COMPARISON OF QUASI-ISOTROPIC APPROXIMATIONS OF THE COUPLING RAY THEORY IN 1-D MODEL 'OTC'

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Summary
Commonly used quasi–isotropic approximations of the coupling ray theory are discussed. The coupling ray theory and its quasi–isotropic approximations are then numerically compared with the exact analytical solution for the plane S wave, propagating along the axis of spirality in the 1-D anisotropic "oblique twisted crystal" model.

1. Introduction

There are two different high–frequency asymptotic ray theories, the *isotropic ray theory* assuming equal velocities of both S–wave polarizations and the *anisotropic ray theory* assuming both S–wave polarizations strictly decoupled. In the isotropic ray theory, the S–wave polarization vectors do not rotate about the ray, whereas in the anisotropic ray theory they coincide with the eigenvectors of the Christoffel matrix which may rotate rapidly about the ray.

In "weakly anisotropic" models, at moderate frequencies, the S–wave polarization tends to remain unrotated round the ray but is partly attracted by the rotation of the eigenvectors of the Christoffel matrix. The intensity of the attraction increases with frequency. The isotropic and anisotropic ray theories are thus limiting cases and the gap between them has to be filled. A ray theory providing continuous transition between the isotropic and anisotropic ray theories was proposed by Coates & Chapman (1990) and is called the *coupling ray theory*. The numerical algorithm of calculation of the frequency–dependent complex–valued S–wave polarization vectors of the coupling ray theory has been proposed by Bulant & Klimeš (2002).

The results of the isotropic ray theory, anisotropic ray theory and coupling ray theory have been compared with the exact solution in the "simplified twisted crystal" model analytically by Klimeš (1999) and numerically by Bulant, Klimeš & Pšenčík (1999; 2000).

There are many possible modifications and approximations of the coupling ray theory. For example, the reference ray may be calculated in different ways (Bakker 2002), the Christoffel matrix may be approximated by its quasi–isotropic projections onto the plane perpendicular to the reference ray and onto the line tangent to the reference ray (Pšenčík 1998), the Green tensor may be approximated by its quasi–isotropic projection onto the reference S–wave polarization plane given by two orthonormal reference polarization vectors, travel times corresponding to the anisotropic ray theory may be approximated in several ways, e.g., by linear quasi–isotropic perturbation with respect to the density–normalized elastic moduli (Pšenčík 1998), etc. Commonly used quasi–isotropic approximations of the coupling ray theory are discussed in Section 2.

The effects of the anisotropic common ray approximation by Bakker (2002) and the less accurate isotropic common ray approximation have been studied by Klimeš & Bulant (2002). The effect of the quasi–isotropic projection of the Green tensor has been numerically demonstrated by Bulant & Klimeš (2002) on the example of synthetic seismograms calculated by the coupling ray theory in a 1-D anisotropic model.

The effects of the quasi–isotropic approximation of the Christoffel matrix, the quasi–isotropic projection of the Green tensor and the quasi–isotropic approximation of travel times are numerically demonstrated in this paper. In Section 3, a simple 1-D anisotropic "oblique twisted crystal" model is introduced, and the exact solution for the one–wave plane–wave propagator matrix is briefly summarized. The numerical results of the coupling ray theory and its above–mentioned quasi–isotropic approximations are then compared with the exact solution in Section 4.

The lower–case subscripts take values $i, j, k, ... = 1, 2, 3$, the upper–case subscripts take values $I, J, K, ... = 1, 2$; the Einstein summation over the pairs of identical indices is applied.

2. Quasi–isotropic approximations of the coupling ray theory

The coupling ray theory by Coates & Chapman (1990) is applicable at all degrees of anisotropy, but it is often modified by various *quasi–isotropic approximations.*

2.1. Selection of the reference ray

The isotropic ray theory is always the limiting case of the coupling ray theory for decreasing anisotropy at a fixed frequency. On the other hand, the high–frequency limit of the coupling ray theory at a fixed anisotropy depends on the choice of the reference ray, and even on the choice of the *system* of reference rays, because the amplitudes are determined by the paraxial reference rays.

