Correlation functions of random media

Luděk Klimeš, Department of Geophysics, Charles University, Prague, Czech Republic*

Summary

In geophysics, the correlation functions of random media are of principal importance for understanding and inverting the properties of seismic waves propagating in geological structures. Unfortunately, the kinds of correlation functions inappropriate for the description of geological structures are often assumed and applied. The most frequently used types of correlation functions are thus summarized in this paper, together with explanation of the physical meaning of their parameters.

A stationary random medium is assumed to be realized in terms of a white noise filtered by a spectral filter. The spectral filter is considered isotropic, in a simple general form enabling the random media used in geophysics to be specified. The medium correlation functions, corresponding to the individual special cases of the general random medium (Gaussian, exponential, von Kármán, self–affine, Kummer), are then derived and briefly discussed. The corresponding elliptically anisotropic correlation functions can simply be obtained by linear coordinate transforms.

1 Introduction

In order to estimate the relation of the seismic model to the geological structure and, in consequence, to estimate the relation of the synthetic quantities, calculated in the seismic model, to reality, it is important to have an estimate of the medium covariance function. On large scales, the medium covariance function is of principal importance in refraction travel–time tomographic inversion (Tarantola 1987, Klimeš 1997a), especially when estimating the accuracy of the seismic model, its relation to the geological structure, or the covariance matrix describing the statistics of synthetic travel times. On smaller scales, correlation functions enable to estimate the scattering attenuation (e.g., Wu 1982) or to approximately upscale a rough distribution of material parameters and estimate the relation to a smoother distribution of effective material parameters (e.g., Gold et al. 2000; Müller & Shapiro 2001).

In contrast to the importance of the correlation functions, the kinds of correlation functions inappropriate for the description of geological structures are applied in too many geophysical papers, whereas the kinds of correlation functions suitable for the description are applied rarely in the geophysical literature. For example, the physically reasonable values of the Hurst exponent N, describing the scaling properties of geological structures, are $-\frac{1}{2} < N < 0$, whereas the two values most popular in the geophysical literature are $N = -\frac{3}{2}$ and $N = \frac{1}{2}$. The author thus has come to the conclusion that it is reasonable to briefly summarize the most frequently used types of correlation functions, together with explanation of the physical meaning of their parameters in geophysical applications.

The material parameters of the *geological structure* are assumed to be realizations of random media with known statistical characteristics. This approach enables us to account satisfactorily for the differences between the geological structure and the *seismic model* of the structure, based on our quite incomplete information on the structure. The statistical characteristics represent our knowledge of the deterministic features of the structure and may change with more information on the structure. The randomness corresponds to the unknown details in the geological structure and may decrease with more information on the structure.

We assume each material parameter of the geological structure to be composed of a heterogeneous mean value and a particular representation of the stationary (statistically homogeneous) random medium in this paper. The stationary random medium is assumed to be realized in terms of a white noise filtered by a spectral filter. In this way, the medium covariance function may be expressed in terms of the correlation function.

The aim of this paper is to summarize systematically and discuss briefly the types of random media used in geophysics. The spectral filter is considered isotropic or elliptically anisotropic, in a simple general form described by four constants, enabling the random media used in geophysics to be specified. The medium correlation functions, corresponding to the individual special cases of the general random medium, are then briefly discussed. For detailed derivation and discussion refer to Klimeš (1997b).

2 Correlation functions of material parameters

The geological structure is assumed to be expressed in terms of spatial distribution $u = u(\mathbf{x})$ of several *material parameters* u. For example, the material parameters may represent the density, P and S wave velocities or slownesses, or the elements of the stiffness matrix. Mean value

$$u_0(\mathbf{x}) = \langle u(\mathbf{x}) \rangle \qquad (1)$$

of material parameter $u(\mathbf{x})$ describes the most probable value of the material parameter. The mean value $u_0(\mathbf{x})$ of the material parameter is a very smooth function in most cases.

