Geophys. J. Int. (2008) **175,** 346–355 doi: 10.1111/j.1365-246X.2008.03850.x

Attenuation vector in heterogeneous, weakly dissipative, anisotropic media

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Accepted 2008 May 9. Received 2008 May 4; in original form 2007 November 7

SUMMARY

The attenuation vector plays an important role in investigating waves propagating in dissipative, isotropic or anisotropic media. It is defined as the imaginary part of the complex-valued traveltime gradient. The real part of the complex-valued traveltime gradient is referred to as the propagation vector. In this paper, simple expressions for the attenuation vector are derived for seismic body waves propagating in heterogeneous, weakly dissipative, anisotropic media. The ray-theory perturbation method is used, in which the dissipative medium is considered to be a small perturbation of a perfectly elastic medium. The general perturbation procedure proposed by Klimeš (2002) is very suitable for this purpose, as it does not require ray tracing in dissipative media; the computation of rays in the reference perfectly elastic medium, followed by quadratures along them, is quite sufficient. It is shown that the waves propagating in a heterogeneous, weakly dissipative, anisotropic or isotropic medium are, in general, inhomogeneous. This means that their attenuation vector is not parallel to the propagation vector. This holds even for waves generated by a point-source in a homogeneous anisotropic weakly dissipative medium. An exception is a wave generated by a point-source in a homogeneous isotropic dissipative medium, where the generated wave is homogeneous. Thus, the commonly used concept of homogeneous waves can be applied neither to heterogeneous nor to anisotropic dissipative media. The situation is different for plane waves propagating in homogeneous dissipative isotropic or anisotropic media. In this case, the homogeneity or inhomogeneity of the plane wave may be chosen freely. Besides the attenuation vector, we also study the complex-valued ray-velocity vector. In heterogeneous media, the complex-valued ray-velocity vector is generally inhomogeneous, that is, its real and imaginary parts are not parallel. It is also shown that twice the scalar product of the attenuation vector with the ray-velocity vector in the reference medium yields $1/Q$, where Q is the position- and direction-dependent quality factor. Quality factor Q does not depend on the inhomogeneity of the wave under consideration and offers a convenient measure of intrinsic material dissipation.

Key words: Seismic anisotropy; Seismic attenuation; Theoretical seismology; Wave propagation.

1 INTRODUCTION

In seismic ray methods, perturbation approaches play a very important role. We assume that a model, in which we wish to study wave propagation, differs only slightly from another model, called the *reference* model, in which simpler solutions are known. The solution in the perturbed model can then be sought in the form of a power series in the deviations of the perturbed and reference models. When only the first term of the series is considered, we speak of the first-order perturbation expansion, or briefly the *first-order perturbation*.

Perturbations of traveltime $\tau(x_n)$ and/or perturbations of its spatial derivatives $\tau_{,i} = \partial\tau/\partial x_i$ in perfectly elastic media are often sought. The relevant first-order perturbation formula for the traveltime has been known for a long time. For isotropic reference and perturbed models, see Aki & Richards (1980), where many other references can be found. For an anisotropic, perfectly elastic reference model and an anisotropic, perfectly elastic perturbed model, see Červený (1982), Červený & Jech (1982), Hanyga (1982), Farra (1989), Jech & Pšenčík (1989), Chapman & Pratt (1992), etc. For a review and other references, see Červený (2001, section 3.9). In these approaches, a reference ray is traced in the reference, isotropic or anisotropic, medium, and the first-order traveltime perturbation correction is sought by quadratures along the reference ray. The second-order perturbation of traveltime can also be calculated in a similar way. Farra (1999) proposed a procedure for calculating the second-order traveltime perturbation, based on the calculation of the first-order perturbation of the reference ray. A different procedure by Snieder & Sambridge (1992) is also based on the perturbation of rays. These procedures can also be adopted to compute the first-order perturbation of the slowness vector. For example, the procedure by Farra (1999) was used by Červený (2002) to calculate the first-order perturbation of the slowness vector.

A fully general procedure for computing the higher-order perturbations of traveltime and its derivatives along the reference ray was proposed by Klimeš (2002). This procedure has two great advantages: (a) The procedure is fully based on the quadratures along the reference ray in the reference medium. The complex-valued perturbed ray is not needed. (b) The procedure is quite universal and general, and may be recursively used to calculate the perturbations of any order of the traveltime and its derivatives of any order.

In the procedure, the dynamic ray tracing along the reference ray in the reference medium plays a very important role. For more details, see Section 2 and Klimeš (2002). A brief account of the procedure can also be found in Červený *et al.* (2007, Section 3). For the calculation of the perturbations and spatial derivatives of the ray-theory amplitude along the reference ray, refer to Klimeš (2006a).

In dissipative media, the traveltime and rays are generally complex-valued. Complex-valued ray tracing procedures for dissipative media do not usually belong to standard software at seismological institutions. If we deal with weakly dissipative media, it is thus convenient to use one of the above-discussed procedures, in which the dissipation is considered to be a small perturbation of a perfectly elastic reference medium. We can then calculate the effects of dissipation along real-valued reference rays in the reference, perfectly elastic medium. Standard, generally available routines can be used to perform ray tracing and dynamic ray tracing in the reference, perfectly elastic medium. The results for a weakly dissipative perturbed medium are then obtained by quadratures along the real-valued reference ray in the reference medium. For example, we can describe the reference medium in terms of real-valued elastic moduli and the perturbed medium in terms of complex-valued viscoelastic moduli, in which the real parts are the same as in the reference medium and the imaginary parts, called dissipative moduli, are small. Hence, the perturbation of the medium is imaginary-valued. This approach immediately yields the imaginary-valued first-order perturbation of traveltime, which represents the attenuation along the ray. For isotropic, weakly dissipative media, refer to Moczo *et al.* (1987) and for anisotropic, weakly dissipative media, refer to Gajewski & Pšenčík (1992).

We are interested in the perturbed complex-valued traveltime and its spatial derivatives, but not at all in perturbed, complex rays. The only rays, with which we work in this paper, are real-valued reference rays in the reference perfectly elastic medium.

The traveltime and its gradient in the perturbed dissipative medium are complex-valued. We decompose the complex-valued traveltime gradient into the real and imaginary parts,

$$\tau_{,i} = \mathrm{Re}(\tau_{,i}) + \mathrm{i}\,\mathrm{Im}(\tau_{,i}), \tag{1}$$

where $\mathrm{Re}(\tau_{,i})$ is called the propagation vector and $\mathrm{Im}(\tau_{,i})$ the attenuation vector. For $\mathrm{Re}(\tau_{,i})$ and $\mathrm{Im}(\tau_{,i})$ parallel [or $\mathrm{Im}(\tau_{,i}) = 0$], we speak of *homogeneous waves*; for $\mathrm{Re}(\tau_{,i})$ and $\mathrm{Im}(\tau_{,i})$ nonparallel, we speak of *inhomogeneous waves*. Inhomogeneous waves play a very important role in heterogeneous dissipative, isotropic or anisotropic media, both in the numerical modelling of seismic wavefields and in the inversion of seismic data. One of the purposes of this paper is to show that the waves are always inhomogeneous if the medium, in which they propagate, is heterogeneous and dissipative. Even for a point-source in a homogeneous anisotropic medium, they are in general inhomogeneous. For an up-to-date review of inhomogeneous waves, their properties and generation, see Declercq *et al.* (2005). The main problem in all applications involving inhomogeneous waves consists in the separation of effects of intrinsic dissipation, caused by non-vanishing dissipative moduli of the medium, from the effects of the inhomogeneity of the wave under consideration. This paper is a contribution to the study of this separation. Among others, we present a useful expression for quality factor Q, which does not depend on the inhomogeneity of the wave under consideration.

The first attempt to study the traveltime gradient of waves propagating in weakly dissipative media using perturbation methods was described by Červený & Pšenčík (2002). The authors used Farra's (1999) procedure, modified by Červený (2002), for the imaginary-valued perturbation of elastic moduli. They derived a complex-valued perturbed ray in a perturbed weakly dissipative medium. From the equation for the complex-valued ray, they obtained the complex-valued perturbation of the traveltime gradient. Farra's (1999) method is suitable to find the equation of the complex-valued ray in a weakly dissipative medium. In the study of the perturbation of the traveltime gradient, however, the computation of the perturbed complex-valued ray is an unnecessary complication, which can be avoided by using Klimeš (2002) approach.

In this paper, we study the complex-valued traveltime gradient of a wave propagating in a weakly dissipative, isotropic or anisotropic, heterogeneous medium, using Klimeš's (2002) method. The advantage of this method is that it does not require calculating the complex-valued perturbed ray in the dissipative medium. The procedure is quite straightforward. For heterogeneous media, it requires two quadratures along the reference real-valued ray. For homogeneous media, integration is not required, and the result is obtained analytically in closed form.

