Two S-wave eigenvectors of the Christoffel matrix need not exist in anisotropic viscoelastic media

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Summary

The $3\times3\times3\times3$ frequency–domain stiffness tensor is complex–valued in viscoelastic media. The 3×3 Christoffel matrix is then also complex–valued. We demonstrate that a complex–valued Christoffel matrix need not have all three eigenvectors at an S–wave singularity, and we thus cannot apply the eigenvectors to calculating the phase–space derivatives of the Hamiltonian function.

Keywords

Attenuation, anisotropy, wave propagation, ray theory, ray tracing.

1. Introduction

Attenuation is a very important phenomenon in wave propagation. The $3\times3\times3\times3$ frequency–domain stiffness tensor (elastic tensor, tensor of elastic moduli) is complex–valued in viscoelastic media. The 3×3 Christoffel matrix is then also complex–valued. The Hamiltonian function and its phase–space derivatives are usually calculated in terms of the eigenvectors of the Christoffel matrix (Klimeš, 2006). In this formulation, we need the S–wave eigenvectors of the Christoffel matrix even for calculating the geodesic deviation of P–wave rays.

In this paper, we demonstrate that a complex–valued Christoffel matrix need not have all three eigenvectors at an S–wave singularity, and we thus cannot apply the eigenvectors to calculating the phase–space derivatives of the Hamiltonian function. We present a simple example of a weakly anisotropic viscoelastic medium with three singular directions in Section 2, and calculate the corresponding Christoffel matrix in Section 3. We then show that the Christoffel matrix has just one S–wave eigenvector in each singular direction.

Seismic Waves in Complex 3–D Structures, **30** (2020), 73–76 (ISSN 2336–3827, online at http://sw3d.cz)

2. Example of a stiffness tensor

Let us start with simple example

$$a_0^{ijkl} = \begin{array}{c} \\ {\scriptstyle 11} \\ {\scriptstyle 22} \\ {\scriptstyle 33} \\ {\scriptstyle 23} \\ {\scriptstyle 13} \\ {\scriptstyle 12} \end{array} \begin{array}{c} \begin{array}{cccccc} {\scriptstyle 11} & {\scriptstyle 22} & {\scriptstyle 33} & {\scriptstyle 23} & {\scriptstyle 13} & {\scriptstyle 12} \end{array} \\ \begin{pmatrix} 3-4\mathrm{i}\delta & 1+2\mathrm{i}\delta & 1+2\mathrm{i}\delta & 0 & 0 & 0 \\ & 3-4\mathrm{i}\delta & 1+2\mathrm{i}\delta & 0 & 0 & 0 \\ & & 3-4\mathrm{i}\delta & 0 & 0 & 0 \\ & & & 1-3\mathrm{i}\delta & 0 & 0 \\ & & & & 1-3\mathrm{i}\delta & 0 \\ & & & & & 1-3\mathrm{i}\delta \end{pmatrix} \end{array} \tag{1}$$

of the complex–valued frequency–domain density–reduced stiffness tensor of an isotropic viscoelastic medium. Since the stiffness tensor is symmetric with respect to the first pair of indices and with respect to the second pair of indices, it is expressed in the form of the 6×6 stiffness matrix which lines correspond to the first pair of indices and columns to the second pair of indices. We assume in this paper that the stiffness tensor is symmetric with respect to the exchange of the first pair of indices and the second pair of indices, i.e., that 6×6 stiffness matrix (1) is symmetric. For the sake of simplicity, we have omitted here a multiplication factor which roughly corresponds to the square of S–wave velocity. Parameter δ is assumed real–valued. Considering here the Fourier transform with sign convention according to Červený (2001, eq. A.1.2), parameter δ is taken positive.

This is a frequent type of an isotropic viscoelastic medium, with the square of P–wave velocity roughly three time greater than the square of S–wave velocity. The medium is elastic at volume deformations (Anderson, Ben Menahem & Archambeau, 1965). The medium is weakly attenuating for small δ.

We now modify stiffness tensor (1) by a small anisotropic perturbation,

$$a^{ijkl} = \begin{array}{c} \\ {\scriptstyle 11} \\ {\scriptstyle 22} \\ {\scriptstyle 33} \\ {\scriptstyle 23} \\ {\scriptstyle 13} \\ {\scriptstyle 12} \end{array} \begin{array}{c} \begin{array}{cccccc} {\scriptstyle 11} & {\scriptstyle 22} & {\scriptstyle 33} & {\scriptstyle 23} & {\scriptstyle 13} & {\scriptstyle 12} \end{array} \\ \begin{pmatrix} 3-4\mathrm{i}\delta & 1+2\mathrm{i}\delta & 1+2\mathrm{i}\delta & 0 & 0 & 0 \\ & 3-4\mathrm{i}\delta & 1+2\mathrm{i}\delta & 0 & 0 & 0 \\ & & 3-4\mathrm{i}\delta & 0 & 0 & 0 \\ & & & 1-3\mathrm{i}\delta+2\epsilon & \mathrm{i}\epsilon & 2\mathrm{i}\epsilon \\ & & & & 1-3\mathrm{i}\delta & \mathrm{i}\epsilon \\ & & & & & 1-3\mathrm{i}\delta-2\epsilon \end{pmatrix} \end{array} , \tag{2}$$

where ϵ is a complex–valued parameter. The upper left-hand 3×3 minor of this matrix corresponds to medium (1) elastic at volume deformations. The lower right-hand 3×3 minor of this matrix has eigenvalues

$$1-3\mathrm{i}\delta-\mathrm{i}\epsilon-\epsilon \quad , \qquad 1-3\mathrm{i}\delta-\mathrm{i}\epsilon+\epsilon \quad , \qquad 1-3\mathrm{i}\delta+2\mathrm{i}\epsilon \quad . \tag{3}$$