If we choose the *anisotropic ray theory reference ray* and select the initial polarization corresponding to the reference ray, the coupling ray theory will correctly limit to the anisotropic ray theory for high–frequencies. For other choices of reference rays, the high–frequency limit of the coupling ray theory at a fixed anisotropy is incorrect, although the differences may be small at the finite frequencies under consideration.

In the *anisotropic common ray approximation*, the common reference ray is traced using the averaged Hamiltonian of both S–wave polarizations (Bakker 2002).

In the less accurate *isotropic common ray approximation*, the reference ray is traced in the reference isotropic model. Moreover, the reference isotropic model may be selected in different ways, yielding quasi–isotropic approximations of various accuracy.

The method of estimating the errors due to the isotropic common ray approximation and anisotropic common ray approximation has been proposed by Klimeš & Bulant (2002).

2.2. Quasi–isotropic projection of the Green tensor

The coupling ray theory solution u_m may be approximated by its projection

$$\tilde{u}_i = h_{iM} h_{mM} u_m \tag{1}$$

onto the reference S–wave polarization plane, given by two orthonormal reference polarization vectors h_{k1}, h_{k2}. This approximation may simplify the modification of existing isotropic ray tracing codes for the coupling ray theory. The error of this approximation is obvious and can be calculated simply.

2.3. Quasi–isotropic approximation of the Christoffel matrix

The Christoffel matrix may be approximated by its projections onto the reference S–wave polarization plane given by h_{k1}, h_{k2} and onto the reference P–wave polarization line given by h_{k3}. The eigenvectors g_{k1} and g_{k2} of the Christoffel matrix then become situated in plane h_{k1}, h_{k2} as in the zero-order quasi–isotropic approximation of Pšenčík (1998). This approximation includes the quasi–isotropic projection of the Green tensor described in Section 2.2.

The quasi–isotropic approximation of the Christoffel matrix is probably the worst quasi–isotropic approximation. The alteration of the coupling ray theory synthetic seismograms due to the quasi–isotropic approximation of the Christoffel matrix has been demonstrated by Bulant & Klimeš (2002).

2.4. Quasi–isotropic perturbation of travel times

The anisotropic ray theory travel times used in the coupling ray theory should be calculated by a numerical quadrature along the reference ray. The quasi–isotropic perturbation of travel times just corrects the reference travel time by the linear perturbation with respect to the density–normalized elastic moduli. The quasi–isotropic perturbation of travel times leads to an erroneous time shift in synthetic seismograms but has a negligible impact on the amplitudes.

3. Twisted crystal models

The *twisted crystal model* is created of a homogeneous anisotropic elastic material by uniformly helicoidally twisting the $x_1 x_2$ coordinate plane along the x_3 axis. It is one of the simplest models useful for demonstrating the limits of applicability of the zero–order isotropic and anisotropic ray theories and for testing the coupling ray theory. The great advantage of this model is that the exact analytical solution for the plane S wave propagating along the axis of spirality can be examined analytically.

The elastodynamic equation in the frequency domain reads

$$[\varrho a_{ijkl} u_{k,l}]_{,j} = -\varrho \omega^2 u_i \quad , \tag{2}$$

where a_{ijkl} are the density–normalized elastic moduli (stiffness tensor), u_i is the displacement, ϱ is the density, and ω stands for the frequency.