Briefly to the content of this paper. In Section 2, the general perturbation procedure proposed by Klimeš (2002) is described. Specific equations for the perturbations of the traveltime gradient along the reference ray are presented. Section 3 introduces the perturbation Hamiltonian used in the paper, corresponding to the imaginary-valued perturbation of elastic moduli. Section 4 presents the final perturbation equations for the traveltime gradient in heterogeneous weakly dissipative, isotropic or anisotropic media. Particular attention is devoted to attenuation vector $\mathrm{Im}\,(\tau_{,i})$. Specific analytical results for homogeneous media are discussed in Section 5. In Section 6, conditions under which a wave is homogeneous or inhomogeneous are discussed.

We use Cartesian coordinates x_i and time t throughout. The lower–case Roman indices take values 1,2,3, the upper–case indices 1,2. The Greek indices take any positive integer value. The Einstein summation convention over Roman (but not Greek) indices is used. For time-harmonic waves, we consider the exponential time factor $\exp(-\mathrm{i}\omega t)$, where ω is a fixed, real-valued and positive circular frequency. Symbol $^{\mathrm{T}}$ in the superscript of a matrix means 'transpose' and symbol $^{-\mathrm{T}}$ in the superscript denotes 'inverse transpose'.

2 GENERAL PERTURBATION PROCEDURE

The procedure discussed in this section was proposed by Klimeš (2002), who also described it in detail. Here it is explained only briefly, and the necessary equations are used without deriving them. We mostly use the same notation and terminology as Klimeš (2002).

We assume that perturbation Hamiltonian $\mathcal{H}$ is a function of phase-space coordinates (spatial coordinates x_i, slowness-vector components p_j) and of any number of perturbation parameters f_κ:

$$\mathcal{H} = \mathcal{H}(x_i, p_j, f_\kappa). \tag{2}$$

The traveltime and other quantities in the model are assumed to be functions of spatial coordinates x_i and of perturbation parameters f_κ:

$$\tau = \tau(x_i, f_\kappa). \tag{3}$$

We introduce the following notation for partial derivatives: The partial derivatives with respect to spatial Cartesian coordinates x_i are denoted by lower-case Roman subscripts following the comma and taking values 1, 2 or 3. Similarly, the partial derivatives with respect to the perturbation parameters are denoted by lower-case Greek subscripts, following the comma and taking any integer value. The partial derivatives with respect to the slowness-vector components are denoted by lower-case Roman superscripts, following the comma and taking values 1, 2 or 3. Two examples:

$$\tau_{,ij\ldots n\alpha\beta\ldots\nu} = \frac{\partial}{\partial f_\alpha}\frac{\partial}{\partial f_\beta}\cdots\frac{\partial}{\partial f_\nu}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\cdots\frac{\partial}{\partial x_n}\tau(x_k, f_\kappa), \tag{4}$$

$$\begin{aligned}\mathcal{H}^{,ab\ldots f}_{,ij\ldots n\alpha\beta\ldots\nu} = {} & \frac{\partial}{\partial f_\alpha}\frac{\partial}{\partial f_\beta}\cdots\frac{\partial}{\partial f_\nu}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\cdots\frac{\partial}{\partial x_n}\\ & \frac{\partial}{\partial p_a}\frac{\partial}{\partial p_b}\cdots\frac{\partial}{\partial p_f}\mathcal{H}(x_k, p_l, f_\kappa).\end{aligned} \tag{5}$$

We refer to the Taylor expansion with respect to perturbation parameters f_α as the *perturbation expansion*, see, for example, Klimeš & Bulant (2006, eq. 19). In this paper, we shall approximate the attenuation vector by its first-order perturbation expansion.

We consider a known 'reference ray' in the reference perfectly elastic medium and assume that the dynamic ray tracing along the reference ray has been performed. The dynamic ray tracing system consists of six ordinary differential equations for Q_{ia} and P_{ia} along the ray:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\gamma_3}Q_{ia} &= \mathcal{H}^{,i}_{,j}Q_{ja} + \mathcal{H}^{,ij}P_{ja}\\ \frac{\mathrm{d}}{\mathrm{d}\gamma_3}P_{ia} &= -\mathcal{H}_{,ij}Q_{ja} - \mathcal{H}^{,j}_{,i}P_{ja}.\end{aligned} \tag{6}$$

The dynamic ray tracing system is now included in most computer packages based on the ray method, and its solutions for proper initial conditions cause no difficulties. We define the 3×3 matrices $\mathbf{Q}$ and $\mathbf{P}$, computed by dynamic ray tracing, in the following way: element Q_{ia} represents the partial derivative of x_i, with respect to ray coordinate γ_a, and P_{ia} the partial derivative of the slowness vector component $p_i = \tau_{,i}$ with respect to ray coordinate γ_a,

$$Q_{ia} = \partial x_i/\partial\gamma_a \qquad P_{ia} = \partial\tau_{,i}/\partial\gamma_a. \tag{7}$$

Here ray coordinates γ_1 and γ_2 are ray parameters, for example, the take-off angles, and ray coordinate γ_3 is the independent parameter (variable) along the ray, determined by the form of the Hamiltonian. In this paper, parameter γ_3 corresponds to the unperturbed traveltime along the reference ray. The 3×3 matrix $\mathbf{Q}$ is often called the spreading matrix. Note that Q_{ia} and P_{ia} are real-valued, and that Q_{i3} and P_{i3} are known from ray tracing. It also follows from (7) that

$$P_{ia} = \tau_{,ij}Q_{ja}. \tag{8}$$

A general equation for $\tau_{,ij\ldots n\alpha\beta\ldots\nu}$, that is, for a perturbation of any order ($\alpha\beta\ldots\nu$) of traveltime derivative of any order ($ij\ldots n$), reads (Klimeš 2002, eq. 20),

$$\tau_{,ij\ldots n\alpha\beta\ldots\nu} = T_{ab\ldots f\alpha\beta\ldots\nu}Q^{-1}_{ai}Q^{-1}_{bj}\ldots Q^{-1}_{fn}, \tag{9}$$

where Q^{-1}_{mj} are elements of matrix $\mathbf{Q}^{-1}$. Covariant derivatives $T_{ab\ldots f\alpha\beta\ldots\nu}$ of the traveltime in ray coordinates can be calculated using relation

$$T_{ab\ldots f\alpha\beta\ldots\nu}(\gamma_3) = T_{ab\ldots f\alpha\beta\ldots\nu}(\gamma_3^0) + \int_{\gamma_3^0}^{\gamma_3}K_{ij\ldots n\alpha\beta\ldots\nu}Q_{ia}Q_{jb}\ldots Q_{nf}\,\mathrm{d}\gamma_3'. \tag{10}$$

Kernels $K_{ij\ldots n\alpha\beta\ldots\nu}$ are given in Klimeš (2002, eq. 21). They contain the partial derivatives of τ of a lower order than the computed $\tau_{,ij\ldots n\alpha\beta\ldots\nu}$. Consequently, eqs (9) and (10) must be used recursively, starting with the lower-order partial derivatives. For higher-order derivatives of τ, the algorithm may lead to cumbersome computations. For lower-order derivatives, however, it yields simple results. Symbol γ_3^0 denotes the traveltime at the initial point of the ray.

Here we do not give the general expressions for $K_{ij..n\alpha\beta..\nu}$, as we do not need them. We only present the following two expressions

$$K_\alpha = -\mathcal{H}_{,\alpha}, \tag{11}$$

$$K_{i\alpha} = -\mathcal{H}_{,i\alpha} - \mathcal{H}^{,r}_{,\alpha}\tau_{,ri}. \tag{12}$$

Kernel (11) can be used to determine the first-order perturbation of traveltime, and kernel (12) the first-order perturbation of the traveltime gradient. Note that $T_{3\alpha}(\gamma_3)$ can be computed without any quadrature (Klimeš 2002, eq. 28):

$$T_{3\alpha}(\gamma_3) = -\mathcal{H}_{,\alpha}(\gamma_3). \tag{13}$$

The perturbation of the traveltime gradient, $\tau_{,i\alpha}(\gamma_3)$, can then be computed using formula

$$\tau_{,i\alpha}(\gamma_3) = T_{a\alpha}(\gamma_3)Q^{-1}_{ai}(\gamma_3), \tag{14}$$

where Q^{-1}_{mj} are elements of matrix $\mathbf{Q}^{-1}$, and

$$\begin{aligned}T_{K\alpha}(\gamma_3) &= T_{K\alpha}(\gamma_3^0) - \int_{\gamma_3^0}^{\gamma_3}[\mathcal{H}_{,i\alpha}(\gamma_3')Q_{iK}(\gamma_3') + \mathcal{H}^{,i}_{,\alpha}(\gamma_3')P_{iK}(\gamma_3')]\mathrm{d}\gamma_3',\\ T_{3\alpha}(\gamma_3) &= -\mathcal{H}_{,\alpha}(\gamma_3).\end{aligned} \tag{15}$$

Eq. (14), with (15), represents the final expression for the perturbation derivative of the traveltime gradient. The computation of $\tau_{,i\alpha}(\gamma_3)$ requires two quadratures ($K = 1, 2$) along the reference ray.

