Numerical comparison of the isotropic–common–ray and anisotropic–common–ray approximations of the coupling ray theory

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Summary

In the common–ray approximation of the coupling ray theory, the S–wave traveltimes in the coupling equations are approximated by the first–order perturbation expansion from the common reference ray. The common–ray approximation eliminates problems with ray tracing through S-wave singularities and also considerably simplifies the numerical algorithm of the coupling ray theory for S waves, but may introduce errors in traveltimes due to the perturbation from the common reference ray. The errors of S–wave traveltimes due to the common–ray approximation may be approximated by the second–order terms in the perturbation expansion. The first–order and second–order terms in the perturbation expansion can be calculated by simple numerical quadratures along the common reference ray.

The common reference ray may be represented by the isotropic common ray or by the anisotropic common ray. The equations for the first–order and second–order perturbation expansions of traveltime from both the isotropic and anisotropic common ray to the anisotropic–ray–theory rays have already been derived. We numerically test the equations in three smooth 1–D velocity models of differing degrees of anisotropy.

We first compare the first–order and second–order perturbation expansions of traveltime from the isotropic common rays and from the anisotropic common rays to the anisotropic–ray–theory rays with the correct anisotropic–ray–theory traveltimes in order to illustrate the accuracy of the perturbation expansions and reliability of the error estimates.

Then, the synthetic seismograms obtained by the isotropic–common–ray and anisotropic–common–ray approximations of the coupling ray theory are compared with the more accurate coupling–ray–theory synthetic seismograms simulated by the second–order perturbation expansion of traveltime from the anisotropic common rays. In the numerical examples, the errors of the anisotropic–common–ray approximation of the coupling ray theory are considerably smaller than the errors of the isotropic–common–ray approximation.

Errors due to the common-ray approximations of the coupling ray theory

There are two different high–frequency asymptotic ray theories: the *isotropic ray theory* based on the assumption of equal velocities of both S–wave polarizations, and the *anisotropic ray theory* assuming both S–wave polarizations strictly decoupled. In the isotropic ray theory, the S–wave polarization vectors do not rotate about the ray, whereas in the anisotropic ray theory they coincide with the eigenvectors of the Christoffel matrix which may rotate rapidly about the ray.

In “weakly anisotropic” models, at moderate frequencies, the S–wave polarization tends to remain unrotated round the ray but is partly attracted by the rotation of the eigenvectors of the Christoffel matrix. The intensity of the attraction increases with frequency. This behaviour of the S–wave polarization is described by the *coupling ray theory* proposed by Coates & Chapman (1990). The frequency–dependent coupling ray theory is the generalization of both the zero–order isotropic and anisotropic ray theories and provides continuous transition between them. The coupling ray theory is applicable to S waves at all degrees of anisotropy, from isotropic velocity models to considerably anisotropic ones, and is particularly important at degrees of anisotropy and frequencies typical in seismic exploration and structural seismology on all scales. The numerical algorithm for calculating the frequency–dependent complex–valued S–wave polarization vectors of the coupling ray theory has been designed by Bulant & Klimeš (2002).

There are many commonly used *quasi–isotropic approximations* of the coupling ray theory (e.g., Pšenčík, 1998), which diminish the accuracy of the coupling ray theory both with increasing frequency and increasing degree of anisotropy. Refer to Bulant & Klimeš (2002; 2004) and to Klimeš & Bulant (2004) for the description of individual quasi–isotropic approximations and for the examples of their impact on synthetic seismograms. Most of these quasi–isotropic approximations can be avoided with minimal effort except for the *common–ray approximation* for S waves.

