maslov asymptotic theory. *Stud. Geophys. Geod.* **28**, 237–247.

Klimeš, L., 1994. Transformations for dynamic ray tracing in anisotropic media. *Wave Motion*, **20**, 261 - 272.

Klimeš, L., 1997. Phase shift of the Green function due to caustics in anisotropic media, Expanded Abstracts of 1997 SEG meeting, Dallas, 1834-1837.

Klimeš, L., 2010. Phase shift of the Green function due to caustics in anisotropic media. *Stud. geophys. geod.*, **54**, 268–289.

Klimeš, L., 2012. Zero-order ray-theory Green tensor in a heterogeneous anisotropic elastic medium. *Stud. geophys. geod.*, **56**, 373–382.

Klimeš, L., 2014a. Superposition of Gaussian packets in heterogeneous anisotropic media. In: *Seismic waves in complex 3-D structures*, Report **24**, pp. 127–130. Charles University, Faculty of Mathematics and Physics, Dept. of Geophysics, Praha, online at "http://sw3d.cz".

Klimeš, L., 2014b. Phase shift of a general wavefield due to caustics in anisotropic media. In: *Seismic waves in complex 3-D structures*, Report **24**, pp. 95–109. Charles University, Faculty of Mathematics and Physics, Dept. of Geophysics, Praha, online at "http://sw3d.cz".

Klimeš, L., 2015. Superposition of Gaussian beams and column Gaussian packets in heterogeneous anisotropic media. In: *Seismic waves in complex 3-D structures*, Report **25**, pp. 103–108. Charles University, Faculty of Mathematics and Physics, Dept. of Geophysics, Praha, online at "http: sw3d.cz". ISSN 2336–3827.

Kravtsov, Yu.A., and Berczynski, P., 2007. Gaussian beams in inhomogeneous media; A review. *Stud. Geoph. Geod.,*, **51**, 1 - 36.

Leung, S., Qian, J., and Burridge, R., 2007. Eulerian Gaussian beams for high-frequency wave propagation. *Geophysics*, **72**, SM61 - SM76.

Maslov, V.P., 1965. *Theory of perturbations and asymptotic methods.* Moscow: Moscow State Univ. Press (in Russian).

Popov, M.M., 1982. A new method of computation of wave fields using Gaussian beams. *Wave Motion*, **4**, 85 - 97.

Popov, M.M., Semtchenok, N.H., Verdel, A.R., Popov, P.M., 2007. Seismic migration by Gaussian beams. *Doklady Earth Sciences*, **417**, 1236–1239.

Protasov, M.I., 2015. 2-D Gaussian beam imaging of multicomponent seismic data in anisotropic media. *Geophys. J. Int.*, **203**, 2021–2031.

Protasov, M.I., and Tcheverda, V.A., 2012. True amplitude elastic Gaussian beam imaging of multicomponent walkaway vertical seismic profiling data. *Geophysical Prospecting*, **60**, 1030–1042.

Pšenčík, I., and Farra, V., 2014. First-order P-wave ray synthetic seismograms in inhomogeneous, weakly anisotropic, layered media. *Geophys. J. Int.*, **198**, 298–307.

Pšenčík, I. and Teles, T.N., 1996. Point source radiation in inhomogeneous anisotropic structures. *Pure Appl. Geophys.*, **148**, 591-623.

Ralston, J., 1983. Gaussian beams and the propagation of singularities. In *Studies in partial differential equations*, ed. W. Littman, Math. Ass. of Am. Studies in Mathematics, **23**, 206 - 248.

Thomson, C. J., and Chapman, C. H., 1985. An inroduction to Maslov's asymptotic method. *Geophys. J. R. astr. Soc.*, **83**, 143 – 168.

Vinje, V., Roberts, G., and Taylor, R., 2008. Controlled beam migration: a versabile structure imaging tool. First Break, **26**, 109 - 113.

Weber, M., 1988. Computation of body-wave seismograms in absorbing 2-D media using the Gaussian beam method: comparison with exact methods. Geophys. J., **92**, 9–24.

White, B., Norris, A., Bayliss, A., and Burridge, R., 1987. Some remarks on the Gaussian beam summation method. *Geophys. J.R. astr. Soc.*, **89**, 579 - 636.

Zhu, T., Gray, S.H., and Wang, D., 2005. Kinematic and dynamic ray tracing in anisotropic media: Theory and applications. In: *75th Annual Int. Meeting, SEG, Expanded Abstracts*, 96 - 99.

Zhu, T., Gray, S.H., and Wang, D., 2007. Prestack Gaussian-beam depth migration in anisotropic media. *Geophysics*, **72**, S133 - S138.