In some applications, we are also interested in the perturbation of ray-velocity vector $\mathcal{U}$ (also called the group-velocity vector or the energy-velocity vector). In the reference medium, the components $\mathcal{U}_i$ of ray-velocity vector $\mathcal{U}$ tangent to the reference ray are given by the well-known relation

$$\mathcal{U}_i(x_m) = \mathcal{H}^{,i}(x_m, p_n), \tag{16}$$

where

$$p_i = \tau_{,i}(x_m) \tag{17}$$

are the Cartesian components of slowness vector **p**. In the perturbed medium, we can express the complex-valued ray-velocity vector as follows:

$$\mathcal{U}_i(x_m, f_\alpha) = \mathcal{H}^{,i}(x_m, \tau_{,n}(x_m, f_\alpha), f_\alpha), \tag{18}$$

as $\tau_{,n}$ also depends on x_n and f_α, and should be perturbed. Note that the perturbed ray-velocity vector and perturbed traveltime gradient satisfy identity

$$\mathcal{U}_i(x_m, f_\alpha)\tau_{,i}(x_n, f_\beta) = 1, \tag{19}$$

following from the homogeneity of the Hamiltonian, Euler's theorem and the Hamilton–Jacobi equation. Eq. (19) shows that the relation $\mathcal{U}_i\tau_{,i} = 1$, well known from ray methods in perfectly elastic heterogeneous, isotropic or anisotropic media, remains valid also for perturbed dissipative media.

The perturbation derivative of $\mathcal{U}_i$ reads

$$\mathcal{U}_{i,\alpha} = \mathcal{H}^{,i}_{,\alpha} + \mathcal{H}^{,ij}\tau_{,j\alpha}. \tag{20}$$

Inserting (14) into (20), we obtain, at any point γ_3 of the reference ray,

$$\mathcal{U}_{i,\alpha}(\gamma_3) = \mathcal{H}^{,i}_{,\alpha}(\gamma_3) + \mathcal{H}^{,ij}(\gamma_3)T_{a\alpha}(\gamma_3)Q^{-1}_{aj}(\gamma_3). \tag{21}$$

Here $T_{a\alpha}(\gamma_3)$ is given by (15). Eq. (21) is the expression for the perturbation derivative of ray-velocity vector $\boldsymbol{\mathcal{U}}(\gamma_3)$.

3 HAMILTONIANS

In the zero-order approximation of the ray method, the displacement vector of time-harmonic waves $\mathbf{u}(\mathbf{x}, t)$ may be expressed in the following form:

$$\mathbf{u}(\mathbf{x}, t) = a(\mathbf{x})\mathbf{U}(\mathbf{x})\exp\{-\mathrm{i}\omega[t - \tau(\mathbf{x})]\}. \tag{22}$$

Here ω is a real-valued, positive, fixed frequency, $\tau(\mathbf{x})$ represents the traveltime, $\mathbf{U}(\mathbf{x})$ the normalized polarization vector ($\mathbf{U}^{\mathrm{T}}\,\mathbf{U} = 1$) and $a(\mathbf{x})$ the scalar amplitude factor. Functions $\tau(\mathbf{x})$, $\mathbf{U}(\mathbf{x})$ and $a(\mathbf{x})$ should be determined to satisfy the equation of motion and boundary conditions. For heterogeneous anisotropic media, the source-free equation of motion reads

$$(c_{ijkl}u_{k,l})_{,j} = \rho\ddot{u}_i, \tag{23}$$

where $\rho = \rho(\mathbf{x})$ is the density, $c_{ijkl} = c_{ijkl}(\mathbf{x})$ are the elastic or viscoelastic moduli, and the dots denote derivatives with respect to time. See Fedorov (1968), Musgrave (1970), Helbig (1994), and many others. For perfectly elastic media, c_{ijkl} are real-valued and frequency-independent, but for viscoelastic media they are complex-valued and depend on ω (Thomson 1997),

$$c_{ijkl} = c^R_{ijkl} - \mathrm{i}c^I_{ijkl}. \tag{24}$$

Circular frequency ω is not used explicitly as an argument of c_{ijkl}, as we consider ω fixed throughout the paper. The minus sign in (24) is related to the minus sign in exponential factor $\exp(-\mathrm{i}\omega t)$, see (22). We consider only such 'physical' models, for which c^R_{ijkl} in the Voigt notation are positive-definite and c^I_{ijkl} positive-definite or zero. The case of $c^I_{ijkl} = 0$ corresponds to the perfectly elastic medium.

Inserting (22) into (23) yields

$$-N_i(\mathbf{U}) + \mathrm{i}\omega^{-1}M_i(\mathbf{U}) + \omega^{-2}L_i(\mathbf{U}) = 0, \tag{25}$$

where

$$\begin{aligned} N_i(\mathbf{U}) &= c_{ijkl}\tau_{,j}\tau_{,l}U_k - \rho U_i, \\ M_i(\mathbf{U}) &= c_{ijkl}\tau_{,j}U_{k,l} + (c_{ijkl}\tau_{,l}U_k)_{,j}, \\ L_i(\mathbf{U}) &= (c_{ijkl}U_{k,l})_{,j}. \end{aligned} \tag{26}$$

Here $\tau_{,i}$ are the Cartesian components of the traveltime gradient.

In perfectly elastic media, $\tau(\mathbf{x})$, $\tau_{,i}(\mathbf{x})$ and $U_i(\mathbf{x})$ are frequency independent. Consequently, N_i, M_i and L_i in (25) are also frequency independent. Since (25) should be satisfied for any frequency ω, N_i, M_i and L_i should equal zero. We shall consider $N_i(\mathbf{U}) = 0$, that is,

$$a_{ijkl}\tau_{,j}\tau_{,l}U_k = U_i. \tag{27}$$

Here

$$a_{ijkl}(\mathbf{x}) = c_{ijkl}(\mathbf{x})/\rho(\mathbf{x}) \tag{28}$$

are the density-normalized elastic moduli. Eq. (27) is the Christoffel equation and plays a basic role in our treatment.

For viscoelastic media, c_{ijkl} depend on frequency, and the above reasoning leading to $N_i(\mathbf{U}) = 0$ cannot be strictly used. Nevertheless, for smoothly varying media and if c_{ijkl} are weakly frequency dependent, we may use (27) as a good approximation also for viscoelastic media, particularly if we consider fixed $\omega \gg 0$.

We can introduce the generalized Christoffel matrix $\Gamma_{ik}(\mathbf{x}, \mathbf{p})$:

$$\Gamma_{ik}(\mathbf{x}, \mathbf{p}) = a_{ijkl}(\mathbf{x})p_j p_l. \tag{29}$$

The Christoffel equation (27) can then be expressed in the following form:

$$\Gamma_{ik}U_k = U_i. \tag{30}$$

Hereinafter, we shall use the generalized Christoffel matrix (29), which should strictly be distinguished from the Christoffel matrix

$$\overline{\Gamma}_{ik} = a_{ijkl}n_j n_l, \tag{31}$$

where **n** is a *given* real-valued unit vector perpendicular to the wave front.

We now derive an important equation from (30). We multiply (30) by U_i and obtain

$$\Gamma_{ik}U_iU_k = a_{ijkl}p_jp_lU_iU_k = 1. \tag{32}$$

Thus, function $\Gamma_{ik}U_iU_k$ is a function of x_n and p_m and equals 1 at any point of the model, for any slowness vector **p**. Consequently, we may take this function to represent the Hamiltonian. Actually, we can also multiply it by any real-valued positive constant. For convenience, we use the multiplicative constant 1/2 and introduce the Hamiltonian

$$\mathcal{H}(x_n, p_m) = \frac{1}{2}\Gamma_{ik}U_iU_k = \frac{1}{2}a_{ijkl}p_jp_lU_iU_k. \tag{33}$$

Let us emphasize that we have not solved the eigenvalue problem for Γ_{ik} to derive Hamiltonian (33). Eq. (33) for the Hamiltonian is valid in both perfectly elastic media (a_{ijkl}, p_i, U_i real-valued) and dissipative media (a_{ijkl}, p_i, U_i complex-valued).

Now we construct perturbation Hamiltonian $\mathcal{H}(x_i, p_j, f_\kappa)$, where f_κ are perturbation parameters. We introduce only one perturbation parameter f_α by modifying the density-normalized elastic/viscoelastic moduli in the following way:

$$a_{ijkl}(x_n, f_\alpha) = a^R_{ijkl}(x_n) - \mathrm{i}f_\alpha a^I_{ijkl}(x_n). \tag{34}$$

For $f_\alpha = 0$, density-normalized moduli $a_{ijkl}(x_n, f_\alpha)$ correspond to a perfectly elastic medium, with $a_{ijkl} = a^R_{ijkl}$. For $f_\alpha = 1$, density-normalized moduli $a_{ijkl}(x_n, f_\alpha)$ correspond to a viscoelastic medium, with $a_{ijkl} = a^R_{ijkl} - \mathrm{i}\,a^I_{ijkl}$ corresponding to eq. (24).