In the common–ray approximation, only one reference ray is traced for both anisotropic–ray–theory S waves, and both S–wave anisotropic–ray–theory travel times are approximated by the perturbation expansion from the common reference ray. Tracing of the common ray (isotropic or anisotropic) does not suffer from the problems with touching slowness surfacess which occure in anisotropic ray theory. The common–ray approximation thus considerably simplifies the coding of the coupling ray theory and also simplifies numerical calculations and makes them feasible even for the directions of equal S-wave eigenvalues, but may introduce errors in travel times due to the perturbation. These travel–time errors can deteriorate the coupling–ray–theory solution at high frequencies. It is thus of principal importance for numerical applications to estimate the travel–time errors due to the common–ray approximation, and then the related error of the wavefield.

In the common–ray approximation, the S–wave travel times are usually approximated by the first–order perturbation expansion from the common reference ray. The second–order terms in the perturbation expansion may be infinitely large in the vicinity of caustics, and thus cannot be used for the approximation of the S–wave travel times, but they may be used for the estimation of the errors due to the common–ray approximation.

The common reference rays have routinely been represented by the *isotropic common rays* calculated in the isotropic reference model. Klimeš & Bulant (2004) derived the equations for estimating the travel–time errors due to the *isotropic–common–ray approximation* by numerical quadrature along the isotropic common rays in smooth velocity models without interfaces. The accuracy of the isotropic–common–ray approximation can thus routinely be checked. In addition, the accuracy of the anisotropic–common–ray approximation can approximately be studied along isotropic common rays, without tracing the anisotropic common rays. If the error of the isotropic–common–ray approximation exceeds an acceptable limit, we can immediately decide whether the anisotropic–common–ray approximation would be sufficiently accurate, or whether the common–ray approximation is not acceptable and the two anisotropic–ray–theory S–wave rays should be traced as reference rays for the coupling ray theory.

The numerical results by Klimeš & Bulant (2004) suggested that the *anisotropic–common–ray approximation* is much more accurate than the isotropic–common–ray approximation. The *anisotropic common rays* are traced in the anisotropic model, using the averaged Hamiltonian of both anisotropic–ray–theory S waves (Bakker, 2002). The travel–time errors due to the anisotropic–common–ray approximation can be estimated by numerical quadrature along the anisotropic common rays in smooth velocity models without interfaces. The derivation for estimating the errors along the anisotropic common rays (Klimeš & Bulant, 2006) is based on the general equations for the second–order perturbations of travel time by Klimeš (2002), and is analogous to the derivation by Klimeš & Bulant (2004) for estimating the errors along the isotropic common rays. The equations for estimating the travel–time errors due to the common–ray approximations of the coupling ray theory for S waves are derived in a form suitable for dynamic ray tracing in ray–centred coordinates attached to the anisotropic common rays. The accuracy of the anisotropic–common–ray approximation can thus routinely be checked using the equations.

Numerical example

Two–point anisotropic common rays are traced in three 1-D models QI, QI2 and QI4 of differing degree of anisotropy. A vertically heterogeneous 1–D anisotropic model QI was provided by Pšenčík & Dellinger (2001, model WA rotated by 45°). To emphasize the effects of perturbations of travel time, new models with an increased degree of anisotropy have been derived from the QI model. The differences of the elastic moduli of models QI2 and QI4 from the elastic moduli of the reference isotropic model are exactly twice and four times larger than the differences of model QI, respectively. Two–point isotropic common rays are traced in the corresponding reference isotropic models. For the source-receiver configuration refer to Figure 1.

The first–order and second–order isotropic–common–ray and anisotropic–common–ray approximations of the anisotropic–ray–theory travel times are compared with the correct anisotropic–ray–theory travel times in order to demonstrate the errors of both the approximations and the applicability of the perturbation expansion, see Tables 1 and 2.

The coupling–ray–theory synthetic seismograms, corresponding to vertical force $\mathbf{F} = (0, 0, 100)^{\mathrm{T}}$, calculated using the isotropic–common–ray approximation and anisotropic–common–ray approximation in models QI, QI2 and QI4 are then compared with the synthetic seismograms calculated by the second–order approximation from the anisotropic common rays, see Figures 2 and 3. The source time function is the Gabor signal with reference frequency $f = 50$ Hz, band–pass filtered by a cosine filter given by frequencies 0 Hz, 5 Hz, 60 Hz and 100 Hz.

