Lyapunov exponents for 2-D ray tracing without interfaces

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Summary

The Lyapunov exponents asymptotically quantify the exponential divergence of rays. The "Lyapunov exponent" for a finite 2-D ray and the average "Lyapunov exponents" for a set of finite 2-D rays or lines and for a 2-D velocity model are introduced. The equations for the estimation of the average "Lyapunov exponents" in a given smooth 2-D velocity model without interfaces are proposed and illustrated by a numerical example. The equations allow the average exponential divergence of rays and exponential growth of the number of travel–time branches in the velocity model to be estimated prior to ray tracing.

1 Introduction

If heterogeneities in a velocity model (macro model) exceed a certain degree, the average geometrical spreading exponentially increases with length of the rays and, in consequence, the average number of travel times exponentially increases with distance from the source. This behaviour of rays strictly limits the possibility of calculating two–point rays and travel times in overly complex models because:
(a) The geometrical spreading is so large that two–point rays cannot be found within the numerical accuracy. Similarly, the ray tubes cannot be sufficiently narrow for travel–time interpolation.
(b) The number of two–point travel times at each point is so large that the travel times cannot be calculated within reasonable computational time and costs.
(c) The number of two–point travel times at each point is so large that they can hardly be useful for any application, independently of the applicability of the ray theory which is not considered here.

It is thus of principal interest to quantify the exponential divergence of rays with respect to the complexity of the model, and to formulate explicit criteria enabling models suitable for ray tracing to be constructed. The exponential divergence of rays is quantified by the Lyapunov exponents. Refer to the overview by Matyska (1999) for the introduction to the deterministic chaos and Lyapunov exponents with relevant references. The aim of this paper is to estimate the "average Lyapunov exponent", describing the average spreading of ray tubes and average number of travel times, in smooth 2-D models without interfaces.

2 Paraxial–ray propagator matrix

In this section, we still consider a 3-D space, but the form of the quantities and equations in 2-D (or N-D) space is obvious. Let us denote by $\mathbf{w} = (x^1, x^2, x^3, p_1, p_2, p_3)^{\mathrm{T}}$ the phase-space coordinates. The ray tracing equations may be expressed in the form of

$$\frac{\mathrm{d}\mathbf{w}_\alpha}{\mathrm{d}\vartheta} = \Sigma_{\alpha\beta}\frac{\partial H}{\partial w^\beta} \quad , \tag{1}$$

where

$$\boldsymbol{\Sigma} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \quad , \tag{2}$$

with $\mathbf{0}$ and $\mathbf{1}$ being the 3×3 (in 3-D space) zero and identity matrices, respectively. Here parameter ϑ along a ray is determined by the form of Hamiltonian $H = H(w_\alpha)$. The paraxial approximation of deviation $\delta\mathbf{w}(\vartheta)$ between the phase–space coordinates of the points of two infinitesimally close rays is

$$\delta\mathbf{w}(\vartheta) = \boldsymbol{\Pi}(\vartheta,\vartheta_0)\ \delta\mathbf{w}(\vartheta_0) \quad , \tag{3}$$

where $\delta\mathbf{w}(\vartheta_0)$ is the initial value of the deviation. Here $\boldsymbol{\Pi}$ is the *paraxial–ray propagator matrix*,

$$\Pi_{\alpha\beta}(\vartheta,\vartheta_0) = \frac{\partial w^\alpha(\vartheta)}{\partial w^\beta(\vartheta_0)} \quad . \tag{4}$$

The derivative of the paraxial–ray propagator matrix along a ray is given by the *dynamic ray tracing* equation

$$\frac{\mathrm{d}\boldsymbol{\Pi}(\vartheta,\vartheta_0)}{\mathrm{d}\vartheta} = \boldsymbol{\Sigma}\ \mathbf{H}(\vartheta)\,\boldsymbol{\Pi}(\vartheta,\vartheta_0) \quad , \tag{5}$$

where

$$H_{\alpha\beta} = \frac{\partial^2 H}{\partial w^\alpha \partial w^\beta} \tag{6}$$

are the second phase–space partial derivatives of the Hamiltonian. Dynamic ray tracing equation (5) directly follows from ray tracing equations (1) differentiated with respect to the initial conditions.