Perturbation Hamiltonian $\mathcal{H}(x_i, p_j, f_\alpha)$ may then be introduced in several ways. We introduce it simply by replacing a_{ijkl} in (33) with

$a_{ijkl}(x_n, f_\alpha)$ given by (34). This also implies that U_k is a function of f_α, $U_k = U_k(x_n, p_m, f_\alpha)$, see (27). We obtain

$$\mathcal{H}(x_n, p_m, f_\alpha) = \frac{1}{2} a_{ijkl}(x_q, f_\alpha) p_j p_l U_i(x_n, p_m, f_\alpha) U_k(x_r, p_s, f_\alpha). \tag{35}$$

Eq. (35) is the final form of the perturbation Hamiltonian to be used hereinafter.

4 PERTURBATION FROM PERFECTLY ELASTIC TO VISCOELASTIC MEDIA

In this section, we specify the general perturbation equations for the traveltime and its gradient, derived in Section 2, for perturbations from perfectly elastic to viscoelastic media. For this purpose, we use perturbation Hamiltonian (35). We consider the reference and perturbed media to be heterogeneous and anisotropic.

In this paper, we approximate the traveltime gradient by its first-order perturbation expansion

$$\tau_{,i}(x_m, f_\alpha) \approx \tau_{,i}(x_m) + \tau_{,i\alpha}(x_m) f_\alpha, \tag{36}$$

where traveltime gradient $\tau_{,i}(x_m)$ and its perturbation derivative $\tau_{,i\alpha}(x_m)$ with arguments f_α omitted correspond to the reference ray in the reference elastic model $a^R_{ijkl}(x_n)$. For $f_\alpha = 1$, we obtain the first-order perturbation of the complex-valued traveltime gradient $\tau_{,i}(x_m, 1)$ in the viscoelastic model $a_{ijkl} = a^R_{ijkl} - \mathrm{i}a^I_{ijkl}$. We denote the first-order approximation of the attenuation vector Im $[\tau_{,i}(x_m, 1)]$ by A_i. Since traveltime gradient $\tau_{,i}(x_m)$ in the reference model is real-valued, the first-order attenuation vector reads

$$A_i = \mathrm{Im}[\tau_{,i\alpha}(x_m)]. \tag{37}$$

In the following, we shall often refer to the first-order attenuation vector $\mathbf{A}$ briefly as the attenuation vector.

We now derive the expressions for the partial derivatives $\mathcal{H}_{,\alpha}$, $\mathcal{H}^{,i}_{,\alpha}$ and $\mathcal{H}_{,i\alpha}$ of Hamiltonian (35) at $f_\alpha = 0$. We shall need them in eqs (14) and (15) for $\tau_{,i\alpha}$. We first derive the equations for $\mathcal{H}^{,i}$, $\mathcal{H}_{,i}$ and $\mathcal{H}_{,\alpha}$. For $\mathcal{H}^{,i}$, we obtain

$$\begin{aligned}\mathcal{H}^{,i} &= \partial\mathcal{H}/\partial p_i = a^R_{ijkl} p_l U_j U_k + \Gamma_{nk} U_k \partial U_n/\partial p_i \\ &= a^R_{ijkl} p_l U_j U_k + U_n \partial U_n/\partial p_i = a^R_{ijkl} p_l U_j U_k.\end{aligned}$$

Here we have taken into account that $\Gamma_{nk} U_k = U_n$, see (30), and that $U_n \partial U_n/\partial p_i = \frac{1}{2}\partial(U_n U_n)/\partial p_i = 0$, as $\mathbf{U}$ is a unit vector. Similarly, we obtain partial derivatives $\mathcal{H}_{,i}$ and $\mathcal{H}_{,\alpha}$. Together:

$$\begin{aligned}\mathcal{H}^{,i} &= \partial\mathcal{H}/\partial p_i = a^R_{ijkl} p_l U_j U_k, \\ \mathcal{H}_{,i} &= \partial\mathcal{H}/\partial x_i = \frac{1}{2} a^R_{njkl,i} p_j p_l U_n U_k, \\ \mathcal{H}_{,\alpha} &= \partial\mathcal{H}/\partial f_\alpha = -\frac{1}{2}\mathrm{i}a^I_{ijkl} p_j p_l U_i U_k.\end{aligned} \tag{38}$$

Here $\mathcal{H}^{,i}$ give the Cartesian components of the ray-velocity vector $\mathcal{U}_i$, see (16). We introduce the phase-space and perturbation derivatives

$$\Gamma_{jk,m} = a^R_{ijkl,m} p_i p_l, \tag{39}$$

$$\Gamma^{,i}_{jk} = (a^R_{ijkl} + a^R_{ikjl}) p_l, \tag{40}$$

$$\Gamma_{jk,\alpha} = -\mathrm{i}a^I_{ijkl} p_i p_l \tag{41}$$

of the Christoffel matrix. The 'perturbation' partial derivatives $\mathcal{H}^{,i}_{,\alpha}$ and $\mathcal{H}_{,i\alpha}$ then read (Klimeš 2006c, eq. 23)

$$\begin{aligned}\mathcal{H}^{,i}_{,\alpha} &= -\mathrm{i}a^I_{ijkl} p_l U_j U_k + \sum_{a=2}^{3} U_j \Gamma^{,i}_{jk} U^a_k\ F^a_\alpha, \\ \mathcal{H}_{,i\alpha} &= -\frac{1}{2}\mathrm{i}a^I_{njkl,i} p_j p_l U_n U_k + \sum_{a=2}^{3} U_j \Gamma_{jk,i} U^a_k\ F^a_\alpha,\end{aligned} \tag{42}$$

where

$$F^a_\alpha = U_m \Gamma_{mn,\alpha} U^a_n / [1 - (\mathcal{C}^a/\mathcal{C}^1)^2]. \tag{43}$$

In (42) and (43), we denoted components of the polarization vectors of the three wave modes U^1_i, U^2_i, U^3_i, numbered so that $U_i = U^1_i$. Note that the sums in (42) are related to the perturbation of polarization vectors. Eqs (42) give the expressions for the partial derivatives needed in (14) and (15). In (14) and (15) and in the following, all quantities including U_i are taken for $f_\alpha = 0$, that is, in the reference medium. All perturbation derivatives $\mathcal{H}_{,\alpha}$, $\mathcal{H}^{,i}_{,\alpha}$ and $\mathcal{H}_{,i\alpha}$ are then purely imaginary.

Now we insert eqs (38) and (42) into eqs (14) and (15). We take into account that the Cartesian components of first-order attenuation vector $\mathbf{A}$ are given by the relation $A_i = \mathrm{Im}\tau_{,i\alpha}$ and obtain from (14):

$$A_i(\gamma_3) = \mathrm{Im}[T_{a\alpha}(\gamma_3)]\ Q^{-1}_{ai}(\gamma_3), \tag{44}$$

where $T_{a\alpha}(\gamma_3)$ are given by (15). Consequently,

$$\begin{aligned}\mathrm{Im}T_{K\alpha}(\gamma_3) &= \mathrm{Im}T_{K\alpha}(\gamma^0_3) \\ &\quad + \int_{\gamma^0_3}^{\gamma_3} \left[\mathcal{B}_i(\gamma'_3) Q_{iK}(\gamma'_3) + \mathcal{W}_i(\gamma'_3) P_{iK}(\gamma'_3)\right] \mathrm{d}\gamma'_3, \\ \mathrm{Im}T_{3\alpha}(\gamma_3) &= \frac{1}{2}\mathcal{A}^{in}(\gamma_3).\end{aligned} \tag{45}$$

Here $\mathcal{A}^{in}$ is the so-called intrinsic attenuation factor (Červený & Pšenčík 2008a), given by relation

$$\mathcal{A}^{in} = a^I_{ijkl} p_j p_l U_i U_k. \tag{46}$$

Cartesian components $\mathcal{B}_i$ of vector $\boldsymbol{\mathcal{B}}$ are given by relation

$$\mathcal{B}_i = \frac{1}{2} a^I_{njkl,i} p_j p_l U_n U_k + \mathrm{i}\sum_{a=2}^{3} U_j \Gamma_{jk,i} U^a_k\ F^a_\alpha. \tag{47}$$

They depend on the spatial derivatives of a^I_{njkl}, that is, on the spatial variations of density-normalized dissipative moduli a^I_{njkl}. For media in which density-normalized dissipative moduli a^I_{njkl} do not vary with position, $\boldsymbol{\mathcal{B}} = \mathbf{0}$. Finally, Cartesian components $\mathcal{W}_i$ of vector $\boldsymbol{\mathcal{W}}$ are given by relation

$$\mathcal{W}_i = a^I_{ijkl} p_l U_j U_k + \mathrm{i}\sum_{a=2}^{3} U_j \Gamma^{,i}_{jk} U^a_k\ F^a_\alpha. \tag{48}$$

Vector $\mathcal{W}_i$ is related to the perturbation of ray-velocity vector $\mathcal{H}^{,i}$, see (38). Scalar $\mathcal{A}^{in}$ and vectors $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{W}}$ are real valued and become zero in perfectly elastic media. Factors i in (47) and (48) are eliminated by multiplication with $-$i in (41). Quantity $\mathcal{A}^{in}$ is well known from previous papers. It was used by Gajewski & Pšenčík (1992), who also showed that $1/\mathcal{A}^{in}$ provides a good representation of the direction-dependent quality factor $Q = 1/\mathcal{A}^{in}$ at any point of the ray in weakly dissipative media. It was also derived by Červený & Pšenčík (2008a), who proposed $\mathcal{A}^{in}$ be called the intrinsic attenuation factor, as it did not depend on the inhomogeneity of the wave under consideration.

