RAY-CENTRED COORDINATE SYSTEMS IN ANISOTROPIC MEDIA

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ABSTRACT

Whereas the ray–centred coordinates for isotropic media by Popov and Pšenčík are uniquely defined by the selection of the basis vectors at one point along the ray, there is considerable freedom in selecting the ray–centred coordinates for anisotropic media. We describe the properties common to all ray–centred coordinate systems for anisotropic media and general conditions, which may be imposed on the basis vectors. We then discuss six different particular choices of ray–centred coordinates in an anisotropic medium. This overview may be useful in choosing the ray–centred coordinates best suited for a particular application. The equations are derived for a general homogeneous Hamiltonian of an arbitrary degree and are thus applicable both to the anisotropic–ray–theory rays and anisotropic common S–wave rays.

Key words: Hamiltonian ray tracing, dynamic ray tracing, ray–centred coordinates, anisotropy, heterogeneous media

1. INTRODUCTION

Rays may be interpreted as geodesics with respect to the "wave–propagation" metric. In Riemannian normal coordinates, a geodesic represents one coordinate axis. At each point of this axial geodesic, the other coordinates are chosen along the surface generated by the geodesics perpendicular to the axial geodesic. In ray–centred coordinates, a geodesic with respect to the wave–propagation metric again represents one coordinate axis. The only difference is that the perpendicular geodesics are replaced by geodesics with respect to another, Cartesian metric, which are still perpendicular to the axial geodesic with respect to the wave–propagation metric.

Riemannian normal coordinates are used in the general theory of relativity, and were applied by *Babich (1968)* and *Babich and Buldyrev (1972)* to Gaussian beams.

Ray–centred coordinates were applied to the polarization of electromagnetic waves in isotropic media by *Luneburg (1944)*, to Gaussian beams in isotropic media by *Kirpichnikova (1971)*, to dynamic ray tracing in isotropic media by *Popov and Pšenčík (1978a, 1978b)*, and then to dynamic ray tracing in anisotropic media by *Hanyga (1982)*, *Kendall et al. (1992)* and *Klimeš (1994)*.

Whereas the ray–centred coordinates for isotropic media by *Popov and Pšenčík (1978a, 1978b)* are uniquely defined by the selection of the basis vectors at one point along the ray, there is considerable freedom in selecting the ray–centred coordinates for anisotropic media. The choice of the ray–centred coordinate system considerably influences dynamic ray tracing in ray–centred coordinates. In this paper, we discuss various possible choices of the ray–centred coordinate system corresponding to a general homogeneous Hamiltonian of an arbitrary degree in an anisotropic medium. Depending on the choice of the Hamiltonian, ray–centred coordinates may be centred at anisotropic–ray–theory rays or at anisotropic common S–wave rays (*Klimeš, 2006*).

After introducing the Hamiltonian and its partial phase–space derivatives in Section 2, we define the ray–centred coordinates in Section 3. In Section 4, the properties common to all ray–centred coordinate systems for anisotropic media and general conditions, which may be imposed on the basis vectors, are discussed. Six different particular choices of ray–centred coordinates in an anisotropic medium are then described in Section 5. This overview may be useful in choosing the ray–centred coordinates best suited for a particular application.

The Einstein summation over the pairs of identical indices $a, b, c, ... = 1, 2, 3$ or $A, B, C, ... = 1, 2$ is used throughout this paper.

2. HAMILTONIAN AND ITS PARTIAL PHASE–SPACE DERIVATIVES

We consider a general homogeneous Hamiltonian of arbitrary degree N. According to Euler's theorem on homogeneous functions, the parameter along rays is then proportional to the travel time. We thus choose the value of the Hamiltonian in the stationary Hamilton–Jacobi equation so that the parameter along rays is equal to the travel time,

$$H(x^m, \tau_{,n}) = \frac{1}{N} \quad , \tag{1}$$

where N is the degree of Hamiltonian $H(x^m, p_n)$, homogeneous with respect to slowness vector p_i.

For the calculation of the first–order and second–order partial phase–space derivatives $H_{,i}$, $H^{,i}$, $H_{,ij}$, $H_{,i}^{,j}$, $H^{,ij}$ of this Hamiltonian, both for anisotropic–ray–theory rays and for anisotropic common S–wave rays, refer to *Klimeš (2006)*. Throughout this paper, a subscript following a comma denotes the partial spatial derivative and a superscript following a comma denotes the partial derivative with respect to the slowness vector,

$$H_{,i} \equiv \frac{\partial H}{\partial x^i} \quad , \qquad H^{,i} \equiv \frac{\partial H}{\partial p_i} \quad . \tag{2}$$

The Hamiltonian ray tracing equations then read

$$\dot{x}^k = H^{,k} \quad , \tag{3}$$

$$\dot{p}_k = -H_{,k} \quad . \tag{4}$$

3. RAY–CENTRED COORDINATES IN ANISOTROPIC MEDIA

The q^3–coordinate line of ray–centred coordinates q^m is the axial ray parametrized by parameter

$$q^3 = q_0^3 + \int_{\tau_0}^{\tau} w \, \mathrm{d}\tau \quad . \tag{5}$$

For example, the parameter may be travel time $q^3 = \tau$ if $w = 1$, or arclength $q^3 = s$ if w is the ray velocity, $w = \sqrt{H^{,i}H^{,i}}$. The q^A coordinates lie, for fixed q^3, in the wavefront tangent plane. The Cartesian coordinates corresponding to q^m are

$$x^i = x_0^i(q^3) + t_M^i(q^3)\, q^M \quad , \tag{6}$$

where $x_0^i(q^3)$ are points of the axial ray. This is both the definition of the ray–centred coordinates, and the transformation equation from the ray–centred to Cartesian coordinates. The transformation matrices, taken at the central ray, are

$$t_m^i = \frac{\partial x^i}{\partial q^m} \quad , \qquad \hat{t}_i^m = \frac{\partial q^m}{\partial x^i} \quad . \tag{7}$$

Transformation matrices (7) are related by equations

$$t_M^i = h_M^i \quad , \qquad t_3^i = w^{-1} h_3^i \quad , \qquad \hat{t}_i^M = \hat{h}_i^M \quad , \qquad \hat{t}_i^3 = w \hat{h}_i^3 \tag{8}$$

to analogous transformation matrices h_m^i and $\hat{h}_i^m$, formally introduced for $w=1$. In this paper, dynamic ray tracing in ray–centred coordinates is restricted to coordinates q^1 and q^2 only. The choice of q^3 then does not influence dynamic ray tracing. We shall thus substitute h_m^i and $\hat{h}_i^m$ for t_m^i and $\hat{t}_i^m$ hereinafter.