The unknown difference between the material parameter and its mean value may be characterized in terms of the *medium covariance function*

$$C(\mathbf{x}_1, \mathbf{x}_2) = \langle [u(\mathbf{x}_1) - u_0(\mathbf{x}_1)]\ [u(\mathbf{x}_2) - u_0(\mathbf{x}_2)] \rangle \ . \qquad (2)$$

Random medium $u(\mathbf{x})$ is *stationary* if mean value $u_0(\mathbf{x})$ is constant and covariance function $C(\mathbf{x}_1, \mathbf{x}_2)$ is dependent only on distance $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$,

$$C(\mathbf{x}_1, \mathbf{x}_2) = c(\mathbf{x}_1 - \mathbf{x}_2) \quad , \qquad (3)$$

where $c(\mathbf{x})$ is the *correlation function* of the stationary random medium (Tarantola 1987). Note that we do assume mean value $u_0(\mathbf{x})$ to be constant in this paper.

A numerical realization of a stationary random medium may be generated in the following way:

• On a regular rectangular grid, we generate a pseudo-random realization $W(\mathbf{x})$ of the *white noise* of the desired statistical distribution of the functional values, with the *unit standard deviation.*

• Calculate the d-dimensional Fourier transform $\widehat{W}(\mathbf{k})$ of the white noise.

• Multiply the Fourier transform by *spectral filter* $\widehat{F}(\mathbf{k})$.

• Calculate the d-dimensional inverse Fourier transform $U(\mathbf{x})$ of product $\widehat{U}(\mathbf{k}) = \widehat{F}(\mathbf{k})\ \widehat{W}(\mathbf{k})$.
• Add the appropriate mean value $u_0(\mathbf{x})$ to arrive at the desired pseudorandom realization, $u(\mathbf{x}) = u_0(\mathbf{x}) + U(\mathbf{x})$.

The Fourier transform of the correlation function is then
$$\widehat{c}(\mathbf{k}) = (2\pi)^{-d}\ \widehat{F}^*(\mathbf{k})\ \widehat{F}(\mathbf{k}) \quad , \tag{4}$$
where d is the Euclidean dimension of the space (e.g., $d = 3$ in 3-D) and $\widehat{F}^*(\mathbf{k})\widehat{F}(\mathbf{k})$ is the power spectrum of the filter. The corresponding correlation function is
$$\boxed{c(\mathbf{x}) = (2\pi)^{-d} \int_{-\infty}^{+\infty} \mathrm{d}\mathbf{k}\ \exp(\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{x})\ \widehat{F}^*(\mathbf{k})\ \widehat{F}(\mathbf{k}) \quad ,} \tag{5}$$
where we have denoted
$$\int_{-\infty}^{+\infty} \mathrm{d}\mathbf{k} = \int_{-\infty}^{+\infty} \mathrm{d}k_1 \int_{-\infty}^{+\infty} \mathrm{d}k_2 \ldots \int_{-\infty}^{+\infty} \mathrm{d}k_d \quad . \tag{6}$$

3 Isotropic and elliptically anisotropic correlation functions

The realizations $u(\mathbf{x})$ of a random medium with the *isotropic correlation function* or *elliptically anisotropic correlation function* may be obtained by multiplying the Fourier transform of the realizations of a white noise of unit standard deviation by spectral filter
$$\widehat{F}(\mathbf{k}) = \widehat{f}(k) \tag{7}$$
and inversely Fourier transforming the products back into the space domain. Here
$$k = (\mathbf{k}^{\mathrm{T}}\mathbf{L}\mathbf{k})^{\frac{1}{2}} \quad , \qquad x = (\mathbf{x}^{\mathrm{T}}\mathbf{L}^{-1}\mathbf{x})^{\frac{1}{2}} \quad , \tag{8}$$
where the positive–definite symmetric scaling matrix $\mathbf{L}$ may account for the elliptical anisotropy of the correlation functions. For example, $L_{11} = 1$, $L_{12} = 0$, $L_{22} = \frac{1}{4}$ are chosen for Figures 2–6 of this paper.