It may be useful to express (44) in terms of the covariant basis vectors connected with the reference ray. We define the 3 × 3 matrices $\mathbf{Q}$ and $\mathbf{S}$ of the contravariant basis vectors and of the covariant basis vectors as

$$\mathbf{Q} = [\mathbf{Q}_1, \mathbf{Q}_2, \boldsymbol{\mathcal{U}}], \quad \mathbf{S} = [\mathbf{f}_1, \mathbf{f}_2, \mathbf{p}]. \tag{49}$$

Here $\boldsymbol{\mathcal{U}}$ is the ray-velocity vector given by (16), tangent to the reference ray, and $\mathbf{p}$ is the slowness vector, perpendicular to the wave front in the reference medium. Both these vectors are calculated during the ray tracing of the reference ray. The 3×3 matrix $\mathbf{Q}$ equals the spreading matrix, computed by dynamic ray tracing along the reference ray. Vectors $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are tangent to the wave front in the reference perfectly elastic medium. The covariant basis vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ can be determined from contravariant vectors $\mathbf{Q}_1, \mathbf{Q}_2, \boldsymbol{\mathcal{U}}$ as follows:

$$\mathbf{f}_1 = \frac{\mathbf{Q}_2 \times \boldsymbol{\mathcal{U}}}{\boldsymbol{\mathcal{U}}^{\mathrm{T}}(\mathbf{Q}_1 \times \mathbf{Q}_2)}, \quad \mathbf{f}_2 = \frac{\boldsymbol{\mathcal{U}} \times \mathbf{Q}_1}{\boldsymbol{\mathcal{U}}^{\mathrm{T}}(\mathbf{Q}_1 \times \mathbf{Q}_2)}. \tag{50}$$

Vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ are perpendicular to the reference ray. The 3×3 matrices $\mathbf{Q}$ and $\mathbf{S}$ are related as follows:

$$\mathbf{S}(\gamma_3) = \mathbf{Q}^{-\mathrm{T}}(\gamma_3). \tag{51}$$

Note that all basis vectors are real valued and generally not unit.

Using (51), eq. (44) can be expressed as follows:

$$A_i(\gamma_3) = S_{i\alpha}(\gamma_3)\,\mathrm{Im}[T_{\alpha\alpha}(\gamma_3)]. \tag{52}$$

The explicit form of (52) is

$$\mathbf{A}(\gamma_3) = \mathbf{f}_1(\gamma_3)\mathrm{Im}T_{1\alpha}(\gamma_3) + \mathbf{f}_2(\gamma_3)\mathrm{Im}T_{2\alpha}(\gamma_3) + \frac{1}{2}\mathcal{A}^{in}(\gamma_3)\mathbf{p}(\gamma_3). \tag{53}$$

Eq. (53), with $\mathrm{Im}T_{K\alpha}(\gamma_3)$ given by (45), (47) and (48), and with $\mathcal{A}^{in}(\gamma_3)$ given by (46), represents the final result of this paper. It expresses the first-order attenuation vector $\mathbf{A}$ at any point of the reference ray, specified by γ_3, in terms of covariant basis vectors $\mathbf{f}_1(\gamma_3)$, $\mathbf{f}_2(\gamma_3)$ and $\mathbf{p}\,(\gamma_3)$, and of $\mathrm{Im}T_{1\alpha}(\gamma_3)$, $\mathrm{Im}T_{2\alpha}(\gamma_3)$ and $\mathrm{Im}T_{3\alpha}(\gamma_3) = \frac{1}{2}\mathcal{A}^{in}(\gamma_3)$ given by (45)–(48). All quantities in (53) can be simply computed along the reference ray.

In certain applications, it may be useful to consider an alternative expression in which attenuation vector $\mathbf{A}$ is expressed in terms of slowness vector $\mathbf{p}$ and of vectors $\mathbf{Q}_1$ and $\mathbf{Q}_2$, perpendicular to $\mathbf{p}$. The general expression for $\mathbf{A}$ is then rather complicated, but simplifies considerably in special cases. Let us consider unit vector $\mathbf{n}(\gamma_3) = \mathbf{p}(\gamma_3)/|\mathbf{p}(\gamma_3)|$. Considering identity $\boldsymbol{\mathcal{U}} = \mathbf{n}(\boldsymbol{\mathcal{U}}^{\mathrm{T}}\mathbf{n}) + \mathbf{n} \times (\boldsymbol{\mathcal{U}} \times \mathbf{n})$, we obtain from (50),

$$\begin{aligned}\mathbf{f}_1 &= -\mathbf{n}[\boldsymbol{\mathcal{U}}^{\mathrm{T}}(\mathbf{Q}_2 \times \mathbf{n})]/\Delta - (\mathbf{n} \times \mathbf{Q}_2)\mathcal{C}/\Delta,\\ \mathbf{f}_2 &= -\mathbf{n}[\boldsymbol{\mathcal{U}}^{\mathrm{T}}(\mathbf{n} \times \mathbf{Q}_1)]/\Delta + (\mathbf{n} \times \mathbf{Q}_1)\mathcal{C}/\Delta,\end{aligned} \tag{54}$$

where $\mathcal{C} = \boldsymbol{\mathcal{U}}^{\mathrm{T}}\mathbf{n} = 1/|\mathbf{p}|$ is the phase velocity in the reference medium and

$$\Delta = \det\mathbf{Q} = \boldsymbol{\mathcal{U}}^{\mathrm{T}}(\mathbf{Q}_1 \times \mathbf{Q}_2). \tag{55}$$

The square of the phase velocity can also be calculated as the eigenvalue of the Christoffel matrix (31). All quantities in (54) and (55) are taken at γ_3. Inserting (54) into (53) yields an alternative expression for $\mathbf{A}$:

$$\begin{aligned}\mathbf{A}(\gamma_3) = {}& [(\mathbf{n} \times \mathbf{Q}_1)\mathrm{Im}T_{2\alpha} - (\mathbf{n} \times \mathbf{Q}_2)\mathrm{Im}T_{1\alpha}]\mathcal{C}/\Delta\\ & + \mathbf{p}\Bigg[\frac{1}{2}\mathcal{A}^{in} - \mathcal{C}\boldsymbol{\mathcal{U}}^{\mathrm{T}}(\mathbf{n} \times \mathbf{Q}_1)\mathrm{Im}T_{2\alpha}/\Delta\\ & \quad - \mathcal{C}\boldsymbol{\mathcal{U}}^{\mathrm{T}}(\mathbf{Q}_2 \times \mathbf{n})\mathrm{Im}T_{1\alpha}/\Delta\Bigg].\end{aligned} \tag{56}$$

All quantities in (56) are again taken at γ_3. Note that vectors $\mathbf{n} \times \mathbf{Q}_1$ and $\mathbf{n} \times \mathbf{Q}_2$ are perpendicular to $\mathbf{p}$.

There are three 'perturbation quantities', which play an important role in determining attenuation vector $\mathbf{A}$ along the reference ray in a heterogeneous, weakly dissipative, anisotropic medium, namely $\mathcal{A}^{in}$, $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{W}}$. The most important of them is the scalar, real-valued, intrinsic attenuation factor $\mathcal{A}^{in}$. It plays a basic role in computing the quality factor in dissipative media. Multiplying (53) by $\boldsymbol{\mathcal{U}}^{\mathrm{T}}(\gamma_3)$ from the left-hand side and taking into account that $\boldsymbol{\mathcal{U}}^{\mathrm{T}}\mathbf{f}_I = 0$ and $\boldsymbol{\mathcal{U}}^{\mathrm{T}}\mathbf{p} = 1$ at any point of the reference ray, we obtain

$$\boldsymbol{\mathcal{U}}^{\mathrm{T}}(\gamma_3)\mathbf{A}(\gamma_3) = \frac{1}{2}\mathcal{A}^{in}(\gamma_3). \tag{57}$$

Thus, the scalar product of attenuation vector $\mathbf{A}$ with ray-velocity vector $\boldsymbol{\mathcal{U}}$ equals $\frac{1}{2}\mathcal{A}^{in}$.

Červený & Pšenčík (2008a,b) showed that quantity $(\mathcal{A}^{in})^{-1}$ can be used as quality factor Q for weakly dissipative anisotropic media. Their derivation was based on the definition of Q given by Buchen (1971). They also showed that the expression for Q, derived by Gajewski & Pšenčík (1992) using different concepts than we use, fully coincides with the definition $Q^{-1} = \mathcal{A}^{in}$. An analogous expression for Q was also derived by Červený (2001, eq. 5.5.28).