The "contravariant basis vectors" h_1^i, h_2^i, h_3^i and the "covariant basis vectors" $\hat{h}_i^1$, $\hat{h}_i^2$, $\hat{h}_i^3$ of the ray–centred coordinate system are defined by equations (*Klimeš, 1994, eq. 23*)

$$h_3^i = H^{,i} \quad , \tag{9}$$

$$\hat{h}_i^3 = p_i \quad , \tag{10}$$

$$h_a^i \hat{h}_i^b = \delta_a^b \quad . \tag{11}$$

Equation (11) may also be expressed as

$$h_a^i \hat{h}_j^a = \delta_i^j \quad . \tag{12}$$

The first two contravariant basis vectors h_1^i, h_2^i of the ray–centred coordinate system are tangent to the wavefront, the third contravariant basis vector h_3^i is tangent to the ray. The first two covariant basis vectors $\hat{h}_i^1$, $\hat{h}_i^2$ of the ray–centred coordinate system are perpendicular to the ray, the third covariant basis vector $\hat{h}_i^3$ is perpendicular to the wavefront. Whereas basis vectors h_3^i and $\hat{h}_i^3$ are uniquely determined by equations (9) and (10), we may either choose the contravariant basis vectors h_1^i and

h_2^i arbitrarily in the wavefront tangent plane, or choose the covariant basis vectors $\hat{h}_i^1$ and $\hat{h}_i^2$ arbitrarily in the plane perpendicular to the ray velocity vector h_3^i.

If the contravariant basis vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ are known, the covariant basis vectors $\hat{\mathbf{h}}^1$ and $\hat{\mathbf{h}}^2$ are given by relations

$$\hat{\mathbf{h}}^1 = \frac{\mathbf{h}_3 \times \mathbf{h}_2}{\mathbf{h}_1^{\mathrm{T}}(\mathbf{h}_3 \times \mathbf{h}_2)} \quad , \tag{13}$$

$$\hat{\mathbf{h}}^2 = \frac{\mathbf{h}_3 \times \mathbf{h}_1}{\mathbf{h}_2^{\mathrm{T}}(\mathbf{h}_3 \times \mathbf{h}_1)} \quad . \tag{14}$$

If the covariant basis vectors $\hat{\mathbf{h}}^1$ and $\hat{\mathbf{h}}^2$ are known, the contravariant basis vectors $\mathbf{h}_1$ and $\mathbf{h}_1$ are given by relations

$$\mathbf{h}_1 = \frac{\mathbf{p} \times \hat{\mathbf{h}}^2}{\hat{\mathbf{h}}_1^{\mathrm{T}}(\mathbf{p} \times \hat{\mathbf{h}}^2)} \quad , \tag{15}$$

$$\mathbf{h}_2 = \frac{\mathbf{p} \times \hat{\mathbf{h}}^1}{\hat{\mathbf{h}}_2^{\mathrm{T}}(\mathbf{p} \times \hat{\mathbf{h}}^1)} \quad . \tag{16}$$

The 4×4 dynamic ray tracing system in ray–centred coordinates reads (*Klimeš, 1994, eq. 60*)

$$\dot{Q}^M_{(q)} = H^{(q),M}_{,N} Q^N_{(q)} + H^{,MN}_{(q)} P^{(q)}_N \quad , \tag{17}$$

$$\dot{P}^{(q)}_M = -H^{(q)}_{,MN} Q^N_{(q)} - H^{(q),N}_{,M} P^{(q)}_N \quad , \tag{18}$$

where

$$Q^M_{(q)} = \frac{\partial q^M}{\partial \gamma} \quad , \tag{19}$$

$$P^{(q)}_M = \frac{\partial}{\partial \gamma}\left(\frac{\partial \tau}{\partial q^M}\right) \quad . \tag{20}$$

The second–order phase–space derivatives of the Hamiltonian in ray–centred coordinates are given by relations (*Klimeš, 1994, eqs. 51a, 51b, 51c*; *Červený, 2001, eq. 4.2.78*)

$$H^{(q)}_{,AB} = h^j_A [H_{,jk} - (N-1) H_{,j} H_{,k}] h^k_B \quad , \tag{21}$$

$$H^{(q),N}_{,M} = h^j_M H_{,j}^{,k} \hat{h}^N_k - d^N_M \quad , \tag{22}$$

$$H^{,MN} = \hat{h}^M_j H^{,jk} \hat{h}^N_k \quad . \tag{23}$$

Nine coefficients

$$d^a_b = \hat{h}^a_i \dot{h}^i_b = -h^i_b \dot{\hat{h}}^a_i \tag{24}$$

describe the evolution of the ray–centred coordinates along the ray,

$$\dot{h}^i_b = h^i_a d^a_b \quad , \tag{25}$$

$$\dot{\hat{h}}^a_i = -\hat{h}^b_i d^a_b \quad . \tag{26}$$

The four coefficients d^N_M in equation (22) thus depend on the choice of basis vectors h_1^i, h_2^i (or $\hat{h}_i^1$, $\hat{h}_i^2$) along the ray, which will be discussed in the next sections.

