Approximate expression for the group velocity vector in weakly anisotropic media [1]

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Abstract

Determination of the direction of the group velocity vector or the ray vector (unit vector in the direction of the group velocity vector) corresponding to a given phase normal (unit vector in the direction of the phase velocity or slowness vector) in an arbitrary anisotropic medium can be performed using the well-known ray tracing formula. Determination of the phase normal from the direction of the group velocity vector (ray vector) is a more complicated task, which is usually solved iteratively. We review the derivation of an approximate, first-order ray perturbation formula solving this problem and test it on several examples. The formula can be used for the determination of the ray vector from a given phase normal and vice versa. The formula is applicable to qP as well as qS waves in arbitrary weakly anisotropic media, in regions, in which the waves can be dealt with separately (i.e. outside singular regions of qS waves). We show that the formula for the determination of the ray vector from phase normal can roughly describe even triplications in regions corresponding to concave sections of slowness surface. The reverse formula for the determination of the phase normal from the ray vector yields satisfactory results for waves with convex sections of slowness surfaces but fails for waves with concave slowness surface sections. In addition to the described formula approximate expressions for the phase and group velocities of qP and qS waves are also presented and tested. Although anisotropy of some test samples is rather high, the approximate formulae yield surprisingly good results.

1 Introduction

It is well known that the phase velocity and the group velocity are equal in the first-order approximation for anisotropic media, see Backus (1965). The directions of the phase velocity vector (phase normal) and of the group velocity vector (ray vector), however, generally differ. The determination of the ray vector corresponding to a given phase normal is an easy task. The ray tracing formulae for anisotropic media can be used for this purpose, see, e.g., Červený, Molotkov and Pšenčík (1977). The determination of the phase

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normal from the direction of the ray vector is more complicated. It is because the phase normal and the ray vector are related by a complicated nonlinear relation, which must be, in most cases, solved iteratively, and which has, in certain situations (triplication of qS wave surface) three or more different solutions. An alternative to the iterative solution is to use an approximate formula (Pšenčík, 1996) based on the first-order perturbation method for anisotropic media (Jech and Pšenčík, 1989).

In Sec.2, we summarize basic results of the first-order perturbation method for weakly anisotropic media. We use them in rederivation of the approximate formula relating the phase normal n_i and the ray vector N_i (both are unit vectors) and vice versa in Sec.3. In Sec.3, we also give an approximate relation for the group velocity vector in a weakly anisotropic medium of arbitrary symmetry. In Sec.4, accuracy of the mentioned formulae is tested on two numerical examples. One represents a transversely isotropic medium with vertical axis of symmetry (VTI medium). The other represents an orthorhombic medium with symmetry planes coinciding with coordinate planes. Although the approximate formulae are simple and anisotropy of studied examples rather strong, especially in the case of the VTI medium, the formulae yield suprisingly accurate results.

The lowercase indices attain values 1, 2, 3, the uppercase indices only values 1 and 2. Einstein summation convention is used over repeating indices.

2 Approximate formulae for the phase velocity and polarization vectors

Let us consider a weakly anisotropic medium of an arbitrary symmetry specified by the tensor of the density normalized elastic parameters a_{ijkl} and a nearby isotropic reference medium specified by the velocity c_0, where $c_0 = \alpha$ for P waves and $c_0 = \beta$ for S waves. Let us specify a unit vector of phase normal n_i and in a plane perpendicular to it two mutually perpendicular unit vectors $e_i^{(1)}$ and $e_i^{(2)}$. The vectors $e_i^{(1)}$ and $e_i^{(2)}$ can be defined in the following way:

$$e_i^{(1)} = i_i^{(1)} \cos\phi + i_i^{(2)} \sin\phi, \qquad e_i^{(2)} = -i_i^{(1)} \sin\phi + i_i^{(2)} \cos\phi. \tag{1}$$