For the plane wave $u_i = u_i(x_3)$ propagating along the x_3 axis in the 1-D anisotropic model $a_{ijk3} = a_{ijk3}(x_3)$ with constant density ϱ, elastodynamic equation (2) simplifies to

$$[a_{i3k3} u_{k,3}]_{,3} = -\omega^2 u_i \quad . \tag{3}$$

3.1. Simplified twisted crystal model

The general plane–wave solution for the general initial conditions expressed in terms of displacement and stress was derived by Lakhtakia (1994), who also presented explicit analytical equations for the *simplified twisted crystal model* with vanishing a_{1333} and a_{2333}, in which the u_1 and u_2 displacement components are strictly separated from the longitudinal u_3 component. Klimeš (1999) derived the 2×2 one–way propagator matrices in the simplified twisted crystal model, suitable for comparison with the coupling ray theory (Bulant, Klimeš & Pšenčík 1999; 2000).

In the simplified twisted crystal model, S–wave polarization vectors g_{iK} of the anisotropic ray theory are situated in the same plane as the reference S–wave polarization vectors h_{iK}. Then neither the quasi–isotropic projection of the Green tensor, nor the quasi–isotropic approximation of the Christoffel matrix have any impact on the results of the coupling ray theory. We thus modify the simplified twisted crystal model towards the *oblique twisted crystal model* in order to demonstrate these quasi–isotropic approximations.

3.2. Oblique twisted crystal model

In the oblique twisted crystal model, we select elastic moduli

$$a_{i3k3} = E_{im} E_{kn} \overline{a}_{m3n3} \quad , \tag{4}$$

where $\overline{a}_{i3k3}$ are the elastic moduli corresponding to the simplified twisted crystal model.

Constant 3×3 unitary matrix E_{im}, describing rotation in 3-D space, has three independent components and may be expressed in terms of vector E_k. The direction of vector E_k specifies the axis of rotation, while the length

$$E = \sqrt{E_k E_k} \tag{5}$$

of vector E_k specifies the angle of rotation about the half–axis given by vector E_k.

The exact 3×3 one–way propagator matrix U_{ij} of elastodynamic equation (3) in the oblique twisted crystal model is

$$U_{ij} = E_{im} E_{jn} \overline{U}_{mn} \quad , \tag{6}$$

where $\overline{U}_{mn}$ are the components of the 3×3 one–way propagator matrix in the simplified twisted crystal model. Indeed, elastodynamic equation (3) with elastic moduli (4) and one–way propagator matrix (6) is equivalent to the same equation with elastic moduli $\overline{a}_{i3k3}$ and one–way propagator matrix $\overline{U}_{ij}$.

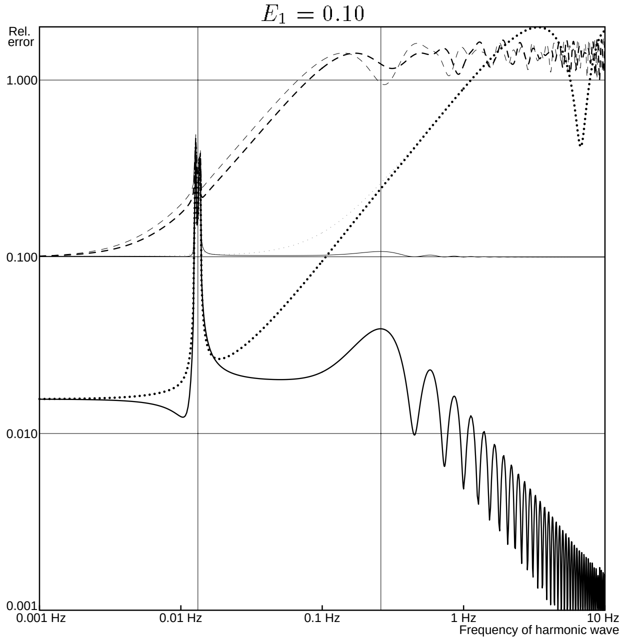

Figure 1: The relative differences of various approximations of the coupling ray theory (Coates & Chapman 1990) from the exact solution in the oblique twisted crystal model. *Bold line*: No quasi–isotropic approximation. *Thin line*: quasi–isotropic projection of the Green tensor. *Dashed line*: quasi–isotropic approximation of the Christoffel matrix. It includes the quasi–isotropic projection of the Green tensor. *Dotted line*: quasi–isotropic approximation of travel times. *Thin dotted line*: quasi–isotropic projection of the Green tensor combined with the quasi–isotropic approximation of travel times. *Thin dashed line*: quasi–isotropic approximation of the Christoffel matrix combined with the quasi–isotropic approximation of travel times. The two vertical lines denote the resonant and coupling frequencies (13), (14). The relative error of 200% occurs, e.g., for an opposite polarization or an opposite phase.