On the other hand, the choice of the four coefficients d^N_M uniquely determines the evolution of the basis vectors of the ray–centred coordinate system along the ray if the basis vectors are given at the initial point of the ray, because the remaining five coefficients d^N_3, d^3_M, d^3_3 are determined by definition (9) and (10). Evolution equations

$$\dot{h}^i_3 = H^{,i}_{,j} H^{,j} - H^{,ij} H_{,j} \quad , \tag{27}$$

$$\dot{\hat{h}}^3_i = -H_{,i} \quad , \tag{28}$$

follow from definition (9) and (10) and from ray tracing system (3) and (4). Coefficients d^a_3 and d^3_b are thus uniquely determined,

$$d^a_3 = \hat{h}^a_i (H^{,i}_{,j} H^{,j} - H^{,ij} H_{,j}) \quad , \tag{29}$$

$$d^3_b = H_{,i} h^i_b \quad , \tag{30}$$

$$d^3_3 = H_{,i} H^{,i} \quad . \tag{31}$$

4. GENERAL NOTES ON THE BASIS VECTORS OF THE RAY–CENTRED COORDINATE SYSTEM

There are many possibilities in choosing basis vectors h^i_1, h^i_2 (or $\hat{h}^1_i$, $\hat{h}^2_i$) along the ray in an anisotropic medium (*Hanyga, 1982*; *Kendall et al., 1992*; *Klimeš, 1994*). Many particular choices have their own advantages for particular applications. Sections 4.1 and 4.2 include some general notes related to the basis vectors. Six particular choices of the basis vectors are then listed in Section 5.

4.1. Notes on the "orthonormality" of the basis vectors

Four "free" coefficients d^A_B characterize a particular choice of the ray–centred coordinate system. Three coefficients d^1_1, d^2_2 and $\frac{1}{2}(d^1_2 + d^2_1)$, forming the symmetric part of the 2×2 matrix d^A_B, are determined by the "orthonormality relations" between h^i_1 and h^i_2. One coefficient $\frac{1}{2}(d^1_2 - d^2_1)$, forming the skew part of the 2×2 matrix d^A_B, characterizes the torsion of the ray–centred coordinate system. Under the orthonormality relations we understand relations

$$h^i_A g_{ij} h^j_B = \delta_{AB} \quad , \tag{32}$$

or

$$\hat{h}^A_i g^{ij} \hat{h}^B_j = \delta^{AB} \quad , \tag{33}$$

where g_{ij} or g^{ij} is some metric tensor. Differentiation of (32) and (33) yields

$$\dot{h}^i_A g_{ij} h^j_B + h^i_A g_{ij} \dot{h}^j_B = -h^i_A \dot{g}_{ij} h^j_B \quad , \tag{34}$$

and

$$\dot{\hat{h}}^A_i g^{ij} \hat{h}^B_j + \hat{h}^A_i g^{ij} \dot{\hat{h}}^B_j = -\hat{h}^A_i \dot{g}^{ij} \hat{h}^B_j \quad . \tag{35}$$

Then

$$d^A_B + d^B_A = -d^3_A h^i_3 g_{ij} h^j_B - d^3_B h^i_3 g_{ij} h^j_A - h^i_A \dot{g}_{ij} h^j_B \quad , \tag{36}$$

and

$$d^A_B + d^B_A = -d^A_3 \hat{h}^3_i g^{ij} \hat{h}^B_j - d^B_3 \hat{h}^3_i g^{ij} \hat{h}^B_j + \hat{h}^A_i \dot{g}^{ij} \hat{h}^B_j \quad . \tag{37}$$

4.2. Notes on the evolution equations for the basis vectors

The evolution differential equations for the basis vectors are often expressed in the form

$$\dot{h}^i_A = D^i_j h^j_A \tag{38}$$

for contravariant basis vectors h^j_1, h^j_2, or in the form

$$\dot{\hat{h}}^A_i = \hat{D}^j_i \hat{h}^A_j \tag{39}$$

for covariant basis vectors $\hat{h}^1_j$, $\hat{h}^2_j$. Since we do not expect to use equations (38) or (39) here for basis vectors h^j_3 or $\hat{h}^3_j$, respectively, coefficients D^i_j and $\hat{D}^j_i$ are not uniquely determined by the basis vectors. Equations (38) or (39) are invariant with respect to gauge transformations

$$D^i_j \longrightarrow D^i_j + \xi^i p_j \quad , \qquad \hat{D}^j_i \longrightarrow \hat{D}^j_i + \hat{\xi}_i H^{,j} \quad , \tag{40}$$

where ξ^i and $\hat{\xi}_i$ are arbitrary vectors given along the ray.

Assume now that we have equation (38) for h^i_A and wish to express equation (39) for $\hat{h}^A_i$, or vice versa. The differentiation of equation (11) along the ray yields

$$\dot{h}^i_a \hat{h}^b_i + h^i_a \dot{\hat{h}}^b_i = 0 \quad , \tag{41}$$

and consequently

$$\dot{h}^i_a = -h^i_b \dot{\hat{h}}^b_j h^j_a \quad , \tag{42}$$

see (12). Inserting (39) for $\dot{\hat{h}}^B_j$,

$$\dot{h}^i_a = -h^i_B \hat{D}^k_j \hat{h}^B_k h^j_a - h^i_3 \dot{\hat{h}}^3_j h^j_a \quad . \tag{43}$$

Considering (9)–(11) and (28),

$$\dot{h}^i_a = -(\delta^i_k - H^{,i} p_k) \hat{D}^k_j h^j_a + H^{,i} H_{,j} h^j_a \quad . \tag{44}$$

The coefficients D^i_j derived from $\hat{D}^i_j$ are then

$$D^i_j = -\hat{D}^i_j + H^{,i}(\hat{D}^k_j p_k + H_{,j}) + \xi^i p_j \quad , \tag{45}$$

where ξ^i is an arbitrary vector given along the ray. Analogously,

$$\dot{\hat{h}}^a_i = -\hat{h}^b_i \dot{h}^j_b \hat{h}^a_j \quad . \tag{46}$$

Inserting (38) for $\dot{h}^j_B$,

$$\dot{\hat{h}}^a_i = -\hat{h}^B_i D^j_k h^k_B \hat{h}^a_j - \hat{h}^3_i \dot{h}^j_3 \hat{h}^a_j \quad . \tag{47}$$