The vectors $i_i^{(1)}$ and $i_i^{(2)}$ are mutually perpendicular unit vectors also situated in the plane perpendicular to the vector n_i. Let us specify them in terms of components of the vector n_i as follows:

$$\vec{i}_1 \equiv D^{-1}(n_1n_3, n_2n_3, n_3^2 - 1), \quad \vec{i}_2 \equiv D^{-1}(-n_2, n_1, 0), \tag{2}$$

where

$$D = (n_1^2 + n_2^2)^{1/2}, \quad n_1^2 + n_2^2 + n_3^2 = 1. \tag{3}$$

For convenience, we also use the notation $n_i \equiv e_i^{(3)}$. For the above specification of the weakly anisotropic medium and of the phase normal, we can write *weak anisotropy matrix* B_{mn}, see Jech & Pšenčík (1989), Pšenčík & Gajewski (1998),

$$B_{mn} = a_{ijkl}n_jn_li_i^{(m)}i_k^{(n)} - c_0^2\delta_{mn}. \tag{4}$$

The matrix B_{mn} plays an important role in all formulae describing weak anisotropy. For the elements of the matrix B_{mn} expressed in terms of weak anisotropy parameters and components of the phase normal n_i, see the Appendix A.

For a fixed phase normal direction, the approximate equations of Jech & Pšenčík (1989) for perturbations of the phase velocity and polarization vectors reduce to the following formulae. The perturbation Δc of the phase velocity from the reference isotropic to a weakly anisotropic medium is given by

$$\Delta c_3 = \frac{1}{2\alpha} B_{33} \tag{5a}$$

for qP waves and by

$$\Delta c_1 = \frac{1}{2\beta}\bar{B}_{11}, \quad \Delta c_2 = \frac{1}{2\beta}\bar{B}_{22} \tag{5b}$$

for qS waves.

The perturbation $\Delta g_i^{(3)}$ of the polarization vector for qP waves is given by

$$\Delta g_i^{(3)} = \frac{B_{13}e_i^{(1)} + B_{23}e_i^{(2)}}{\alpha^2 - \beta^2}. \tag{6a}$$

The perturbations $\Delta g_i^{(K)}$ of the polarization vectors for qS waves are given by

$$\Delta g_i^{(1)} = \frac{\bar{B}_{13}}{\beta^2 - \alpha^2}(n_i + \frac{\bar{B}_{23}e_i^{(2)}}{\bar{B}_{11} - \bar{B}_{22}}), \qquad \Delta g_i^{(2)} = \frac{\bar{B}_{23}}{\beta^2 - \alpha^2}(n_i - \frac{\bar{B}_{13}e_i^{(1)}}{\bar{B}_{11} - \bar{B}_{22}}). \tag{6b}$$

The matrix $\bar{B}_{mn}$ is a symmetric matrix whose elements can be expressed in terms of elements of the weak anisotropy matrix B_{mn} and especially chosen angle ϕ:

$$\bar{B}_{11} = B_{11}\cos^2\phi + 2B_{12}\cos\phi\sin\phi + B_{22}\sin^2\phi,$$

$$\bar{B}_{22} = B_{11}\sin^2\phi - 2B_{12}\cos\phi\sin\phi + B_{22}\cos^2\phi,$$

$$\bar{B}_{12} = (B_{22} - B_{11})\cos\phi\sin\phi + B_{12}(cos^2\phi - \sin^2\phi) = 0,$$

$$\bar{B}_{33} = B_{33}, \quad \bar{B}_{13} = B_{13}\cos\phi + B_{23}\sin\phi, \quad \bar{B}_{23} = -B_{13}\sin\phi + B_{23}\cos\phi. \tag{7}$$

From Eqs.(5) and (6), we can see that the matrix $\bar{B}_{mn}$ appears only in the expressions for qS waves. We can also see from the third equation in (7) that the angle ϕ is chosen so that $\bar{B}_{12} = 0$. From the referred equation, we can find the formula for the determination of the angle ϕ:

$$\tan 2\phi = \frac{2B_{12}}{B_{11} - B_{22}}. \tag{8}$$

Alternatively, the matrix $\bar{B}_{mn}$, can be expressed in the following way

$$\bar{B}_{mn} = a_{ijkl}n_j n_l e_i^{(m)} e_k^{(n)} - c_0^2\delta_{mn}. \tag{9}$$

Here the vectors $e_i^{(I)}$ are given by Eqs.(1) with ϕ specified in Eq.(8).