Then d dimensional Fourier transform (5) may be expressed in terms of the one-dimensional Hankel transform of $k^{\frac{d-1}{2}}\widehat{f}^*(k)\widehat{f}(k)$,
$$\boxed{c(\mathbf{x}) = (2\pi)^{-\frac{d}{2}} x^{\frac{1-d}{2}} \int_0^{+\infty} \mathrm{d}k \sqrt{kx}\ \mathrm{J}_{\frac{d-2}{2}}(kx)\ k^{\frac{d-1}{2}}\ \widehat{f}^*(k)\widehat{f}(k)\ ,} \tag{9}$$
where $\mathrm{J}_\nu(x)$ is the Bessel function and $\Gamma(x)$ is the Gamma function.

4 Examples of isotropic and elliptically anisotropic correlation functions

Most (or all?) of the isotropic or elliptically anisotropic spectral filters (7) recently considered in the geophysical literature are special cases of filter
$$\boxed{\widehat{f}(k) = \kappa\ \left[a^{-2} + k^2\right]^{-\frac{d}{4}-\frac{N}{2}}\ \exp\left(-\frac{a_{\mathrm{G}}^2 k^2}{8}\right) \quad ,} \tag{10}$$
described by four constants κ, N, a_{G} and a. The individual special filters may be fit by the relevant choice of *Hurst exponent* (Hurst parameter, Hurst number) N and two correlation lengths a_{G} (Gaussian correlation length, see Sections 4.1 and 4.4) and a (von Kármán correlation length, see Section 4.2). Function (10) is the product of spectral filter
$$\widehat{f}_{\mathrm{S}_N}(k) = \kappa\ k^{-\frac{d}{2}-N} \quad , \tag{11}$$
corresponding to the self–affine random medium, with high–pass wavenumber filter
$$\widehat{f}_a(k) = \left[1 + (ak)^{-2}\right]^{-\frac{d}{4}-\frac{N}{2}} \tag{12}$$
and low–pass wavenumber filter
$$\widehat{f}_{a_{\mathrm{G}}}(k) = \exp\left(-\frac{a_{\mathrm{G}}^2 k^2}{8}\right) \quad , \tag{13}$$
see Figure 1.

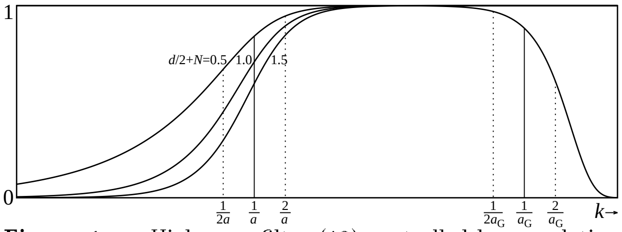

Figure 1. High–pass filter (12) controlled by correlation length a and low–pass Gaussian filter (13) controlled by correlation length a_{G}. The horizontal wavenumber axis has a logarithmic scale.

Note that, with regards to the discussion in Section 4.3, the self–affine random medium with Hurst exponents
$$-\tfrac{1}{2} < N < 0 \tag{14}$$
seems to be suitable for the description of the material properties of the geological structures. One possibility of estimating the mean value $u_0(\mathbf{x})$ and parameters κ and N of the correlation function from travel times in the Western Bohemia region, under the assumption of a self–affine random medium, has been demonstrated by Klimeš (1996). The results suggest that the value of Hurst exponent N, chosen in Figures 3, 5 and 6, is acceptable, e.g., for the Western Bohemia region.