Considering (10)–(13),

$$\dot{\hat{h}}^a_i = -(\delta^k_i - p_i H^{,k}) D^j_k \hat{h}^a_j - p_i \dot{h}^j_3 \hat{h}^a_j \quad . \tag{48}$$

The coefficients $\hat{D}^j_i$ derived from D^j_i are then

$$\hat{D}^j_i = -D^j_i + p_i (D^j_k H^{,k} - \dot{h}^j_3) + \hat{\xi}_i H^{,j} \quad , \tag{49}$$

or, inserting (27),

$$\hat{D}^j_i = -D^j_i + p_i [(D^j_k H^{,k} - H^{,j}_{,k}) H^{,k} + H^{,jk} H_{,k}] + \hat{\xi}_i H^{,j} \quad , \tag{50}$$

where $\hat{\xi}_i$ is an arbitrary vector given along the ray.

The relations between coefficients d^A_B, D^i_j and $\hat{D}^j_i$ follow from definitions (24), (38) and (39),

$$d^A_B = \hat{h}^A_i D^i_j h^j_B = -h^i_B \hat{D}^j_i \hat{h}^A_j \quad , \tag{51}$$

$$D^i_j = h^i_a d^a_B \hat{h}^B_j + \xi^i p_j \quad , \tag{52}$$

$$\hat{D}^j_i = -\hat{h}^b_i d^A_b h^j_A + \hat{\xi}_i H^{,j} \quad , \tag{53}$$

where ξ^i and $\hat{\xi}_i$ are arbitrary vectors given along the ray.

5. PARTICULAR RAY–CENTRED COORDINATE SYSTEMS

In Sections 5.1 to 5.7, we briefly list six particular choices of the basis vectors. In Section 5.1, the basis vectors are determined locally from given constant vectors (*Hanyga, 1982*). The choice described in Sections 5.2 and 5.3 requires the third–order phase–space derivatives $H^{,ij}_{,k}$ and $H^{,ijk}$ of the Hamiltonian to be calculated. The choice described in Section 5.5 requires the second–order phase–space derivatives $H^{,i}_{,j}$ and $H^{,ij}$ of the Hamiltonian to be calculated. The choice described in Section 5.6 requires the second–order phase–space derivatives $H^{,i}_{,j}$ of the Hamiltonian to be calculated. The choices described in Sections 5.4 and 5.7 require the first–order phase–space derivatives of the Hamiltonian only. The choice described in Section 5.4 is the choice preferred by *Klimeš (2006)* for the modification of the isotropic "complete ray tracing" algorithm (*Červený et al., 1988*).

5.1. Wavefront–tangent and ray–normal projections of constant vectors

Hanyga (1982) introduced projection operator

$$P^i_j = \delta^i_j - h^i_3 p_j \tag{54}$$

with respect to the wave–propagation metric, projecting contravariant vectors onto the wavefront tangent plane and covariant vectors onto the plane normal to the ray. Note that projection operator

$$h^i_3 p_j \tag{55}$$

projects contravariant vectors onto the ray tangent and covariant vectors onto the wavefront normal, both with respect to the wave–propagation metric.

Contravariant basis vectors of the ray–centred coordinate system may then be chosen as the projections of two constant vectors e^i_1 and e^i_2 onto the wavefront tangent plane,

$$h^i_A = P^i_j e^j_A \quad . \tag{56}$$

Note that *Hanyga (1982)* suggested choosing e^i_1 and e^i_2 as the unit vectors in the direction of two selected axes of the three Cartesian coordinate axes. The corresponding covariant basis vectors $\hat{\mathbf{h}}^1$ and $\hat{\mathbf{h}}^2$ may be calculated using relations (13) and (14). Differentiating equation (56) with projection operator (54) along the ray, and considering definition (24) of d^a_b and relation (11), we obtain equation

$$d^A_B = -\hat{h}^A_i \dot{h}^i_3 p_j e^j_B \quad . \tag{57}$$

Similarly, we may select two constant vectors $\hat{e}^1_i$ and $\hat{e}^2_i$ and project them onto the plane perpendicular to the ray,

$$\hat{h}^A_i = P^j_i \hat{e}^A_j \quad . \tag{58}$$

The corresponding contravariant basis vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ may be calculated using relations (15) and (16). Differentiating equation (58) with projection operator (54) along the ray, and considering definition (24) of d^a_b and relation (11), we obtain equation

$$d^A_B = h^i_B \dot{p}_i h^j_3 \hat{e}^A_j \quad . \tag{59}$$

The advantage of this option is the possibility of calculating the basis vectors of the ray–centred coordinate system locally, without solving an ordinary differential equation along the ray. Unfortunately, for each couple of constant vectors e^i_1 and e^i_2 ($\hat{e}^1_i$ and $\hat{e}^2_i$), this definition of the basis vectors of the ray–centred coordinate system becomes degenerate if the ray–velocity vector (slowness vector) is not sufficiently deviated from the plane defined by vectors e^i_1 and e^i_2 ($\hat{e}^1_i$ and $\hat{e}^2_i$). We then have to switch to another couple of constant vectors to avoid the degeneracy. The experience with similar switching of constant reference vectors for determining the basis vectors of the ray–centred coordinate system in isotropic media definitely led *Červený et al. (1988)* to abandon this option and prefer the numerical calculation of the basis vectors by numerical integration along the ray. On approaching isotropy, this ray–centred coordinate system does not converge to the ray–centred coordinate system for isotropic media.