3 Lyapunov exponents

The Lyapunov exponents may be defined in several ways (Matyska 1999). Some definitions rely on unspecified norm $\|\bullet\|$ in phase space, which may be chosen arbitrarily. Although the phase–space norm does not affect the values of the Lyapunov exponents defined asymptotically for infinitely long rays, it may considerably affect the estimated values of the Lyapunov exponents along finite rays in finite models.

The estimations of the Lyapunov exponents based on the *characteristic values* of the paraxial–ray propagator matrix are not affected by free parameters. On the other hand, the characteristic values oscillate along rays which makes the estimation of the Lyapunov exponents difficult. Let us denote by μ_1, μ_2, ..., μ_{2N} the *characteristic values* of the $2N\times 2N$ (in N-D space) propagator matrix $\boldsymbol{\Pi}$, i.e. the solutions of the characteristic equation

$$\det\left[\boldsymbol{\Pi}(\vartheta,\vartheta_0) - \mu(\vartheta,\vartheta_0)\mathbf{1}\right] = 0 \quad , \tag{7}$$

sorted according to their absolute values,

$$|\mu_1| \ge |\mu_2| \ge ... \ge |\mu_{2N}| \quad . \tag{8}$$

The complex–valued characteristic values, with the same absolute value, are assumed to be sorted according to their argument $-\pi < \arg\mu \le \pi$. The characteristic values are also sometimes called eigenvalues. We prefer the term characteristic values because the corresponding eigenvectors usually do not exist. The *Lyapunov exponents* along a ray parametrized by monotonically increasing parameter σ may be defined as

$$\lambda_k = \limsup_{\vartheta\to+\infty} \frac{\ln|\mu_k(\vartheta,\vartheta_0)|}{\sigma(\vartheta)-\sigma(\vartheta_0)} \ , \quad k = 1,2,...,N \ , \tag{9}$$

see Matyska (1999). The Lyapunov exponents are thus defined with respect to a particular monotonic parameter

$\sigma = \sigma(\vartheta)$ which may or may not differ from parameter ϑ, determined by the form of the Hamiltonian. Because of the symplectic property of the paraxial–ray propagator matrix, its inverse $\mathbf{\Pi}^{-1}$ has the same set of characteristic values as $\mathbf{\Pi}$. That is why the characteristic values of all Hamiltonian systems form reciprocal pairs $\mu_1\mu_{2N} = 1$, $\mu_2\mu_{2N-1} = 1$, ... $\mu_N\mu_{N+1} = 1$. Each positive Lyapunov exponent of ray tracing (as of other Hamiltonian systems) is thus accompanied by a negative Lyapunov exponent of the same absolute value. It is thus sufficient to study the positive Lyapunov exponents for ray tracing.

4 Ray–centred coordinates

Assume that the Hamiltonian is a homogeneous function of the second order with respect to the slowness vector. Then the corresponding parameter along a ray is the travel time, $\vartheta = \tau$. In the ray–centred coordinates, $\mathbf{w}^{(q)} = (q^1, q^2, q^3, p_1^{(q)}, p_2^{(q)}, p_3^{(q)})^{\mathrm{T}}$, the second phase–space derivatives of the Hamiltonian and the paraxial–ray propagator matrix take the forms