The application of the low–pass filter of correlation length a_{G} removes small–scale heterogeneities. The resulting random medium remains self–affine at distances considerably longer than correlation length a_{G}. The application of the high–pass filter of correlation length a suppresses large–scale heterogeneities. The resulting random medium remains self–affine at distances considerably shorter than correlation length a. If both the low–pass and high–pass filters are applied, the random medium is self–affine at distances x between the correlation lengths,
$$a_{\mathrm{G}} \ll x \ll a \quad . \tag{15}$$

4.1 Gaussian correlation function ($N = -\frac{d}{2}$)

For Hurst exponent $N = -\frac{d}{2}$, equation (9) with spectral filter (10) yields the Gaussian correlation function
$$c_{\mathrm{G}}(\mathbf{x}) = \sigma_{\mathrm{G}}^2\ \exp\left(-a_{\mathrm{G}}^{-2}x^2\right) \tag{16}$$
of a white noise filtered by a Gaussian low–pass filter. Here
$$\sigma_{\mathrm{G}}^2 = \kappa^2\ \pi^{-\frac{d}{2}}\ a_{\mathrm{G}}^{-d} \quad . \tag{17}$$
White noise is approached for $a_{\mathrm{G}} \to 0+$. The Gaussian correlation function is too smooth to represent actual inhomogeneities in the earth (Wu & Aki 1985; Sato & Fehler 1998), and cannot explain observations of both seismic wave scattering and travel–time variations (Frankel & Clayton 1986). The author thus does not see any relation of the Gaussian correlation function to the properties of geological structures. The Gaussian random medium is illustrated by Figure 2.

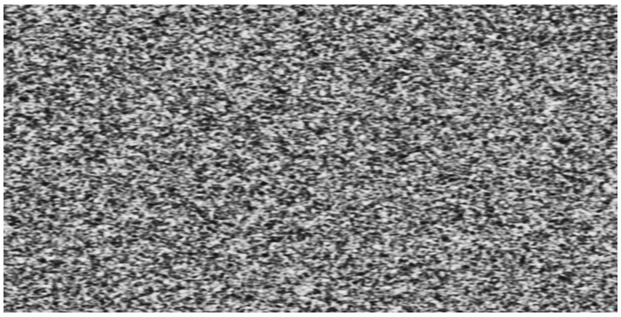

Figure 2. A 2-D representation of the Gaussian random medium ($N=-d/2$, $a=+\infty$) of correlation lengths a_{G} of 0.005 times the respective dimensions of the figure.

4.2 Von Kármán correlation functions ($a_{\mathrm{G}}=0$)

For $a_{\mathrm{G}}=0$, equation (9) with spectral filter (10) yields the von Kármán correlation function

$$c_{\mathrm{K}_N}(\mathbf{x}) = \sigma^2_{\mathrm{K}_0} \left[\frac{x}{2a}\right]^N \mathrm{K}_N\left(\frac{x}{a}\right) \quad , \qquad (18)$$

where K_N is the MacDonald function, $\mathrm{K}_{-N}(\xi)=\mathrm{K}_N(\xi)$, and

$$\sigma^2_{\mathrm{K}_0} = \kappa^2\ 2^{1-d}\ \pi^{-\frac{d}{2}}\ \left[\Gamma(\tfrac{d}{2}+N)\right]^{-1}\ a^{2N} \quad . \qquad (19)$$

For the special case of zero Hurst exponent, $N=0$, we have the *zero von Kármán correlation function* (Frankel & Clayton 1986).

For Hurst exponent $N\neq 0$, the von Kármán correlation function can be normalized,

$$c_{\mathrm{K}_N}(\mathbf{x}) = \sigma^2_{\mathrm{K}}\ \frac{2}{\Gamma(|N|)} \left[\frac{x}{2a}\right]^N \mathrm{K}_N\left(\frac{x}{a}\right) \quad , \qquad (20)$$

where

$$\sigma^2_{\mathrm{K}} = \kappa^2\ 2^{-d}\ \pi^{-\frac{d}{2}}\ \Gamma(|N|)\ \left[\Gamma(\tfrac{d}{2}+N)\right]^{-1}\ a^{2N} \ . \qquad (21)$$

Right–hand side of equation (20) is σ^2_{K} times the von Kármán function of argument $a^{-1}x$ (Frankel & Clayton 1986). The von Kármán medium is self–affine at distances considerably shorter than correlation length a. The von Kármán correlation functions with negative Hurst exponent $N<0$ may be appropriate for the description of the self–affine random medium in which the large heterogeneities (larger than the correlation length) are fit well enough by mean value $u_0(\mathbf{x})$. The von Kármán random medium is illustrated by Figure 3.