4. Numerical comparison with the exact solution

For the numerical comparison, we define the relative (with respect to the initial conditions) difference between one–way propagator matrices $\mathbf{U}$ and $\mathbf{U}_0$ as

$$\Delta = \sqrt{\tfrac{1}{2}\mathrm{tr}\Big([\mathbf{U}_0 - \mathbf{U}]^{+}[\mathbf{U}_0 - \mathbf{U}]\Big)} \quad , \tag{7}$$

where $\overset{+}{\mathbf{U}}$ denotes the matrix Hermitian adjoint to $\mathbf{U}$.

4.1. Model for the numerical comparison

We use the simplified twisted crystal model designed by Vavryčuk (1999). The relation to Vavryčuk's (1999) notation is

$$\varepsilon = \frac{-\gamma\sin^2(\theta)}{1+\gamma\sin^2(\theta)} \ , \quad v_{\mathrm{S}}^2 = a_{44}[1+\gamma\sin^2(\theta)] \ , \quad \varphi = Kx_3 \ . \tag{8}$$

The selected numerical values in (8) are

$$\gamma\sin^2(\theta) = 0.15 \times 0.75 = 0.1125 \quad , \tag{9}$$

$$v_{\mathrm{S}}^2 = 6.0\,\mathrm{km}^2\mathrm{s}^{-2} \times [1+0.15\times 0.75] = 6.675\,\mathrm{km}^2\mathrm{s}^{-2} \ . \tag{10}$$

The relative difference between the faster and slower S–wave velocities is thus about 10%. Parameter K describing the rotation of the crystal axes along the x_3 axis has the value

$$K = 0.032\,\mathrm{km}^{-1} \quad . \tag{11}$$

The source–receiver distance corresponds to the crystal axes rotated by π radians,

$$x_3 = \frac{\pi}{K} \doteq 98.17477\,\mathrm{km} \quad . \tag{12}$$

The *resonant frequency* (Klimeš 1999, section 3.4) is

$$F = \left|\frac{v_{\mathrm{S}}K}{2\pi}\right| \doteq 0.0132\,\mathrm{Hz} \tag{13}$$

and the *coupling frequency* (Klimeš 1999, section 4.1) is

$$\left|\frac{2}{\varepsilon}\right| F \doteq 0.260\,\mathrm{Hz} \quad . \tag{14}$$

The squared P wave velocity is

$$v_{\mathrm{P}}^2 = 3v_{\mathrm{S}}^2 \quad . \tag{15}$$

The rotation vector for the oblique twisted crystal model is

$$E_1 = 0.10 \quad , \qquad E_2 = 0 \quad , \qquad E_3 = 0 \quad . \tag{16}$$

The rotation angle (0.10 radians) thus roughly corresponds to the strength of the anisotropy (10% difference between the S–wave velocities).

4.2. Numerical results

The approximate 3×3 one–way propagator matrices of the plane S wave, corresponding to the coupling ray theory and its quasi–isotropic approximations, have been numerically calculated by means of the Fortran 77 package CRT (Bucha & Bulant 2002). The relative differences of the coupling ray theory, its quasi–isotropic approximations and their combinations from the exact analytical solution are plotted on a log–log scale in Figure 1.

The coupling ray theory yields excellent results in this model, except for the resonant frequencies, which are far outside the validity regions of the ray theories. On the other hand, the quasi–isotropic approximations applied here generate unnecessary additional errors and should be avoided when using the coupling ray theory.