$$\mathbf{H} = \begin{pmatrix} \mathbf{H}_{11} & \begin{smallmatrix}0\\0\end{smallmatrix} & \mathbf{H}_{12} & \begin{smallmatrix}0\\0\end{smallmatrix} \\ \begin{smallmatrix}0 & 0\end{smallmatrix} & 0 & \begin{smallmatrix}0 & 0\end{smallmatrix} & 0 \\ \mathbf{H}_{21} & \begin{smallmatrix}0\\0\end{smallmatrix} & \mathbf{H}_{22} & \begin{smallmatrix}0\\0\end{smallmatrix} \\ \begin{smallmatrix}0 & 0\end{smallmatrix} & 0 & \begin{smallmatrix}0 & 0\end{smallmatrix} & 1 \end{pmatrix} , \quad \mathbf{\Pi} = \begin{pmatrix} \mathbf{Q}_1 & \begin{smallmatrix}0\\0\end{smallmatrix} & \mathbf{Q}_2 & \begin{smallmatrix}0\\0\end{smallmatrix} \\ \begin{smallmatrix}0 & 0\end{smallmatrix} & 1 & \begin{smallmatrix}0 & 0\end{smallmatrix} & \tau \\ \mathbf{P}_1 & \begin{smallmatrix}0\\0\end{smallmatrix} & \mathbf{P}_2 & \begin{smallmatrix}0\\0\end{smallmatrix} \\ \begin{smallmatrix}0 & 0\end{smallmatrix} & 0 & \begin{smallmatrix}0 & 0\end{smallmatrix} & 1 \end{pmatrix} , \qquad (10)$$

see Klimeš (1994). Two characteristic values are thus units, $\mu_3 = 1$ and $\mu_4 = 1$, with the corresponding Lyapunov exponents identical to zero, $\lambda_3 = 0$ and $\lambda_4 = 0$. Two positive Lyapunov exponents λ_1 and λ_2 correspond to 4×4 dynamic ray tracing in the ray–centred coordinates for matrices $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{P}_1, \mathbf{P}_2$. The ray–centred coordinates may be chosen in such a way that $\mathbf{H}_{12} = \mathbf{0}$ and $\mathbf{H}_{21} = \mathbf{0}$ in (10) (Klimeš 1994).

In 2-D, the dynamic ray tracing equation then simplifies to

$$\frac{\mathrm{d}\mathbf{\Pi}}{\mathrm{d}\tau} = \begin{pmatrix} 0 & G \\ -V & 0 \end{pmatrix} \mathbf{\Pi} , \quad \mathbf{\Pi} = \begin{pmatrix} Q_1 & Q_2 \\ P_1 & P_2 \end{pmatrix} , \qquad (11)$$

with

$$V = \frac{\partial^2 H}{\partial q \partial q} , \quad G = \frac{\partial^2 H}{\partial p^{(q)} \partial p^{(q)}} , \qquad (12)$$

and there is just one positive Lyapunov exponent λ. In an isotropic medium with Hamiltonian $H = v^2 \mathbf{p}^{\mathrm{T}} \mathbf{p}$, the second phase–space derivatives of the Hamiltonian are

$$V = v^{-1} \frac{\partial^2 v}{\partial q \partial q} , \quad G = v^2 \qquad (13)$$

where $\frac{\partial^2 v}{\partial q \partial q}$ is the second derivative of the velocity with respect to the in-plane ray–centred coordinate q, perpendicular to the ray.

5 Positive Lyapunov exponent in 2-D

5.1 "Lyapunov exponent" of a finite ray

We decompose the 2×2 paraxial–ray propagator matrix $\mathbf{\Pi}$ into the 2×2 identity matrix $\mathbf{1}$ and the Pauli matrices

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} , \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \qquad (14)$$

The decomposition of the real–valued matrix $\mathbf{\Pi}$ reads

$$\mathbf{\Pi} = \Pi_0 \mathbf{1} + \Pi_1 \boldsymbol{\sigma}_1 + \Pi_2 \mathrm{i} \boldsymbol{\sigma}_2 + \Pi_3 \boldsymbol{\sigma}_3 , \qquad (15)$$

where Π_0, Π_1, Π_2 and Π_3 are real–valued coefficients. Note that coefficients Π_1 and Π_2 are just formal. A reasonable physical meaning and units correspond to the off–diagonal terms $\Pi_1 + \Pi_2$ and $\Pi_1 - \Pi_2$ of matrix $\mathbf{\Pi}$, not just the terms Π_1 or Π_2 on their own. Since $\det \mathbf{\Pi} = 1$, the coefficients satisfy equation