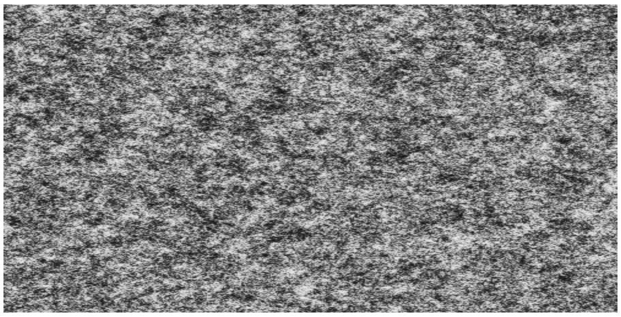

Figure 3. A 2-D representation of the von Kármán random medium ($a_{\mathrm{G}}=0$) described by Hurst exponent $N=-0.2$ and correlation lengths a of 0.020 times the respective dimensions of the figure. Compare with Figure 5 imagined without large heterogeneities.

4.2.1 Exponential correlation function ($a_{\mathrm{G}}=0, N=\frac{1}{2}$)

The asymptotic behaviour of the von Kármán correlation functions (20) for very large arguments is

$$\underset{x\to+\infty}{c_{\mathrm{K}_N}(\mathbf{x})} \approx \sigma^2_{\mathrm{K}} \sqrt{\pi}\ [\Gamma(|N|)]^{-1} \left[\frac{x}{2a}\right]^{N-\frac{1}{2}} \exp\left(-\frac{x}{a}\right) \ . \qquad (22)$$

Relation (22) is exact for $N=\pm\frac{1}{2}$. The special case of Hurst exponent $N=\frac{1}{2}$ is the exponential random medium, which is not able to fit the observations of seismic wave scattering and travel–time variations simultaneously (Frankel & Clayton 1986), because is poor both in small–scale and large–scale heterogeneities. The exponential random medium is illustrated by Figure 4.

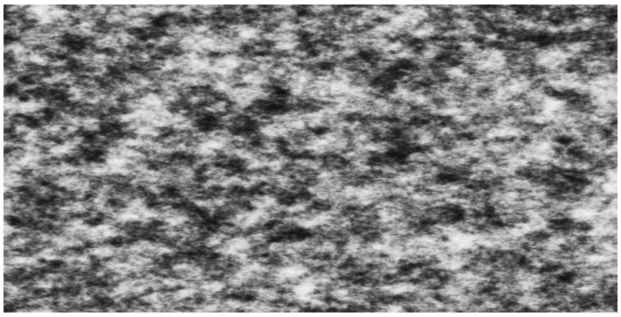

Figure 4. A 2-D representation of the exponential medium ($N=1/2$, $a_{\mathrm{G}}=0$) described by correlation lengths a of 0.020 times the respective dimensions of the figure.

4.3 Self–affine random medium ($a_{\mathrm{G}}=0,\ a=+\infty$)

If $-\frac{d}{2}<2N<0$, equation (9) with spectral filter (10) yields for $a_{\mathrm{G}}=0$ and $a\to+\infty$ the power–law correlation function

$$c_{\mathrm{S}_N}(\mathbf{x}) = \sigma^2_{\mathrm{S}}\ \left[\frac{x}{L}\right]^{2N} \quad , \qquad (23)$$

where

$$\sigma^2_{\mathrm{S}} = \kappa^2\ 2^{-d}\ \pi^{-\frac{d}{2}}\ \Gamma(|N|)\left[\Gamma(\tfrac{d}{2}+N)\right]^{-1}\left[\tfrac{1}{2}L\right]^{2N} \ . \qquad (24)$$

Here L is some reference distance supplemented to preserve the correct physical units in the expressions. It is unnecessary, $L=1$, if equations (8) are used to define x and k, because scaling matrix $\mathbf{L}$ can play the role of L^2.