Conclusions

In using any kind of the common–ray approximation of the coupling ray theory, the travel–time errors due to the applied common–ray approximation should be evaluated. The travel–time errors due to the common–ray approximations can be estimated by numerical quadrature along the isotropic common rays using the equations proposed by Klimeš & Bulant (2004), or by numerical quadrature along the anisotropic common rays using the equations proposed by Klimeš & Bulant (2006). The numerical results suggest that the anisotropic–common–ray approximation is considerably more accurate than the isotropic–common–ray approximation. Whereas the first–order isotropic–common–ray approximation is considerably inaccurate in model QI4, see Figure 2 and Table 1, the first–order anisotropic–common–ray approximation performs well in all three models QI, QI2 and QI4, see Figure 3 and Table 2. For more details refer to Bulant & Klimeš (2007).

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Comparison of common–ray approximations of coupling ray theory

Model	ICR time	ICR linear terms		ICR quadratic terms		ICR remaining terms	
QI	0.443550	-0.002967	0.004117	0.000249	0.000001	0.000009	0.000001
QI2	0.443550	-0.005874	0.011837	0.000960	0.000001	0.000036	0.000005
QI4	0.443550	-0.011517	0.041462	0.003554	0.000711	0.000249	-0.000167

Table 1. Perturbation expansion of travel time from the isotropic common ray to the anisotropic–ray–theory rays for the receiver at the depth 0.29.
The *ICR time* is the reference travel time along the isotropic common ray traced in the reference isotropic model. The *ICR linear terms* are the linear terms in perturbation expansion of travel time from the isotropic common ray to the anisotropic–ray–theory rays and represent the travel–time corrections considered in the isotropic–common–ray approximation of the coupling ray theory. The *ICR quadratic terms* stand for the quadratic terms in the perturbation expansion, and represent the estimates of the errors due to the isotropic–common–ray approximation of the anisotropic–ray–theory travel times. The *ICR remaining terms* stand for the differences between the correct anisotropic–ray–theory travel times and the second–order perturbation expansion from the isotropic common ray, in order to illustrate the reliability of the error estimates. The ICR remaining terms represent both the inaccuracy of numerical ray tracing and the third–order and higher–order terms in the perturbation expansion.

Model	ACR time	ACR linear terms		ACR quadratic terms		ACR remaining terms	
QI	0.444183	-0.003415	0.003415	0.000070	0.000070	0.000003	0.000000
QI2	0.446788	-0.008366	0.008366	0.000245	0.000245	0.000006	-0.000007
QI4	0.460171	-0.024947	0.024947	0.000505	0.000505	0.000108	-0.000067

Table 2. Perturbation expansion of travel time from the anisotropic common ray to the anisotropic–ray–theory rays for the receiver at the depth 0.29.
The *ACR time* is the reference travel time along the anisotropic common ray. The *ACR linear terms* are the linear terms in perturbation expansion of travel time from the anisotropic common ray to the anisotropic–ray–theory rays, and represent the travel–time corrections considered in the anisotropic–common–ray approximation of the coupling ray theory. The *ACR quadratic terms* stand for quadratic terms in the perturbation expansion, and represent the estimates of the errors due to the anisotropic–common–ray approximation of the anisotropic–ray–theory travel times. The *ACR remaining terms* stand for the differences between the correct anisotropic–ray–theory travel times and the second–order perturbation expansion from the anisotropic common ray, in order to illustrate the reliability of the error estimates. The ACR remaining terms represent both the inaccuracy of numerical ray tracing and the third–order and higher–order terms in the perturbation expansion.