It is important to note that the computation of $\mathcal{A}^{in}$ (or quality factor Q) does not require any integration along the reference ray. Moreover, it does not require dynamic ray tracing along the reference ray. The knowledge of slowness vector $\mathbf{p}$, polarization vector $\mathbf{U}$ and the density-normalized dissipative moduli a^I_{ijkl} at a point of the reference ray is fully sufficient to determine $\mathcal{A}^{in}$ (or Q) at that point. Consequently, Q and $\mathcal{A}^{in}$ are local quantities, which depend on the local values of the density-normalized dissipative moduli a^I_{ijkl} and on the local values of slowness vector $\mathbf{p}$ and polarization vector $\mathbf{U}$. The quantities Q and $\mathcal{A}^{in}$ do not depend on the inhomogeneity of the wave.

Let us now investigate the homogeneity/inhomogeneity of a wave propagating in a heterogeneous, weakly dissipative, anisotropic medium. Eq. (53) shows that a wave propagating in a weakly dissipative, anisotropic, heterogeneous medium is, in general, inhomogeneous. Remember that the wave is called inhomogeneous if attenuation vector $\mathrm{Im}(\tau_{,i})$ has non-vanishing component(s) in the direction perpendicular to propagation vector $\mathrm{Re}\,(\tau_{,i})$, see (1). We can investigate these components considering $\mathbf{Q}^{\mathrm{T}}_K\,\mathbf{A}$, as $\mathbf{Q}_K$ are perpendicular to the reference slowness vector $\mathbf{p}$. Multiplying (53) by $\mathbf{Q}^{\mathrm{T}}_K(\gamma_3)$, we obtain

$$\mathbf{Q}^{\mathrm{T}}_K(\gamma_3)\mathbf{A}(\gamma_3) = \mathrm{Im}T_{K\alpha}(\gamma_3). \tag{58}$$

Eq. (58) also follows directly from (44). Here $\mathrm{Im}T_{K\alpha}(\gamma_3)$ are given by (45). In general, $\mathrm{Im}T_{K\alpha}(\gamma_3)$, $K = 1, 2$, are different from zero. This shows that the waves propagating in weakly dissipative anisotropic heterogeneous media are, in general, inhomogeneous. In heterogeneous media, exceptions are very rare. For homogeneous media, see the next section.

Contrary to the determination of $\mathcal{A}^{in}$, the determination of $\mathrm{Im}T_{K\alpha}(\gamma_3)$ requires dynamic ray tracing along the reference ray and integration of $\mathcal{B}_i$ and $\mathcal{W}_i$ along the ray, see (45). Vector $\boldsymbol{\mathcal{W}}$ is related to the perturbation of the ray-velocity vector, and vector $\boldsymbol{\mathcal{B}}$ depends on the spatial derivatives of a^I_{njkl}.

The first-order perturbation eqs (52) or (53), supplemented by (45) and (50), can be used to compute attenuation vector $\mathbf{A}$ in weakly dissipative anisotropic heterogeneous media along any reference ray, along which dynamic ray tracing has also been performed. The equations remain valid for homogeneous media, for isotropic media, for plane waves, for waves generated by a point-source or for waves generated at a curved initial surface. They are valid for inhomogeneous waves, and also predict when the inhomogeneous waves reduce to homogeneous waves.

For completeness, we also give the first-order perturbation expansion of the ray-velocity vector,

$$\mathcal{U}_i(x_m, f_\alpha) \approx \mathcal{U}_i(x_m) + \mathcal{U}_{i,\alpha}(x_m) f_\alpha, \tag{59}$$

where ray-velocity vector $\mathcal{U}_i(x_m)$ and its perturbation derivative $\mathcal{U}_{i,\alpha}(x_m)$, with arguments f_α omitted, correspond to the reference ray in the reference elastic model $a^R_{ijkl}(x_n)$. Here $\mathcal{U}_{i,\alpha}$ is given by (21). At any point of the reference ray parametrized by γ_3, $\mathcal{U}_{i,\alpha}$ reads

$$\mathcal{U}_{i,\alpha}(\gamma_3) = -\mathrm{i}\mathcal{W}_i(\gamma_3) + \mathcal{H}^{,ij}(\gamma_3) T_{a\alpha}(\gamma_3) Q^{-1}_{aj}(\gamma_3). \tag{60}$$

Here $\mathcal{W}_i$ is real valued and is given by (48), and $T_{a\alpha}$ is purely imaginary and is given by (45). Consequently, $\mathcal{U}_{i,\alpha}$ is purely imaginary in this paper.

5 HOMOGENEOUS MEDIUM

It may be useful to derive also certain analytical versions of (53) for some simple cases. Particularly, the analytical expressions for *homogeneous* media may be valuable. There are at least three reasons for this. First: Such analytical solutions contribute considerably to a better understanding of various effects of wave propagation in weakly dissipative media. Second: The analytical perturbation solutions for homogeneous media may be used to compare our results with the analytical perturbation solutions for plane waves propagating in homogeneous viscoelastic media, obtained by Červený & Pšenčík (2008a) using a different method. Third: For homogeneous anisotropic viscoelastic media, the exact expressions for attenuation vector **A** are known, both for plane waves (Červený & Pšenčík 2005, 2006, 2008a) and for waves generated by a point-source (Vavryčuk 2007a,b). Consequently, the accuracy of the perturbation equations can simply be studied in this case.

For homogeneous anisotropic media, the dynamic ray tracing system (6) simplifies considerably, as $\mathcal{H}^{,i}_{,j} = 0$ and $\mathcal{H}_{,ij} = 0$. The dynamic ray tracing system then reads

$$\mathrm{d}Q_{ia}/\mathrm{d}\gamma_3 = \mathcal{H}^{,ij} P_{ja}, \quad \mathrm{d}P_{ia}/\mathrm{d}\gamma_3 = 0. \tag{61}$$

The solution of this system is straightforward:

$$\begin{aligned} Q_{ia}(\gamma_3) &= Q_{ia}\left(\gamma_3^0\right) + \mathcal{H}^{,ij}\left(\gamma_3^0\right) P_{ja}\left(\gamma_3^0\right)\left(\gamma_3 - \gamma_3^0\right), \\ P_{ia}(\gamma_3) &= P_{ia}\left(\gamma_3^0\right). \end{aligned} \tag{62}$$

For index $a = 3$, we obtain

$$Q_{i3}(\gamma_3) = \mathcal{U}_i(\gamma_3), \quad P_{i3}(\gamma_3) = 0. \tag{63}$$

The second equation in (63) follows from the fact that slowness vector **p** is constant along the ray in a homogeneous medium. All the quantities in (62) and (63) are taken in the reference medium. Moreover, for a homogeneous medium, we also obtain

$$\mathcal{B}_i = 0. \tag{64}$$

Consequently, attenuation vector $\mathbf{A}(\gamma_3)$ at any point of the reference ray in a homogeneous medium is given by (44), where **Q** is given by (62) and (63), and $\mathrm{Im}[T_{a\alpha}(\gamma_3)]$ by relations

$$\begin{aligned} \mathrm{Im}T_{K\alpha}(\gamma_3) &= \mathrm{Im}T_{K\alpha}\left(\gamma_3^0\right) + \mathcal{W}_i\left(\gamma_3^0\right) P_{iK}\left(\gamma_3^0\right)\left(\gamma_3 - \gamma_3^0\right), \\ \mathrm{Im}T_{3\alpha}(\gamma_3) &= \frac{1}{2}\mathcal{A}^{in}\left(\gamma_3^0\right), \end{aligned} \tag{65}$$

see (45).

We now consider two special cases of the initial conditions. In Section 5.1, we treat the plane wave initial conditions, and in Section 5.2, we consider the point-source initial conditions.

5.1 Plane waves

Consider a plane wave propagating in an arbitrary direction in a homogeneous weakly dissipative anisotropic medium. Denote the unit normal perpendicular to the wave front in the reference medium by **n**. Then (62) and (65) yield

$$\begin{aligned} &P_{iK}(\gamma_3) = P_{iK}\left(\gamma_3^0\right) = 0, \quad Q_{iK}(\gamma_3) = Q_{iK}\left(\gamma_3^0\right), \\ &\mathrm{Im}T_{K\alpha}(\gamma_3) = \mathrm{Im}T_{K\alpha}\left(\gamma_3^0\right), \quad \mathrm{Im}T_{3\alpha}(\gamma_3) = \frac{1}{2}\mathcal{A}^{in}\left(\gamma_3^0\right). \end{aligned} \tag{66}$$

Vectors $\mathbf{Q}_1(\gamma_3^0)$ and $\mathbf{Q}_2(\gamma_3^0)$ are tangent to the wave front, but otherwise they can be chosen freely. We choose $\mathbf{Q}_1(\gamma_3^0)$, $\mathbf{Q}_2(\gamma_3^0)$, $\mathbf{n}(\gamma_3^0)$ to form an orthogonal right-handed triplet of unit vectors.