The self–affine random medium seems to be the most appropriate description of the material properties of the geological structures, because the structures contain heterogeneities of all scales. Since the correlation function should decrease with distance x, the Hurst exponent should be negative, $N<0$, in the self–affine random medium.

In the inversion of travel–times, correlation functions are required for the calculation of the geometrical travel–time covariances. The line integrals of correlation function (23) across $\mathbf{x}=\mathbf{0}$, representing travel–time variances along infinitely thin rays, are finite for $-\frac{1}{2}<N$ (Klimeš 1996). Although "thick" rays, corresponding to finite frequencies, allow also of $-\frac{d}{2}<N\leq-\frac{1}{2}$, the travel–time variances would increase considerably with frequency for these values of Hurst exponent N. The author does not consider such a strong velocity dispersion to be realistic. The self–affine random medium, with Hurst exponent N acceptable for the Western Bohemia region (Klimeš 1996), is illustrated by Figure 5.

On smaller scales, correlation functions enable to approximately upscale a rough distribution of material parameters and estimate the relation to a smoother distribution of effective material parameters. Müller & Shapiro (2001) derived the expressions for the mean travel–time delay and amplitude decay due to random perturbations of the squared slowness in the weak fluctuation regime. Assuming the self–affine random medium, the integrals in their expressions for the mean logarithmic acoustic wavefield are finite for $-\frac{1}{2}<N<\frac{1}{2}$ in 2-D and for $-\frac{1}{2}<N<0$ in 3-D. Restriction (14) on the Hurst exponent is thus useful also on small scales.

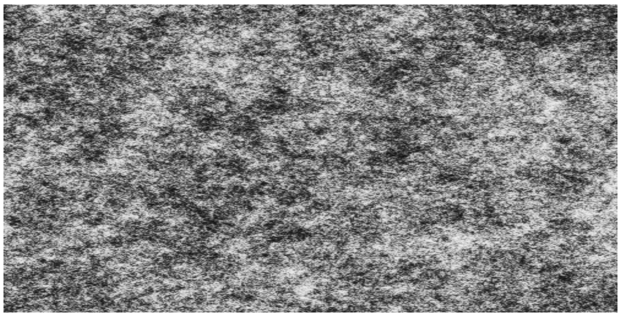

Figure 5. A 2-D representation of the self–affine random medium (a_G=0, a=+∞) described by Hurst exponent N=−0.2.

4.4 Kummer correlation functions (a=+∞)

For $a \to +\infty$, equation (9) with spectral filter (10) yields the "Kummer" correlation function

$$c_{F_N}(\mathbf{x}) = \sigma_F^2\ {}_1F_1(-N;\tfrac{d}{2};-x^2 a_G^{-2}) \quad , \qquad (25)$$

where ${}_1F_1(\alpha;\beta;\xi)$ is the *confluent hypergeometric function*, called also the *Kummer function*, and

$$\sigma_F^2 = \kappa^2\ 2^{-d-2N}\ \pi^{-\frac{d}{2}}\ \Gamma(|N|)\ \left[\Gamma(\tfrac{d}{2})\right]^{-1}\ a_G^{2N} \quad . \qquad (26)$$

The Kummer medium is self–affine at distances considerably longer than correlation length a_G. On the other hand, the Kummer correlation function is smooth at short distances, with $c_{F_N}(\mathbf{0}) = \sigma_F^2$. The Kummer correlation functions may thus be very useful for the discrete numerical approximation of the self–affine correlation function (23), in order to avoid the wavenumber aliasing due to the singularity related to infinitely small heterogeneities when using, e.g., the fast Fourier transform. The random medium with the Kummer correlation function is illustrated by Figure 6.