$$\Pi_0^2 - (\Pi_1 + \Pi_2)(\Pi_1 - \Pi_2) - \Pi_3^2 = 0 . \qquad (16)$$

Since $\det \mathbf{\Pi} = 1$, the greater characteristic value of matrix $\mathbf{\Pi}$ is

$$\mu_1 = |\Pi_0| + \sqrt{\Pi_0^2 - 1} . \qquad (17)$$

Since the characteristic value oscillates along the ray, we shall look for a different "norm" which could substitute the characteristic value. We require the "norm" to be independent of free parameters and of the length and time units. It is not acceptable to have different measure of ray divergence for the same ray expressed once in kilometres and then in feet. Both the off–diagonal components of $\mathbf{\Pi}$ are dependent on the units (kilometres, feet), only their product is independent. However, since $\det \mathbf{\Pi} = 1$, the product of the off–diagonal components contains no additional information to the information carried by the diagonal components. The "norm" can thus depend only on the diagonal components of $\mathbf{\Pi}$, i.e. on coefficients Π_0 and Π_3. Characteristic value (17) is of little use at the points along a ray where Π_0 is small with respect to the other coefficients. We thus replace characteristic value (17) by "norm"

$$M = \sqrt{\Pi_0^2 + \Pi_3^2} + \sqrt{\Pi_0^2 + \Pi_3^2 - 1} , \qquad (18)$$

which is also good for small Π_0 and large Π_3. If both Π_0 and Π_3 are small, product $(\Pi_1 - \Pi_2)(\Pi_1 + \Pi_2)$ of the off–diagonal components is small, too, because of (16). "Norm" (18) can thus reasonably reflect the exponential ray divergence along the whole ray. We hope that "norm" (18) enables "Lyapunov exponents"

$$\lambda(\tau, \tau_0) = \frac{L(\tau, \tau_0)}{\sigma(\tau) - \sigma(\tau_0)} \qquad (19)$$

to be defined for rays of finite lengths, and to replace the limes superior in (9) by a simple limit,

$$\lambda = \lim_{\tau \to +\infty} \lambda(\tau, \tau_0) . \qquad (20)$$

Here we have denoted

$$L(\tau, \tau_0) = \ln [M(\tau, \tau_0)] \qquad (21)$$

the function characterizing propagator matrix $\mathbf{\Pi}(\tau, \tau_0)$. The magnitude of the variation of the "Lyapunov exponents" from ray to ray is considerable and the "Lyapunov exponent" for a single ray obviously cannot be linked to the overall properties of the velocity model.

Fortunately, equation (19) allows the average "Lyapunov exponent"

$$\overline{\lambda} = \sum_{\mathrm{ray}} L_{\mathrm{ray}}(\tau, \tau_0) \left\{ \sum_{\mathrm{ray}} [\sigma_{\mathrm{ray}}(\tau) - \sigma_{\mathrm{ray}}(\tau_0)] \right\}^{-1} \qquad (22)$$

over all rays to be introduced. This average "Lyapunov exponent" expresses the global properties of the velocity model. However, for a single source, it is still dependent on the source geometry and position with respect to the model boundaries. The average "Lyapunov exponent" over various sources should describe the global properties of the velocity model and should be dependent on the model boundaries only if the statistical properties of the model are anisotropic.