We now insert (66) into (56). Taking into account that $\mathbf{n} \times \mathbf{Q}_1 = \mathbf{Q}_2$, $\mathbf{n} \times \mathbf{Q}_2 = -\mathbf{Q}_1$ and $\Delta = \boldsymbol{\mathcal{U}}^{\mathrm{T}}\mathbf{n} = \mathcal{C}$, we obtain

$$\begin{aligned} \mathbf{A}(\gamma_3) = {} & (\mathbf{Q}_2\mathrm{Im}T_{2\alpha} + \mathbf{Q}_1\mathrm{Im}T_{1\alpha}) + \mathbf{p}[\tfrac{1}{2}\mathcal{A}^{in} - (\boldsymbol{\mathcal{U}}^{\mathrm{T}}\mathbf{Q}_2)\mathrm{Im}T_{2\alpha} \\ & -(\boldsymbol{\mathcal{U}}^{\mathrm{T}}\mathbf{Q}_1)\mathrm{Im}T_{1\alpha}]. \end{aligned} \tag{67}$$

This is the final expression for the attenuation vector of a plane wave, propagating in a homogeneous weakly dissipative anisotropic medium. It clearly demonstrates that the plane wave may be homogeneous or inhomogeneous, depending on the choice of the initial conditions, $\mathrm{Im}T_{1\alpha}$ and $\mathrm{Im}T_{2\alpha}$. If we choose $\mathrm{Im}T_{1\alpha} = \mathrm{Im}T_{2\alpha} = 0$, the plane wave is homogeneous. If at least one of the quantities $\mathrm{Im}T_{1\alpha}$ or $\mathrm{Im}T_{2\alpha}$ is non-vanishing, the plane wave is inhomogeneous.

Weakly inhomogeneous and homogeneous plane waves propagating in homogeneous weakly dissipative anisotropic media were also studied by Hayes & Rivlin (1974), Deschamps & Assouline (2000), Zhu & Tsvankin (2006, 2007), Červený & Pšenčík (2008a,b), etc. Červený & Pšenčík (2008a) derived the first-order attenuation vector directly from exact solutions. We will now show that both results are the same. We choose initial conditions

$$\mathrm{Im}T_{1\alpha} = D, \quad \mathrm{Im}T_{2\alpha} = 0, \tag{68}$$

and call D the inhomogeneity parameter. Then, using the notation $\mathbf{m} = \mathbf{Q}_1$, we obtain from (67):

$$\mathbf{A} = D\mathbf{m} + \left[\frac{1}{2}\mathcal{A}^{in} - (\boldsymbol{\mathcal{U}}^{\mathrm{T}}\mathbf{m})D\right]\mathbf{p}. \tag{69}$$

This coincides with the equation derived by Červený & Pšenčík (2008a, eq. 31).

5.2 Waves generated by a point-source

Consider a wave generated by a point-source situated at point S and propagating to receiver R in a homogeneous weakly dissipative anisotropic medium. The reference ray is then a straight line connecting points S and R, parallel to ray-velocity vector $\boldsymbol{\mathcal{U}}$. We shall not treat the problem of determining the relevant slowness vector **p** from $\boldsymbol{\mathcal{U}}$, and assume that the slowness vector **p**, corresponding to the given $\boldsymbol{\mathcal{U}}$, is known. There may be several such vectors **p**, and we consider anyone of them. The initial conditions for dynamic ray tracing are then $Q_{ik}(\gamma_3^0) = 0$, $P_{ik}(\gamma_3^0) \neq 0$. Vectors $\mathbf{P}_1 = (P_{11}, P_{21}, P_{31})^{\mathrm{T}}$ and $\mathbf{P}_2 = (P_{12}, P_{22}, P_{32})^{\mathrm{T}}$ are tangent to the slowness surface and are perpendicular to the reference ray, $P_{iK}\mathcal{U}_i = 0$. They thus represent the covariant basis vectors of the ray-centred coordinate system in a homogeneous medium, but not in a heterogeneous medium, where the basis vectors should be chosen differently, see Klimeš (2006b). Vectors $\mathbf{P}_1$, $\mathbf{P}_2$ can be chosen arbitrarily in the plane perpendicular to $\boldsymbol{\mathcal{U}}$. Eqs (62) and (63) then yield

$$Q_{iK}(\gamma_3) = \mathcal{H}^{,ij}\left(\gamma_3^0\right) P_{jK}\left(\gamma_3^0\right)\left(\gamma_3 - \gamma_3^0\right), \quad Q_{i3}(\gamma_3) = \mathcal{U}_i\left(\gamma_3^0\right). \tag{70}$$

The initial value $\mathrm{Im}T_{K\alpha}(\gamma_3^0)$ at the point-source vanishes (see Klimeš 2002). Then (65) yields

$$\mathrm{Im}T_{K\alpha}(\gamma_3) = \mathcal{W}_i\left(\gamma_3^0\right) P_{iK}\left(\gamma_3^0\right)\left(\gamma_3 - \gamma_3^0\right), \quad \mathrm{Im}T_{3\alpha}(\gamma_3) = \frac{1}{2}\mathcal{A}^{in}\left(\gamma_3^0\right). \tag{71}$$

If vector $\boldsymbol{\mathcal{W}}$ is parallel to $\boldsymbol{\mathcal{U}}$, $\mathrm{Im}T_{K\alpha}(\gamma_3)$ vanishes.

To calculate $\mathbf{A}(\gamma_3)$, it would, of course, be possible to use (44) directly, with $\mathbf{Q}(\gamma_3)$ given by (70) and $\mathrm{Im}T_{k\alpha}(\gamma_3)$ by (71). We shall, however, use an alternative approach, which will provide a clearer physical insight. We express $\mathbf{A}(\gamma_3)$ in terms of covariant basis vectors $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{p}$ connected with the reference ray, where $\mathbf{P}_1$ and $\mathbf{P}_2$ are perpendicular to the ray, and $\mathbf{p}$ is the slowness vector:

$$A_i = a_1 P_{i1} + a_2 P_{i2} + \frac{1}{2}\mathcal{A}^{in} p_i. \tag{72}$$

Here a_1 and a_2 have to be determined. We determine them from the known expressions for $\mathrm{Im}T_{K\alpha}(\gamma_3)$ given by (71), using relation (58). In componental form, we can express (58) as follows:

$$\mathrm{Im}T_{K\alpha} = Q_{iK}(a_1 P_{i1} + a_2 P_{i2}). \tag{73}$$

Inserting (70) and (71) into (73) yields

$$\mathcal{W}^{(K)} = D_{KJ} a_J, \tag{74}$$

where

$$\mathcal{W}^{(K)} = \mathcal{W}_i P_{iK} \tag{75}$$

and

$$D_{MK} = P_{iM}\mathcal{H}^{,ij} P_{jK}. \tag{76}$$

The real-valued scalars $\mathcal{W}^{(1)}$ and $\mathcal{W}^{(2)}$ represent the projections of vector $\boldsymbol{\mathcal{W}}$ onto basis vectors $\mathbf{P}_1$ and $\mathbf{P}_2$. The 2×2 matrix $\mathbf{D}$ is related to the curvature matrix of the slowness surface. The solution of equations (74) for a_1 and a_2 is as follows:

$$a_J = D_{JK}^{-1}\mathcal{W}^{(K)}, \tag{77}$$

where D_{JK}^{-1} are the elements of matrix $\mathbf{D}^{-1}$. The final expression for attenuation vector $\mathbf{A}$ in terms of covariant basis vectors $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{p}$ reads

$$\mathbf{A} = \mathcal{W}^{(J)} D_{JK}^{-1}\mathbf{P}_K + \frac{1}{2}\mathcal{A}^{in}\mathbf{p}, \tag{78}$$

see (72).

As in (56), we can also express attenuation vector $\mathbf{A}$ in terms of slowness vector $\mathbf{p}$ and two vectors perpendicular to $\mathbf{p}$. This provides us with a simple possibility of discussing the inhomogeneity of the wave under consideration. We again denote the real-valued unit vector along $\mathbf{p}$ in the reference medium by $\mathbf{n}$ and introduce vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ using relation:

$$\mathbf{h}_I = \mathbf{n} \times (\mathbf{P}_I \times \mathbf{n}). \tag{79}$$

Here $\mathbf{h}_1$ and $\mathbf{h}_2$ are perpendicular to slowness vector $\mathbf{p}$. We then use the obvious decomposition

$$\mathbf{P}_I = \mathbf{n}\left(\mathbf{P}_I^{\mathrm{T}}\mathbf{n}\right) + \mathbf{h}_I \tag{80}$$

of vectors $\mathbf{P}_1$ and $\mathbf{P}_2$. Inserting (80) into (78) yields

$$\mathbf{A} = \mathcal{W}^{(I)} D_{IK}^{-1}\mathbf{h}_K + \mathbf{p}\left(\frac{1}{2}\mathcal{A}^{in} + \mathcal{C}^2\mathcal{W}^{(I)} D_{IK}^{-1}\mathbf{P}_K^{\mathrm{T}}\mathbf{p}\right), \tag{81}$$

where $\mathcal{C}$ is the phase velocity. As we can see, the inhomogeneity of the wave generated by a point-source in a homogeneous weakly dissipative medium is influenced by $\mathcal{W}^{(I)}$ and the curvature of the slowness surface. The wave is always homogeneous if $\mathcal{W}^{(I)} = 0$. As $\mathcal{W}^{(I)} = \boldsymbol{\mathcal{W}}^{\mathrm{T}}\mathbf{P}_I$, the wave is homogeneous if $\boldsymbol{\mathcal{W}}$ is parallel to $\boldsymbol{\mathcal{U}}$. For homogeneous, isotropic, weakly dissipative media, expression (48) for $\mathcal{W}_i$ simplifies considerably. It reads

$$\mathcal{W}_i = V^2 p_i/(2Q), \tag{82}$$

where V is the velocity and Q is the quality factor of the wave under consideration (P or S). Consequently, $\boldsymbol{\mathcal{W}}$ is parallel to $\mathbf{p}$ and $\boldsymbol{\mathcal{U}}$, and the waves generated by a point-source in homogeneous, isotropic, weakly dissipative media are always homogeneous. Contrary to this, the waves generated by a point-source in homogeneous, anisotropic, weakly dissipative media or in heterogeneous, isotropic, weakly dissipative media are, in general, inhomogeneous.