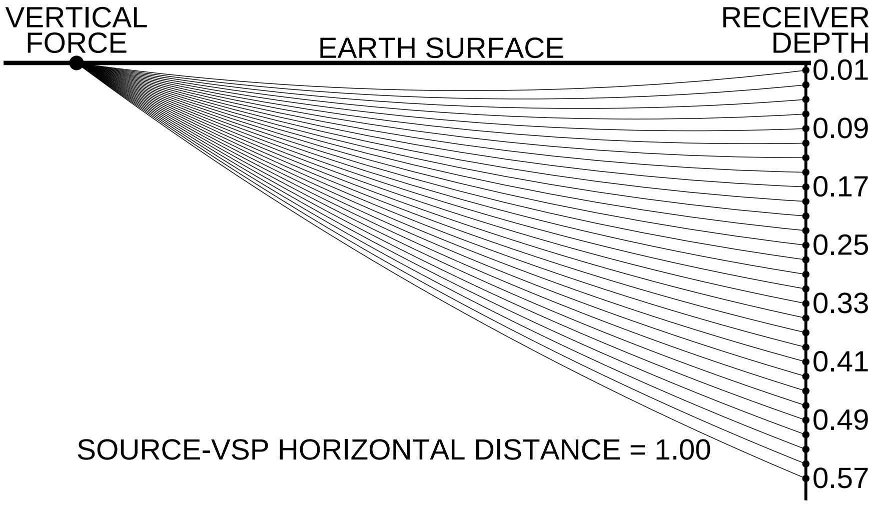

Figure 1. The source and receiver configuration in models QI, QI2 and QI4.