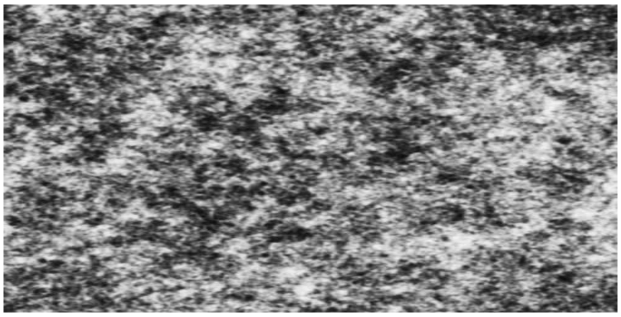

Figure 6. A 2-D representation of the Kummer random medium ($a = +\infty$) described by Hurst exponent N=−0.2 and correlation lengths a_G of 0.005 times the respective dimensions of the figure. Compare with Figure 5 imagined without small heterogeneities.

5 Conclusions

Several simple types of medium correlation functions, their mutual relation and physical meaning of their parameters has briefly been discussed. Among them, the *self–affine random medium* is the most appropriate for description of the material properties of geological structures, because the earth contains heterogeneities on all scales.

The *Kummer correlation functions*, with the Gaussian correlation length as small as the computing equipment allows, are recommended for numerical realizations of the self–affine random medium if we need to avoid the singularity related to infinitely small heterogeneities.

The *von Kármán correlation functions* with Hurst exponent $-\frac{1}{2} < N < 0$ may be useful for the description of small–scale random heterogeneities if the large–scale heterogeneities are known, i.e., are not random.

It is difficult to justify the application of the frequently used *Gaussian correlation function* and *von Kármán correlation functions* with a positive Hurst exponent (e.g., *exponential correlation function*).

References

Frankel, A. & Clayton, R.W. (1986): Finite difference simulations of seismic scattering: Implications for the propagation of short–period seismic waves in the crust and models of crustal heterogeneity. J. geophys. Res., **91 B**, 6465–6489.

Gold, N., Shapiro, S.A., Bojinski, S. & Müller, T.M. (2000): An approach to upscaling for seismic waves in statistically isotropic heterogeneous elastic media. Geophysics, **65**, 1837–1850.

Klimeš, L. (1996): Correlation function of a self–affine random medium. In: Seismic Waves in Complex 3-D Structures, Report 4, pp. 25–38, Dep. Geophys., Charles Univ., Prague.*

Klimeš, L. (1997a): Accuracy and resolution of the travel–time tomography. In: Seismic Waves in Complex 3-D Structures, Report 6, pp. 13–24, Dep. Geophys., Charles Univ., Prague.*

Klimeš, L. (1997b): Correlation functions of random media. In: Seismic Waves in Complex 3-D Structures, Report 6, pp. 25–40, Dep. Geophys., Charles Univ., Prague.*

Müller, T.M. & Shapiro, S.A. (2001): Most probable seismic pulses in single realizations of two– and three–dimensional random media. Geophys. J. int., **144**, 83–95.

Sato, H. & Fehler, M. (1998): Seismic Wave Propagation and Scattering in the Heterogeneous Earth. Springer-Verlag, New York.

Tarantola, A. (1987): Inverse Problem Theory. Elsevier, Amsterdam.

Wu, R.S. (1982): Attenuation of short period seismic due to scattering. Geophys. Res. Lett., **9**, 9–12.

Wu, R-S. & Aki, K. (1985): Elastic wave scattering by a random medium and the small–scale inhomogeneities in the litosphere. J. geophys. Res., **90B**, 10261–10273.

Acknowledgements

I am indebted to Paul Spudich who provided me with his perfect code I used to calculate the representations of the random media presented here. This research has been supported by the Grant Agency of the Czech Republic under Contracts 205/95/1465 and 205/01/0927, by the Ministry of Education of the Czech Republic within Research Project J13/98 113200004, and by the members of the consortium "Seismic Waves in Complex 3-D Structures" (see "http:/seis.karlov.mff.cuni.cz/consort/main.htm").

* Available online at "http://seis.karlov.mff.cuni.cz/consort/main.htm".