For completeness, we also derive certain properties of ray-velocity vector $\boldsymbol{\mathcal{U}}$ of a wave generated by a point-source in a homogeneous, weakly dissipative, anisotropic medium. First, we show that the first-order perturbation of the ray-velocity vector is tangent to the reference ray. Using (21), we obtain

$$\mathcal{U}_{i,\alpha} P_{iK} = \mathcal{H}^{,i}_{,\alpha} P_{iK} + \mathcal{H}^{,ij} P_{iK} Q_{mj}^{-1} T_{m\alpha}, \tag{83}$$

where Q_{mj}^{-1} are elements of matrix $\mathbf{Q}^{-1}$. We now decompose the summation over $m = 1, 2, 3$ into the summation over $M = 1, 2$ and $m = 3$,

$$\mathcal{U}_{i,\alpha} P_{iK} = \mathcal{H}^{,i}_{,\alpha} P_{iK} + \mathcal{H}^{,ij} P_{iK} Q_{Mj}^{-1} T_{M\alpha} + \mathcal{H}^{,ij} P_{iK} Q_{3j}^{-1} T_{3\alpha}. \tag{84}$$

As $\mathcal{H}^{,ij} Q_{3j}^{-1} = \mathcal{H}^{,ij} p_j = \mathcal{U}_i$, the last term in (84) reads $\mathcal{U}_i P_{iK} T_{3\alpha}$. Since $\mathcal{U}_i P_{iK} = 0$, the last term in (84) vanishes.

We now take into account that $T_{m\alpha}$ is purely imaginary, see the discussion following (43). The second term in (84) can then be modified using the first equations of (70) and (71) so that

$$\mathcal{U}_{i,\alpha} P_{iK} = \mathcal{H}^{,i}_{,\alpha} P_{iK} + \mathrm{i} Q_{jK} Q_{Mj}^{-1}\mathcal{W}_r P_{rM}. \tag{85}$$

Finally, taking into account that $\mathcal{H}^{,i}_{,\alpha} = -\mathrm{i}\mathcal{W}_i$ and $Q_{Mj}^{-1} Q_{jK} = \delta_{MK}$, we obtain

$$\mathcal{U}_{i,\alpha} P_{iK} = -\mathrm{i}\mathcal{W}_i P_{iK} + \mathrm{i}\mathcal{W}_r P_{rK} = 0. \tag{86}$$

Consequently, the ray-normal components of the perturbation derivative of the ray-velocity vector vanish in this case. As $\mathcal{U}_i$ in the reference medium are real-valued and $\mathcal{U}_{i,\alpha}$ are imaginary-valued, this means that the real part of the complex-valued ray-velocity vector $\boldsymbol{\mathcal{U}}$ in the perturbed medium is parallel to its imaginary part. This yields the important conclusion that the ray-velocity vector of a wave generated by a point-source in a *homogeneous*, weakly dissipative, anisotropic medium is always *homogeneous*. This conclusion is well known from the literature, see, for example, Vavryčuk (2007a,b). The wave under consideration, however, is not homogeneous, as the homogeneity of the wave is defined with respect to the homogeneity of its traveltime gradient, not of its ray-velocity vector. It should, however, be emphasized that the above conclusion about the homogeneity of the ray-velocity vector holds only for homogeneous media. It is not valid for heterogeneous media.

Let us now compute scalar product $\mathcal{U}_{i,\alpha} p_i = \mathcal{U}_{i,\alpha}\tau_{,i}$: Equation (19) yields

$$\mathcal{U}_{i,\alpha}\tau_{,i} = -\mathcal{U}_i\tau_{,i\alpha}. \tag{87}$$

Considering that $\tau_{,i\alpha} = \mathrm{i}A_i$ and inserting (57) into (87), we arrive at

$$\mathcal{U}_{i,\alpha} p_i = -\frac{1}{2}\mathrm{i}\mathcal{A}^{in}, \tag{88}$$

where $\mathcal{A}^{in}$ is the intrinsic attenuation factor given by (46). Thus, the ray-velocity considerations for waves, generated by a point-source in a homogeneous weakly dissipative anisotropic medium, also yield the intrinsic attenuation factor $\mathcal{A}^{in}$, reciprocal to quality factor Q. Eq. (88), together with (86), also yields the final expression for the perturbation $\mathcal{U}_{i,\alpha}$ of the ray-velocity vector:

$$\mathcal{U}_{i,\alpha} = -\frac{1}{2}\mathrm{i}\mathcal{A}^{in}\mathcal{U}_i. \tag{89}$$

Consequently, the first-order perturbation expansion of the ray-velocity vector corresponding to a point source in a homogeneous, isotropic or anisotropic, weakly dissipative medium reads

$$\mathcal{U}_i(f_\alpha) \approx \left(1 - \frac{1}{2}\mathrm{i}\mathcal{A}^{in} f_\alpha\right) \mathcal{U}_i, \tag{90}$$

where ray-velocity vector $\mathcal{U}_i$, with arguments f_α omitted, corresponds to the reference ray in the homogeneous reference elastic model a^R_{ijkl}.

6 CONCLUDING REMARKS

In this paper, the perturbation method is used to investigate the attenuation vector $\mathrm{Im}(\tau_{,i})$ of high-frequency seismic body waves, propagating in heterogeneous, weakly dissipative, isotropic or anisotropic media. The dissipative medium is considered to be a small perturbation of the perfectly elastic medium. First-order attenuation vector **A** is computed along a reference ray, constructed in the reference medium. The explicit expression (44) for first-order attenuation vector **A** is derived, which requires dynamic ray tracing along the reference ray and computing two simple integrals (45) along the reference ray.

We will now briefly recapitulate the homogeneity/inhomogeneity of traveltime gradient (1) in weakly dissipative anisotropic media. The results show that attenuation vector $\mathrm{Im}(\tau_{,i})$ is not, in general, parallel to propagation vector $\mathrm{Re}(\tau_{,i})$, so that the traveltime gradient is inhomogeneous. In this case, the wave under consideration is also called inhomogeneous. Consequently, the high-frequency waves propagating in heterogeneous, weakly dissipative, isotropic or anisotropic media are, in general, *inhomogeneous*. The exceptions are the plane waves, propagating in a *homogeneous* dissipative, isotropic or anisotropic medium, which may be homogeneous or inhomogeneous, depending on the initial conditions. Another exception is the wave generated by a point-source in a *homogeneous*, dissipative, but *isotropic* medium, which is always homogeneous.

In our future investigation, we wish to concentrate on the numerical study of the inhomogeneity of the waves, to see quantitatively how the inhomogeneity of the wave depends on velocity anisotropy, heterogeneity, the dissipative properties of the medium, and the configuration of the experiment.

In addition to the attenuation vector, also the complex-valued ray-velocity vector $\boldsymbol{\mathcal{U}}$ in heterogeneous, weakly dissipative, anisotropic or isotropic medium has been studied. It has been shown that this vector is, in general, also inhomogeneous. The exceptions are the same as in the case of the traveltime gradient, with one additional exception: The ray-velocity vector $\boldsymbol{\mathcal{U}}$ of the wave generated by a point-source in a homogeneous dissipative anisotropic medium is homogeneous (although the wave itself is inhomogeneous).

The derived equations offer a simple possibility of calculating the local quality factor Q at any point of the reference ray. Integrating $1/Q$ with respect to the reference traveltime along the whole ray from the point-source to the receiver, we can also compute the global absorption factor (Červený 2001, eq. 5.5.12).

ACKNOWLEDGMENTS

The authors are grateful to Mathias Alerini and to another anonymous reviewer, whose suggestions made it possible to improve the paper considerably. This research has been supported by the Consortium Project 'Seismic Waves in Complex 3-D Structures' (see 'http://sw3d.cz'), by Research Projects 205/05/2182, 205/07/0032 and 205/08/0332 of the Grant Agency of the Czech Republic and by Research Project MSM0021620860 of the Ministry of Education of the Czech Republic.

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