5.2. Basis vectors orthonormal with respect to the wave–propagation metric

The Finslerian contravariant wave–propagation metric tensor is given by the second–order derivatives of the homogeneous Hamiltonian

$$\widetilde{H} = \tfrac{1}{2}\,(NH)^{\frac{2}{N}} \tag{60}$$

of the second degree with respect to the slowness vector,

$$G^{ij} = \widetilde{H}^{,ij} = \tfrac{1}{2}[(NH)^{\frac{2}{N}}]^{,ij} = (NH)^{\frac{2-N}{N}}[H^{,ij} + \tfrac{2-N}{NH}H^{,i}H^{,j}] \quad . \tag{61}$$

Since $H = 1/N$ along the rays,

$$G^{ij} = H^{,ij} + (2-N)H^{,i}H^{,j} \tag{62}$$

along the Hamiltonian hypersurface in phase space. If relation

$$h^i_a = G^{ij}\hat{h}^a_j \quad , \tag{63}$$

which always holds for $a = 3$, also applies to $a = 1,2$, the orthonormality relation

$$\hat{h}^a_i G^{ij}\hat{h}^b_j = \delta^{ab} \quad , \tag{64}$$

following from (11) and (63), applies to all a, b, not only to $a = 3$ or $b = 3$. Then also

$$\hat{h}^a_i = G_{ij}h^j_a \tag{65}$$

and

$$h^i_a G_{ij} h^j_b = \delta_{ab} \quad , \tag{66}$$

where the covariant metric tensor G_{ij} is the matrix inverse to the contravariant metric tensor G^{ij},

$$G_{ij}G^{jk} = \delta^k_i \quad . \tag{67}$$

Differentiating equation (64) along the ray and considering definition (24) of d^a_b, we obtain equation

$$d^a_b + d^b_a = \hat{h}^a_i \dot{G}^{ij}\hat{h}^b_j \quad , \tag{68}$$

where

$$\dot{G}^{ij} = G^{ij}_{,k}H^{,k} - G^{ij,k}H_{,k} \quad . \tag{69}$$

For $a,b = 1,2$, equation (68) is a special case of equation (37) with $g^{ij} = G^{ij}$ and orthonormality relations (64). Coefficient $\frac{1}{2}(d^1_2 - d^2_1)$ may be obtained from additional conditions, e.g., from the additional postulates of Cartan imposed on the *parallel displacement* along a ray (*Rund, 1959, sec. 3.1., conditions III*). Refer to Section 5.3 for a brief summary of the parallel displacement.

Note that tensor

$$G^{ij,k} = \widetilde{H}^{,ijk} = \tfrac{1}{2}[(NH)^{\frac{2}{N}}]^{,ijk} \tag{70}$$

in equation (69) is symmetric with respect to all three superscripts, and defines the contravariant *Cartan torsion tensor*

$$C^{ijk} = -\tfrac{1}{2}G^{ij,k} \tag{71}$$

(*Rund, 1959, eq. 3.1.11*). The Cartan torsion tensor satisfies identity

$$C^{ijk} p_k = 0 \tag{72}$$

(*Rund, 1959, eq. 3.1.12*), which follows from the zero–degree homogeneity of the metric tensor with respect to the slowness vector.

Relation for $G^{ij,k}$ in terms of the derivatives of a general homogeneous Hamiltonian may be obtained by differentiating equation (61) with respect to p_k,

$$G^{ij,k} = (NH)^{\frac{2-N}{N}} \left[H^{,ijk} + \tfrac{2-N}{NH} \left(H^{,i} H^{,jk} + H^{,j} H^{,ik} + H^{,k} H^{,ij} + \tfrac{2-2N}{NH} H^{,i} H^{,j} H^{,k} \right) \right] . \tag{73}$$

Since $H = 1/N$ along the rays,

$$G^{ij,k} = H^{,ijk} + (2-N)[H^{,i} H^{,jk} + H^{,j} H^{,ik} + H^{,k} H^{,ij} + (2-2N) H^{,i} H^{,j} H^{,k}] . \tag{74}$$

The analogous relation for $G^{ij}_{,k}$ may be obtained by differentiating equation (61) with respect to x^k,

$$G^{ij}_{,k} = (NH)^{\frac{2-N}{N}} [H^{,ij}_{,k} + \tfrac{2-N}{NH} (H^{,ij} H_{,k} + H^{,i}_{,k} H^{,j} + H^{,i} H^{,j}_{,k} + \tfrac{2-2N}{NH} H^{,i} H^{,j} H_{,k})] . \tag{75}$$

Since $H = 1/N$ along the rays,

$$G^{ij}_{,k} = H^{,ij}_{,k} + (2-N)[H^{,ij} H_{,k} + H^{,i}_{,k} H^{,j} + H^{,i} H^{,j}_{,k} + (2-2N) H^{,i} H^{,j} H_{,k}] . \tag{76}$$

Inserting (69) with (74) and (76) into (68) and considering (11), for the symmetric part of the 2×2 matrix d^A_B we obtain equation

$$\tfrac{1}{2}(d^A_B + d^B_A) = \tfrac{1}{2} \hat{h}^A_i (H^{,ij}_{,k} H^{,k} - H^{,ijk} H_{,k}) \, \hat{h}^B_j \quad , \tag{77}$$

valid for a homogeneous Hamiltonian of an arbitrary degree.

A disadvantage of this option is the necessity to calculate the third–order partial phase–space derivatives $H^{,ij}_{,k}$ and $H^{,ijk}$ of the Hamiltonian. On approaching isotropy, this ray–centred coordinate system does not converge to the ray–centred coordinate system for isotropic media.

5.3. Finslerian parallel displacement along a ray

We shall consider only the parallel displacement along a ray, not along a general curve. We shall adopt the relevant equations of *Rund (1959)* without deriving them.