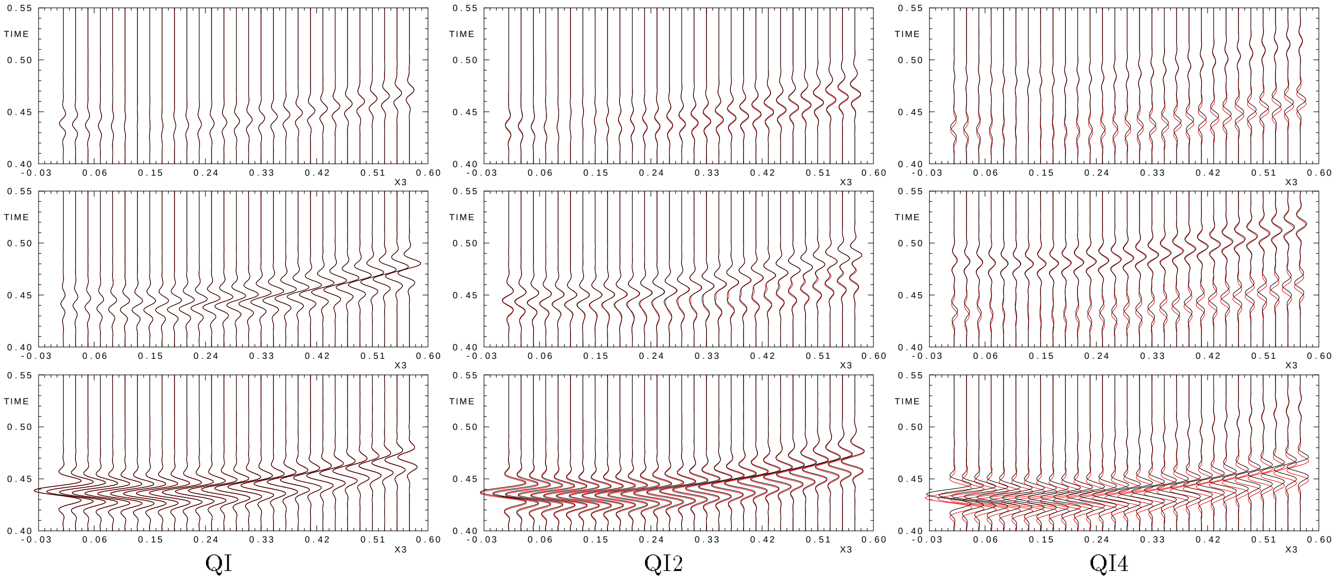

Figure 2. Inaccuracy of the isotropic–common–ray approximation of the coupling ray theory in models QI, QI2 and QI4. From top to bottom: the first (radial) component, the second (transverse) component, the third (vertical) component. **Black:** Coupling–ray–theory synthetic seismograms without common ray approximation, simulated by the second–order perturbation expansion of travel time from anisotropic common rays. For the errors of this simulation refer to the *ACR remaining terms* in Table 2. **Red:** Isotropic common ray approximation. The differences between the seismograms are negligible in model QI, small but already clearly visible in model QI2, and considerable in model QI4.

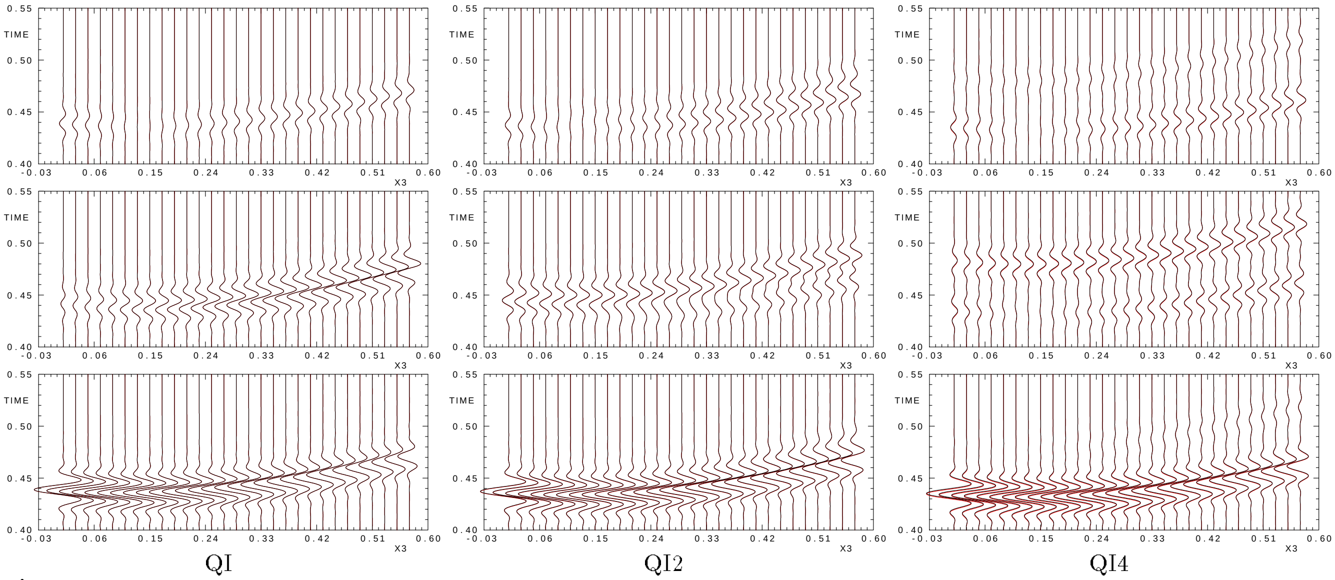

Figure 3. Inaccuracy of the anisotropic–common–ray approximation of the coupling ray theory in models QI, QI2 and QI4. From top to bottom: the first (radial) component, the second (transverse) component, the third (vertical) component. **Black:** Coupling–ray–theory synthetic seismograms without common ray approximation, simulated by the second–order perturbation expansion of travel time from anisotropic common rays. For the errors of this simulation refer to the *ACR remaining terms* in Table 2. **Red:** Anisotropic common ray approximation. The differences between the seismograms are invisible in model QI, negligible in model QI2, and still acceptable in model QI4.