5.2 Approximation of the paraxial–ray propagator matrix

We decompose propagator matrix $\mathbf{\Pi}$ into the product of the infinitesimal propagator matrices along short ray segments, and distinguish the "high velocity" ($VG<0$) and "low velocity" ($VG>0$) zones. We introduce three parameters

$$\Lambda = \int\sqrt{-\min(VG,0)}\,\mathrm{d}\tau\ ,\quad \Phi = \int\sqrt{\max(0,VG)}\,\mathrm{d}\tau\ ,$$
$$\Psi = \tfrac{1}{4}\ln(|VG^{-1}|)\ . \qquad (23)$$

Parameter Λ accounts for the exponential spreading of the ray tube in phase space and parameter Φ describes how the ray tube is twisted in phase space. After several approximations, described in detail by Klimeš (1999), we arrive at the rough estimation of the propagator matrix,

$$\mathbf{\Pi} \approx \prod_{n=1}^{N-1}\cos(\Delta\Phi_n)\left[\mathbf{1}\cosh\left(\sum_{n=1}^{N}\Delta\Lambda_n\right)\cos(\Delta\Phi_N+\Delta\Phi_0)\right.$$
$$\left.+\boldsymbol{\sigma}_3\sinh\left(\sum_{n=1}^{N}\Delta\Lambda_n\right)\sin(\Delta\Phi_N-\Delta\Phi_0)\right]+\boldsymbol{\sigma}_1\ldots+\boldsymbol{\sigma}_2\ldots\ , \qquad (24)$$

where $\Delta\Lambda_n$, $n=1,2,...,N$, correspond to the "high–velocity" ray segments ($VG<0$) and $\Delta\Phi_n$, $n=0,1,2,...,N$, correspond to the "low–velocity" ray segments ($VG>0$), $\Delta\Phi_0$ and $\Delta\Phi_N$ may become zero if the ray starts or terminates in a "high–velocity" zone. Note that (24) is a very bad approximation of the propagator matrix at a single point of a single ray, but is a reasonably good estimate of the "average" propagator matrix.

5.3 Approximation of the Lyapunov exponent

Inserting coefficients Π_0 and Π_3 from (24) into (18) and (21), we approximately obtain

$$L \approx \sum_{n=1}^{N}\Delta\Lambda_n + \sum_{n=1}^{N-1}\ln|\cos(\Delta\Phi_n)|$$
$$+\tfrac{1}{2}\ln\left[\cos^2(\Delta\Phi_0+\Delta\Phi_N)+\sin^2(\Delta\Phi_0-\Delta\Phi_N)\right]\ . \qquad (25)$$

The mean value of $\ln|\cos(\Delta\Phi_n)|$ over all values of angle $\Delta\Phi_n$ is

$$\langle\ln|\cos(\Delta\Phi_n)|\rangle = -\ln 2\ . \qquad (26)$$

Because of the rough approximation of argument $\Delta\Phi_n$ of the cosine, the cosine is considerably inaccurate for Φ_n comparable with $\frac{\pi}{2}$ and greater. We thus replace $\ln|\cos(\Delta\Phi_n)|$ for $\Delta\Phi_n>\frac{\pi}{3}$ with its mean value,

$$\Delta\Phi_n \longrightarrow \min\left(\Delta\Phi_n,\tfrac{\pi}{3}\right)\ ,\ n=1,2,...,N-1\ . \qquad (27)$$

Similarly, the mean value of the last addend in (25) over all angles $\Delta\Phi_0$ and $\Delta\Phi_N$ is

$$\left\langle\tfrac{1}{2}\ln\left[\cos^2(\Delta\Phi_0+\Delta\Phi_N)+\sin^2(\Delta\Phi_0-\Delta\Phi_N)\right]\right\rangle = -\ln 2\ . \qquad (28)$$

The mean value is reached if both $\Delta\Phi_0=\frac{\pi}{6}$ and $\Delta\Phi_N=\frac{\pi}{6}$. We thus replace

$$\Delta\Phi_n \longrightarrow \min\left(\Delta\Phi_n,\tfrac{\pi}{6}\right)\ ,\quad n=0,N\ . \qquad (29)$$

Finally,

$$L \approx \sum_{n=1}^{N}\Delta\Lambda_n + \sum_{n=1}^{N-1}\ln\left\{\cos\left[\min\left(\Delta\Phi_n,\frac{\pi}{3}\right)\right]\right\}$$
$$+\frac{1}{2}\ln\left\{1-\sin\left[2\min\left(\Delta\Phi_0,\frac{\pi}{6}\right)\right]\sin\left[2\min\left(\Delta\Phi_N,\frac{\pi}{6}\right)\right]\right\}. \qquad (30)$$