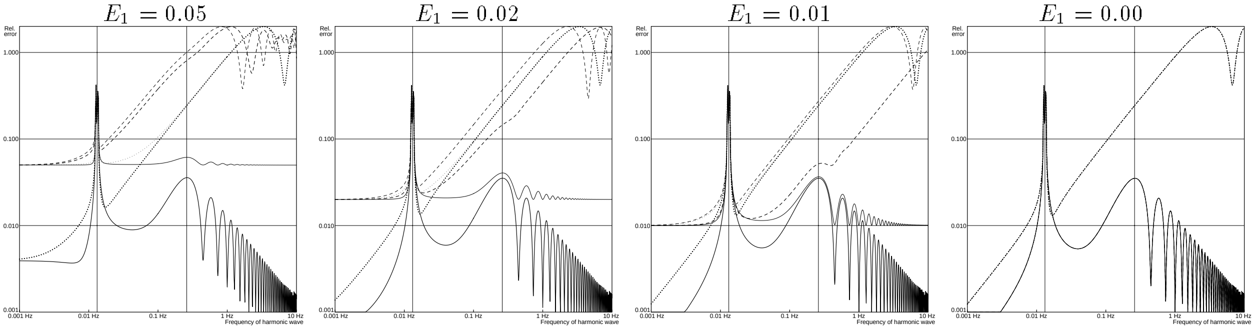

Figure 2: Analogous to Figure 1, but for different oblique twisted crystal models obtained by modifying rotation angle E_1 in data (16). Whereas the model for Figure 1 is given by $E_1 = 0.10$, the differences plotted here correspond to $E_1 = 0.05$ (top left), $E_1 = 0.02$ (top right), $E_1 = 0.01$ (bottom left) and $E_1 = 0.00$ (bottom right). Only the quasi–isotropic approximation of travel times can be demonstrated in the simplified twisted crystal model ($E_1 = 0.00$).

The error of the *quasi–isotropic projection of the Green tensor* (Section 2.2) can be estimated simply, is frequency–independent, and corresponds to the rotation angle in this model, see (16). This frequency–independent error may be expected to be proportional to the degree of anisotropy in generally anisotropic models on average.
The *quasi–isotropic approximation of the Christoffel matrix* (Section 2.3) generates the travel–time error. The wavefield error is then proportional to the frequency. The error is proportional to the square of the angle between the correct S–wave polarization plane and the reference S–wave polarization plane, in this case to the square of the rotation angle. The wavefield error, proportional to the frequency, may thus be expected to be proportional to the square of the degree of anisotropy in generally anisotropic models on average. The quasi–isotropic approximation of the Christoffel matrix seems to be the worst quasi–isotropic approximation.
The *quasi–isotropic perturbation of travel times* (Section 2.4) generates the travel–time error. The wavefield error is then proportional to the frequency. The error is proportional to the square of the difference between the anisotropic model and the reference isotropic model. The quasi–isotropic approximation of travel time thus depends on the reference velocity. The wavefield error, proportional to the frequency, may be expected to be proportional to the square of the degree of anisotropy in generally anisotropic models on average.
The *common ray approximations* (Section 2.1) have no impact on the coupling ray theory in this model. The inaccuracy due to the common ray approximations have been studied by (Klimeš & Bulant 2002).
The behaviour of the errors due to the quasi–isotropic approximations of the coupling ray theory when decreasing rotation angle E_1 in (16) towards the simplified twisted crystal model, used for comparison with the isotropic ray theory and anisotropic ray theory by Bulant, Klimeš & Pšenčík (1999; 2000), is demonstrated in Figure 2.

5. Conclusions

When using the coupling ray theory, the quasi–isotropic projection of the Green tensor, the quasi–isotropic approximation of the Christoffel matrix, and the quasi–isotropic approximation of travel times should be avoided.

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