Since the energy can be dissipated but cannot be created, the imaginary parts of eigenvalues (3) cannot be positive for the Fourier transform with sign convention according to Červený (2001, eq. A.1.2), and we have three conditions

$$-\mathrm{Re}\,(\epsilon) - \mathrm{Im}(\epsilon) \le 3\,\delta \quad , \qquad -\mathrm{Re}\,(\epsilon) + \mathrm{Im}(\epsilon) \le 3\,\delta \quad , \qquad 2\,\mathrm{Re}\,(\epsilon) \le 3\,\delta \quad . \tag{4}$$

These conditions are equivalent to two conditions

$$|\mathrm{Re}\,(\epsilon)| \le \tfrac{3}{2}\,\delta \quad , \qquad |\mathrm{Im}(\epsilon)| \le 3\,\delta + \mathrm{Re}\,(\epsilon) \quad . \tag{5}$$

Positive real–valued parameter δ is small for usual attenuation. Then $\mathrm{Re}\,(\epsilon)$ and $\mathrm{Im}(\epsilon)$ should also be small. Since ϵ is assumed small rather than great, there is no need to study the stability conditions for the stiffness tensor.

3. Christoffel matrix

The 3×3 Christoffel matrix is defined as

$$\Gamma^{ik} = a^{ijkl} p_j p_l \quad , \tag{6}$$

where p_j are the components of the slowness vector. For the real–valued reference slowness vector, we obtain the complex–valued reference Christoffel matrix.

The Christoffel matrix in isotropic viscoelastic medium (1) reads

$$\Gamma_0^{ik} = \left[(1-3\mathrm{i}\delta)\delta^{ik} p_l p_l + (2-\mathrm{i}\delta) p_i p_k\right] \quad . \tag{7}$$

The Christoffel matrix in medium (2) reads

$$\Gamma^{ik} = \Gamma_0^{ik} + \epsilon \begin{pmatrix} \mathrm{i}2p_2p_3 - 2p_2p_2 & \mathrm{i}(p_3p_3 + 2p_2p_3 + p_1p_3) - 2p_1p_2 & \mathrm{i}(p_2p_3 + 2p_2p_2 + p_1p_2) \\ & \mathrm{i}4p_1p_3 + 2p_3p_3 - 2p_1p_1 & \mathrm{i}(p_1p_3 + 2p_1p_2 + p_1p_1) + 2p_2p_3 \\ & & \mathrm{i}2p_1p_2 + 2p_2p_2 \end{pmatrix} . \tag{8}$$

Christoffel matrix (8) has at least three singular directions corresponding to the coordinate axes.

3.1. First singular direction

For $p_j = (p, 0, 0)$, Christoffel matrix (8) reads

$$\Gamma^{ik} = \begin{pmatrix} 3-4\mathrm{i}\delta & 0 & 0 \\ & 1-3\mathrm{i}\delta-2\epsilon & \mathrm{i}\epsilon \\ & & 1-3\mathrm{i}\delta \end{pmatrix} p^2 \quad . \tag{9}$$

This Christoffel matrix has P–wave eigenvalue $(3-4\mathrm{i}\delta)p^2$ with eigenvector $(1,0,0)$, and double S–wave eigenvalue $(1-3\mathrm{i}\delta-\epsilon)p^2$ with just a single eigenvector $(0,1,-\mathrm{i})$ corresponding to right–handed circular polarization.

3.2. Second singular direction

For $p_j = (0, p, 0)$, Christoffel matrix (8) reads

$$\Gamma^{ik} = \begin{pmatrix} 1-3\mathrm{i}\delta-2\epsilon & 0 & 2\mathrm{i}\epsilon \\ & 3-4\mathrm{i}\delta & 0 \\ & & 1-3\mathrm{i}\delta+2\epsilon \end{pmatrix} p^2 \quad . \tag{10}$$

This Christoffel matrix has P–wave eigenvalue $(3-4\mathrm{i}\delta)p^2$ with eigenvector $(0,1,0)$, and double S–wave eigenvalue $(1–3\mathrm{i}\delta)p^2$ with just a single eigenvector $(1,0,-\mathrm{i})$ corresponding to left–handed circular polarization.

3.3. Third singular direction

For $p_j = (0, 0, p)$, Christoffel matrix (8) reads

$$\Gamma^{ik} = \begin{pmatrix} 1-3\mathrm{i}\delta & \mathrm{i}\epsilon & 0 \\ & 1-3\mathrm{i}\delta+2\epsilon & 0 \\ & & 3-4\mathrm{i}\delta \end{pmatrix} p^2 \quad . \tag{11}$$

This Christoffel matrix has P–wave eigenvalue $(3-4\mathrm{i}\delta)p^2$ with eigenvector $(0,0,1)$, and double S–wave eigenvalue $(1-3\mathrm{i}\delta+\epsilon)p^2$ with just a single eigenvector $(1,-\mathrm{i},0)$ corresponding to right–handed circular polarization.

4. Conclusions

The ray tracing equations and the equations of geodesic deviation in heterogeneous anisotropic media are often formulated using the eigenvectors of the Christoffel matrix (Klimeš, 2006). The presented example demonstrates that two S–wave eigenvectors of the Christoffel matrix need not exist in weakly anisotropic viscoelastic media, even for arbitrarily weak attenuation.

It is thus necessary to formulate the ray tracing equations and the corresponding equations of geodesic deviation using the eigenvalues of the Christoffel matrix, without the eigenvectors of the Christoffel matrix (Klimeš, 2020).

Acknowledgements

The research has been supported by the Czech science foundation under contract 20-06887S, and by the members of the consortium “Seismic Waves in Complex 3–D Structures” (see “http://sw3d.cz”).

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