The parallel displacement of the contravariant basis vectors along the ray can be expressed in the form

$$\dot{h}^j_a = -C^{mjn} G_{mi} G_{nk} h^i_a \dot{h}^k_3 - P^j_{ik} h^i_a \dot{x}^k \tag{78}$$

(*Rund, 1959, eq. 3.1.3*). The analogous equation for the covariant basis vectors is

$$\dot{\hat{h}}^a_i = G_{im} C^{mjn} G_{nk} \hat{h}^a_j \dot{h}^k_3 + P^j_{ik} \hat{h}^a_j \dot{x}^k \quad . \tag{79}$$

Inserting (27) and considering (62), (65), (67) and (72), equations (78) and (79) read

$$\dot{h}^j_a = C^{mjk} G_{mi} h^i_a H_{,k} - (P^j_{ik} + C^{mjn} G_{mi} G_{nr} H^{,r}_{,k}) h^i_a \dot{x}^k \quad , \tag{80}$$

$$\dot{\hat{h}}_i^a = -G_{im}C^{mjk}\hat{h}_j^a H_{,k} + (P_{ik}^j + G_{im}C^{mjn}G_{nr}H_{,k}^{\;,r})\hat{h}_j^a \dot{x}^k \quad . \tag{81}$$

A relatively complex expression for P_{ik}^j is given by *Rund (1959, eq 3.1.30)*, but the expression for $P_{ik}^j \dot{x}^k$ can be considerably simplified using identity (72). *Rund*'s *(1959)* equation 3.1.30 then yields

$$P_{ik}^j \dot{x}^k = \gamma_{ik}^j \dot{x}^k \quad , \tag{82}$$

where γ_{ik}^j are defined for fixed $\dot{x}^m$ analogously as the Christoffel symbols in Riemannian geometry,

$$\gamma_{ik}^j = \frac{1}{2} G^{jm} \left(\left.\frac{\partial G_{km}}{\partial x^i}\right|_{\dot{x}^a} + \left.\frac{\partial G_{im}}{\partial x^k}\right|_{\dot{x}^a} - \left.\frac{\partial G_{ik}}{\partial x^m}\right|_{\dot{x}^a} \right) \tag{83}$$

(*Rund, 1959, eq. 2.2.3*). Since we calculate spatial derivatives for fixed p_m, we have to convert the spatial derivatives of G_{ij} for fixed $\dot{x}^m$ to the spatial derivatives for fixed p_m,

$$\left.\frac{\partial G_{ij}}{\partial x^k}\right|_{p_a} = \left.\frac{\partial G_{ij}}{\partial x^k}\right|_{\dot{x}^a} + \left.\frac{\partial G_{ij}}{\partial \dot{x}^m}\right|_{x^a} \left.\frac{\partial \dot{x}^m}{\partial x^k}\right|_{p_a} \quad . \tag{84}$$

Inserting

$$\left.\frac{\partial G_{ij}}{\partial \dot{x}^m}\right|_{x^a} = 2\, G_{ir}G_{js}G_{mn}C^{rsn} \tag{85}$$

(*Rund, 1959, eq. 3.1.11*) and (3) into (84), we arrive at transformation equation

$$\left.\frac{\partial G_{ij}}{\partial x^k}\right|_{\dot{x}^a} = \left.\frac{\partial G_{ij}}{\partial x^k}\right|_{p_a} - 2\, G_{ir}G_{js}G_{mn}C^{rsn}H_{,k}^{\;,m} \quad . \tag{86}$$

Inserting (86) into (83), we express γ_{ik}^j as

$$\gamma_{ik}^j = \Gamma_{ik}^j - G_{km}C^{mjr}G_{rs}H_{,i}^{\;,s} - G_{im}C^{mjr}G_{rs}H_{,k}^{\;,s} + G_{im}G_{kn}C^{mnr}G_{rs}H_{,l}^{\;,s}G^{lj} \;, \tag{87}$$

where Γ_{ik}^j are defined for fixed p_m analogously to the Christoffel symbols in Riemannian geometry,

$$\Gamma_{ik}^j = -\tfrac{1}{2}\left(G_{,i}^{jr}G_{rk} + G_{,k}^{jr}G_{ri} - G^{jl}G_{,l}^{rs}G_{ri}G_{sk} \right) \quad . \tag{88}$$

Considering (72) and $\dot{x}^i = G^{ij}p_j$, we may express (82) as

$$P_{ik}^j \dot{x}^k = \Gamma_{ik}^j \dot{x}^k - G_{im}C^{mjr}G_{rs}H_{,k}^{\;,s}\dot{x}^k \quad . \tag{89}$$

Using (89), we may simplify equations (80) and (81) to

$$\dot{h}_a^j = G_{im}C^{mjk}H_{,k}h_a^i - \Gamma_{ik}^j \dot{x}^k h_a^i \quad , \tag{90}$$

$$\dot{\hat{h}}_i^a = -G_{im}C^{mjk}H_{,k}\hat{h}_j^a + \Gamma_{ik}^j \dot{x}^k \hat{h}_j^a \quad . \tag{91}$$

Equation (88) multiplied by $\dot{x}^k = G^{ks}p_s$ reads

$$\Gamma_{ik}^j \dot{x}^k = -\tfrac{1}{2}\left(G_{,k}^{jr}G_{ri}G^{ks} + G_{,i}^{js} - G^{jl}G_{,l}^{rs}G_{ri} \right) p_s \quad . \tag{92}$$

Considering the homogeneity of the Hamiltonian of the second degree used to define metric tensor G^{ij},

$$\Gamma^j_{ik}\dot{x}^k = -\tfrac{1}{2}\left(G^{jr}_{,k}G_{ri}H^{,k} + \widetilde{H}^{,j}_{,i} - G^{jl}\widetilde{H}^{,r}_{,l}G_{ri}\right) \quad . \tag{93}$$

Inserting (90) or (91) into (24), we obtain

$$d^a_b = h^i_b G_{im} C^{mjk} H_{,k} \hat{h}^a_j - h^i_b \Gamma^j_{ik}\dot{x}^k \hat{h}^a_j \quad . \tag{94}$$

Inserting (71) and (93) into (94) and considering (65), we obtain

$$d^a_b = -\tfrac{1}{2}\hat{h}^a_i\hat{h}^b_j G^{ij,k}H_{,k} + \tfrac{1}{2}[G^{ij}_{,k}\hat{h}^a_j\hat{h}^b_i H^{,k} + \widetilde{H}^{,j}_{,i}(\hat{h}^a_j h^i_b - \hat{h}^b_j h^i_a)] \quad . \tag{95}$$