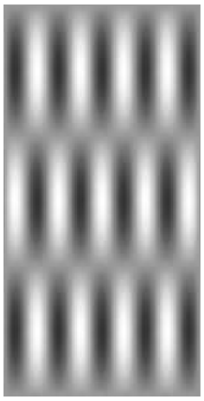

Figure 1. The model is formed by the homogeneous background of velocity 1.0 km/s, perturbed by a stretched bi-sine egg–box of amplitude 0.2 km/s.

Figure 2. The numbers of travel times corresponding to the point source at the bottom of the model box, 1.55 km from the left–hand corner. The maximum number of travel times is 49.

Figure 3. The numbers of travel times corresponding to the point source at the bottom of the model box, 1.85 km from the left–hand corner. The maximum number of travel times is 59.

6 Average "Lyapunov exponent" for the model

Our estimation of the "Lyapunov exponent" is dependent on the position and direction of a particular ray. Let us now average the "Lyapunov exponent" over the whole model volume. We cover the model by a dense system of parallel straight lines. For each direction of the lines, we calculate the average "Lyapunov exponent" (22) using equation (30), similarly as for a system of rays. We then average the calculated "Lyapunov exponent" over the directions, applying a selected directional weighting function. For instance, the weighting function may have the shape of a model box, with the origin at the centre of the box or at the mean position of the intended point sources. The weighting function may also correspond to the probability of the ray directions estimated for ray tracing in the model. Refer to Figure 4 for examples of the directional dependence of the average "Lyapunov exponent" and of the directional weighting function.

7 Numerical example

The model designed by Jean–David Benamou is formed by the homogeneous background of velocity $1.0\,\mathrm{km\,s^{-1}}$, perturbed by a stretched bi–sine egg–box of amplitude $0.2\,\mathrm{km\,s^{-1}}$, see Figure 1. The horizontal dimension of the model box is 3 km, vertical 6 km. There are low–velocity zones close to the four corners. The average travel times in seconds closely correspond to the average lengths in kilometres in this model.

Figure 2 displays the numbers of travel times corresponding to the point source situated at the bottom of the model box, 1.55 km from the left–hand corner. The travel times are calculated in a grid of 121×241 points covering the model box. Figure 3 shows the numbers of travel times for the point source shifted to 1.85 km from the left–hand corner.

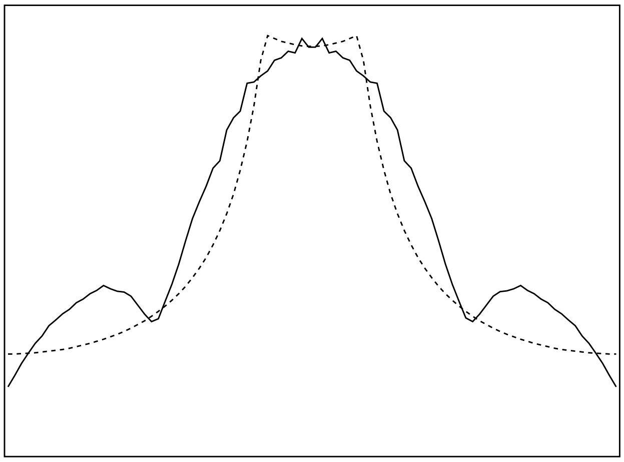

Figure 4. The angular dependence of the average "Lyapunov exponents" for the model [*solid line*] and the selected directional weighting function [*dashed line*]. The extent of the horizontal axis is 180 degrees, with the vertical direction at its centre. The average "Lyapunov exponents" vary between $0.170\,s^{-1}$ and $1.019\,s^{-1}$. The selected directional weighting function corresponds to the model box with the origin at the centre of the bottom edge, see Figure 1. Let us emphasize that the solid line describes the velocity heterogeneities whereas the dashed line characterizes the shape of the model box.