Converting the derivatives of the homogeneous Hamiltonian of the second degree to the derivatives of the homogeneous Hamiltonian of an arbitrary degree using expressions (74), (76) and

$$\widetilde{H}^{,i}_{,j} = H^{,i}_{,j}\,(NH)^{\frac{2}{N}-1} + (2-N)\;H^{,i}H_{,j}\;(NH)^{\frac{2}{N}-2} \quad , \tag{96}$$

we arrive at

$$\begin{aligned} d^A_B = &-\tfrac{1}{2}[\hat{h}^A_i\hat{h}^B_j H^{,ijk} + (2-N)\delta^A_B H^{,k}]H_{,k} \\ &+\tfrac{1}{2}[H^{,ij}_{,k}\hat{h}^A_i\hat{h}^B_j H^{,k} + (2-N)\delta^A_B H_{,k}H^{,k} + H^{,j}_{,i}(\hat{h}^A_j h^i_B - \hat{h}^B_j h^i_A)] \quad . \end{aligned} \tag{97}$$

Finally,

$$d^A_B = \tfrac{1}{2}\hat{h}^A_i\hat{h}^B_j(H^{,ij}_{,k}H^{,k} - H^{,ijk}H_{,k}) + \tfrac{1}{2}H^{,j}_{,i}(\hat{h}^A_j h^i_B - \hat{h}^B_j h^i_A) \quad . \tag{98}$$

The symmetric part of the 2×2 matrix d^A_B is already known from equation (77), but the skew part could not be derived without the parallel displacement. Note that the skew part of matrix (98) renders the 2×2 matrix of the mixed second–order derivatives (22) of the Hamiltonian in ray–centred coordinates symmetric. This choice, being a special case of options introduced in Section 5.2, requires the third–order partial phase–space derivatives $H^{,ij}_{,k}$ and $H^{,ijk}$ of the Hamiltonian to be calculated. On approaching isotropy, this ray–centred coordinate system does not converge to the ray–centred coordinate system for isotropic media.

5.4. Contravariant basis vectors $\mathbf{h}_1$, $\mathbf{h}_2$ orthonormal with respect to the Cartesian metric

Another option is to choose the contravariant basis vectors h^i_1 and h^i_2, which are perpendicular to the slowness vector, mutually orthonormal with respect to the Cartesian metric,

$$h^i_A h^i_B = \delta_{AB} \tag{99}$$

(*Klimeš, 2002, eq. 50*). The simplest way of keeping condition (99) satisfied along the ray is to choose the derivatives of both vectors in the direction of the slowness vector (*Červený, 2001, eq. 4.2.23*),

$$\dot{h}^i_A = -\frac{p_i}{p_r p_r}\dot{p}_j h^j_A \quad . \tag{100}$$

In terms of the Hamiltonian,

$$\dot{h}^i_A = \frac{p_i}{p_r p_r} H_{,j} h^j_A \quad . \tag{101}$$

Note that the right–hand side of equation (101) equals the wavefront–normal projection

$$\frac{p_i p_k}{p_r p_r}[H^{,k}_{,j} - (N-1)H^{,k}H_{,j}]h^j_A \tag{102}$$

(with respect to the Cartesian metric) of the right–hand side of equation (113), because $p_k H^{,k}_{,j} = N H_{,j}$ and $p_k H^{,k} = 1$.

It is sufficient to calculate contravariant basis vector $\mathbf{h}_1$ by solving equation (101) numerically. Vector $\mathbf{h}_2$ is then given by relation

$$\mathbf{h}_2 = \frac{\mathbf{p} \times \mathbf{h}_1}{|\mathbf{p} \times \mathbf{h}_1|} \quad , \tag{103}$$

and vectors $\hat{\mathbf{h}}^1$ and $\hat{\mathbf{h}}^2$ by relations (13) and (14).

The equation for the covariant basis vectors may be obtained from (101) using relation (50),

$$\dot{\hat{h}}^A_i = -(H_{,i} - p_i H^{,k} H_{,k}) \frac{p_j}{p_r p_r} \hat{h}^A_j - p_i (H^{,j}_{,k} H^{,k} - H^{,jk} H_{,k}) \hat{h}^A_j \quad . \tag{104}$$

The $2{\times}2$ submatrix of the $3{\times}3$ matrix (24) is

$$d^A_B = \frac{p_i \hat{h}^A_i}{p_r p_r} H_{,j} h^j_B \quad . \tag{105}$$

This is a very comfortable option for coding, because equation (101) contains only the first–order derivatives of the Hamiltonian. On approaching isotropy, this ray–centred coordinate system with $w = \sqrt{H^{,i} H^{,i}}$ in (5) converges to the ray–centred coordinate system for isotropic media.

This choice is very suitable for modifying the isotropic "complete ray tracing" algorithm by *Červený et al. (1988)* for anisotropic media.

5.5. Covariant basis vectors $\hat{\mathbf{h}}^1$, $\hat{\mathbf{h}}^2$ orthonormal with respect to the Cartesian metric

Another option is to choose the covariant basis vectors $\hat{h}^1_i$ and $\hat{h}^2_i$, which are perpendicular to the ray, mutually orthonormal with respect to the Cartesian metric,

$$\hat{h}^A_i \hat{h}^B_i = \delta^{AB} \tag{106}$$

(*Klimeš, 2002, eq. 39*). The simplest way of keeping condition (106) satisfied along the ray is to choose the derivatives of both vectors in the direction of ray–velocity vector $h^i_3 = H^{,i}$,

$$\dot{\hat{h}}^A_i = -\frac{H^{,i}}{H^{,r} H^{,r}} \dot{h}^j_3 \hat{h}^A_j \quad . \tag{107}$$