Ninety directions with an angular increment of 2 degrees have been chosen to estimate the average "Lyapunov exponent" for the model. For each direction, the model has been covered by 45 equally spaced straight lines. The "Lyapunov exponent" according to equations (19) and (30) has been numerically calculated with a step corresponding to 45 steps along the longest line for the direction. The average "Lyapunov exponent" for each direction has been calculated according to equation (22), with respect to the travel time, $\sigma(\tau) = \tau$. Since the statistical properties of the model are strongly anisotropic, the average "Lyapunov exponents" vary between $0.170\,s^{-1}$ and $1.019\,s^{-1}$ for different directions, see the solid line in Figure 4. The selected directional weighting function corresponds to the model box with the origin at the centre of the bottom edge, see the dashed line in Figure 4. This directional weighting function is suitable for the point sources situated at the bottom or at the top of the model box. The average "Lyapunov exponent" for the model, calculated with this directional weighting function, is

$$\overline{\lambda}_{\mathrm{model}} = 0.698\,\mathrm{s}^{-1} \quad . \tag{31}$$

The average "Lyapunov exponent" does not noticeably vary with the horizontal translation of the origin of the directional weighting function within the middle third of the horizontal model dimension. Figure 5 displays the natural logarithms of the average and maximum numbers of travel times along the individual horizontal gridlines of Figures 2 and 3 in comparison with the average "Lyapunov exponent" (31) for the model.

The average "Lyapunov exponent", calculated for the rays traced from the first source (1.55 km) according to equations (18), (21) and (22), using the paraxial–ray propagator matrix, is $0.679\,s^{-1}$. The average "Lyapunov exponent" estimated along the same rays using equation (30) is $0.676\,s^{-1}$. The average "Lyapunov exponent" calculated for the rays traced from the second source (1.85 km) according to equations (18), (21) and (22), using the paraxial–ray propagator matrix, is $0.613\,s^{-1}$. The average "Lyapunov exponent" estimated along the same rays using equation (30) is $0.697\,s^{-1}$.

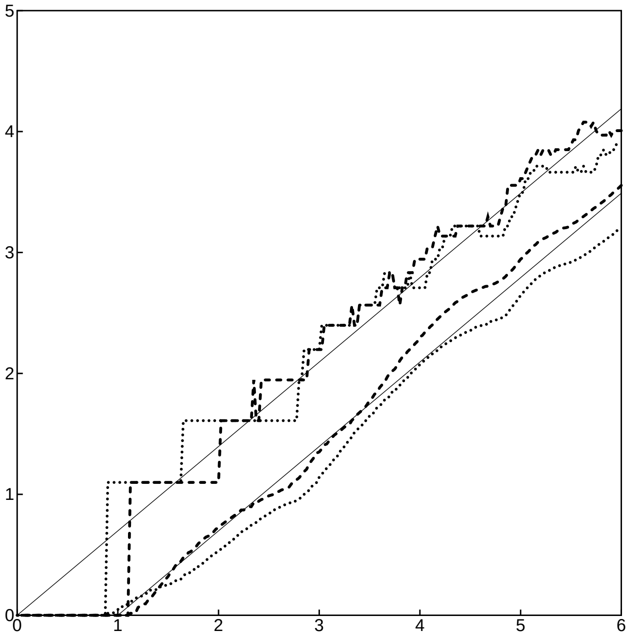

Figure 5. The natural logarithms of the average and maximum numbers of travel times along the individual horizontal gridlines of Figure 2 [*bold dotted line*] and Figure 3 [*bold dashed line*]. The horizontal axis represents the distance of a gridline from the bottom of the model box in kilometres, and serves as a rough approximation of the travel time in seconds. The slope of the *thin solid lines* is given by the average "Lyapunov exponent" (31) for the model.

8 Conclusions

The numerical example demonstrates a good correspondence between the average logarithms of the numbers of ray–theory travel times, the average "Lyapunov exponents" for the rays and the estimate of the average "Lyapunov exponent" for the model.

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