Inserting (27), we may express this equation in terms of the Hamiltonian,

$$\dot{\hat{h}}_i^A = -\frac{H^{,i}}{H^{,r}H^{,r}}(H^{,k}H^{,j}_{,k} - H_{,k}H^{,kj})\,\hat{h}_j^A \quad . \tag{108}$$

Note that the right–hand side of equation (108) is the ray–tangent projection

$$-\frac{H^{,i}H^{,k}}{H^{,r}H^{,r}}(H^{,j}_{,k} - p_k H_{,m}H^{,mj})\hat{h}_j^A \tag{109}$$

(with respect to the Cartesian metric) of the right–hand side of equation (114) because $H^{,k}p_k = 1$. It is sufficient to calculate covariant basis vector $\hat{\mathbf{h}}^1$ by solving equation (108) numerically. Vector $\hat{\mathbf{h}}^2$ is then given by relation

$$\hat{\mathbf{h}}_2 = \frac{\mathbf{h}_3 \times \hat{\mathbf{h}}_1}{|\mathbf{h}_3 \times \hat{\mathbf{h}}_1|} \quad , \tag{110}$$

and contravariant vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ by relations (15) and (16).

The equation for the contravariant basis vectors may be obtained from (107) using relation (45),

$$\dot{h}_A^i = \left[(\dot{h}_3^i - H^{,i}p_k\dot{h}_3^k)\frac{H^{,j}}{H^{,r}H^{,r}} + H^{,i}H_{,j}\right]h_A^j \quad . \tag{111}$$

The 2×2 submatrix of 3×3 matrix (24) is

$$d_B^A = \frac{H^{,i}h_B^i}{H^{,r}H^{,r}}\dot{h}_3^j\hat{h}_j^A \quad . \tag{112}$$

This option requires the first–order and second–order derivatives of the Hamiltonian, see (27). On approaching isotropy, this ray–centred coordinate system with $w = \sqrt{H^{,i}H^{,i}}$ in (5) converges to the ray–centred coordinate system for isotropic media.

5.6. Basis vectors making $H^{(q),N}_{,M}$ zero

Klimeš (1994, eqs. 52a and 52b) and *Červený (2001, eq. 4.2.86)* suggested the option making the mixed second–order derivatives (22) of the Hamiltonian in ray–centred coordinates zero,

$$\dot{h}_A^i = [H^{,i}_{,j} - (N-1)H^{,i}H_{,j}]h_A^j \quad , \tag{113}$$

$$\dot{\hat{h}}_i^A = -(H^{,j}_{,i} - p_i H_{,m}H^{,mj})\hat{h}_j^A \quad . \tag{114}$$

The 2×2 submatrix of the 3×3 matrix (24) is

$$d_B^A = \hat{h}_i^A H^{,i}_{,j}h_B^j \quad , \tag{115}$$

as follows from (22) with $H^{(q),N}_{,M} = 0$. Similarly as the preceding option, this option requires the first–order and second–order derivatives of the Hamiltonian. Since no orthonormality relations between vectors h_1^i and h_2^i (or $\hat{h}_j^1$ and $\hat{h}_j^2$) are yet known, we would have to calculate both contravariant vectors (or both covariant vectors) by numerical integration. On approaching isotropy, this ray–centred coordinate system with $w = \sqrt{H^{,i}H^{,i}}$ in (5) converges to the ray–centred coordinate system for isotropic media.

5.7. Basis vectors making d^A_B zero

Putting

$$d^A_B = 0 \quad , \tag{116}$$

equations (52) and (53) simplify to

$$D^i_j = H^{,i} d^3_B \hat{h}^B_j + \xi^i p_j \tag{117}$$

and

$$\hat{D}^j_i = -p_i d^A_3 h^j_A + \hat{\xi}_i H^{,j} \quad , \tag{118}$$

where d^A_3 and d^3_B are given by (29) and (30), and where ξ^i and $\hat{\xi}_i$ are arbitrary vectors given along the ray. Inserting (117) into (38), we obtain

$$\dot{h}^i_A = H^{,i} d^3_A \quad . \tag{119}$$

Inserting (118) into (39), we obtain

$$\dot{\hat{h}}^A_i = -p_i\, d^A_3 \quad . \tag{120}$$

Inserting (30) into (119), we arrive at

$$\dot{h}^i_A = H^{,i} H_{,j}\, h^j_A \quad , \tag{121}$$

which is the ray–tangent projection (with respect to the wave–propagation metric) of the right–hand side of equation (113) using projection operator (55). Inserting (29) into (120), we arrive at

$$\dot{\hat{h}}^A_i = -p_i \left(H^{,m} H^{,j}_{,m} - H_{,m} H^{,mj} \right) \hat{h}^A_j \quad , \tag{122}$$

which is the wavefront–normal projection (with respect to the wave–propagation metric) of the right–hand side of equation (114) using projection operator (55).

Equation (121) for the contravariant basis vectors contains only the first–order derivatives of the Hamiltonian, similarly as equation (101). Since no orthonormality relations between vectors h^i_1 and h^i_2 are yet known, we would have to calculate both contravariant basis vectors by numerical integration. On approaching isotropy, this ray–centred coordinate system with $w = \sqrt{H^{,i} H^{,i}}$ in (5) converges to the ray–centred coordinate system for isotropic media.

6. CONCLUSIONS

Whereas the ray–centred coordinates for isotropic media by *Popov and Pšenčík (1978a, 1978b)* are uniquely defined by the selection of the basis vectors at one point along the ray, there is considerable freedom in selecting the ray–centred coordinates for anisotropic media. We have discussed six different particular choices of ray–centred coordinates in an anisotropic medium. They have different properties. Four of them converge to the ray–centred coordinate system for isotropic media on approaching isotropy, two of them do not. The ray–centred coordinate system described in Section 5.4 is very suitable for modifying the isotropic "complete ray tracing" algorithm by *Červený et al. (1988)* for anisotropic media.

For additional information, including various papers, computer codes and data, refer to the consortium research project "Seismic Waves in Complex 3–D Structures" ("http://sw3d.mff.cuni.cz").

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