Let us mention that approximate values of phase velocities in the examples shown later in this text were calculated by taking square roots of the following approximate formulae for the square of the qP-wave phase velocity

$$c_3^2 = B_{33} + \alpha^2 \tag{10a}$$

and for the squares of the qS-wave phase velocities

$$c_1^2 = \bar{B}_{11} + \beta^2, \qquad c_2^2 = \bar{B}_{22} + \beta^2. \tag{10b}$$

Eqs.(10) follow straightforwardly from Eqs.(5). Using Eq.(4) or Eq.(9), we can rewrite Eqs.(10) for qP waves

$$c_3^2 = a_{ijkl} n_i n_j n_k n_l \tag{11a}$$

and for qS waves

$$c_1^2 = a_{ijkl} n_j n_l e_i^{(1)} e_k^{(1)}, \qquad c_2^2 = a_{ijkl} n_j n_l e_i^{(2)} e_k^{(2)}. \tag{11b}$$

For similar expressions, see Farra (2001). We can see that the expressions for the square of phase velocity (11) are independent of the choice of the parameters of the reference medium.

3 Relation between phase normal and ray vector

Let us start from the well-known formula for the group velocity vector v_j in an anisotropic medium following from ray tracing equations, see e.g. Červený et al.(1977), Gajewski and Pšenčík (1990):

$$v_j = v N_j = \frac{dx_j}{dt} = c^{-1} a_{ijkl} n_l g_i g_k. \tag{12}$$

Here, v and c are the group and phase velocities and g_i denotes a polarization vector of a considered qP or qS wave. The ray vector N_j and the phase normal n_j generally differ. This can be simply shown if we consider the following expression for the phase velocity, see also the above references:

$$c^2 = a_{ijkl} n_j n_l g_i g_k. \tag{13}$$

Multiplying Eq.(12) by the phase normal n_j and taking into account Eq.(13), we find a well-known and useful identity

$$n_j v_j = n_j N_j v = c. \tag{14}$$

We can see that the vectors n_j and N_j are identical only if $v = c$, which is not generally the case in anisotropic media. The determination of the ray vector N_j from the phase normal n_j can be performed with the use of relation (12). Determination of n_j from N_j, is however, much more complicated task and it is usually solved by an iterative use of the equation (12), see e.g. Jech (1991), Horne and Leaney (2000). As Pšenčík (1996), we use the first-order perturbation method to find an approximate relation between n_j and N_j. We look for the ray vector N_j in an anisotropic medium as a perturbation of the ray vector in the reference isotropic medium. In the isotropic medium, the phase normal and the corresponding ray vector coincide. In the following, we consider the phase normal n_j fixed, i.e., the same in the isotropic and anisotropic medium. We can thus write

$$N_j = n_j + A_1 e_j^{(1)} + A_2 e_j^{(2)}, \tag{15}$$

where A_I are coefficients to be determined.

Using Eq.(15) and neglecting second-order terms, the group velocity vector v_j can be expressed as a sum of the group velocity vector in the reference isotropic medium and its perturbation

$$v_j = (c_0 + \Delta v)n_j + c_0(A_1 e_j^{(1)} + A_2 e_j^{(2)}). \tag{16}$$

By perturbing Eq.(12) from an isotropic reference to a weakly anisotropic medium we can find an alternative expression for the group velocity vector v_j. For any of the three waves propagating in anisotropic media we get (for simplicity, the superscripts denoting the type of the wave are omitted):

$$v_j = c_0 n_j - \Delta c n_j + c_0^{-1}[a_{ijkl}n_l e_i e_k + (\alpha^2 - \beta^2)\Big((n_k e_k)\Delta g_j - (n_k e_k)e_j + (n_k \Delta g_k)e_j\Big)$$

$$-\beta^2\Big(n_j - 2(e_k \Delta g_k)n_j\Big)]. \tag{17}$$

Comparison of projections of group velocity vector v_j given by Eqs.(16) and (17) onto the phase normal n_j yields

$$\Delta v = \Delta c. \tag{18}$$

Eq.(18) confirms well-known fact, see e.g. Backus (1965), that the phase and ray velocities have equal size in weakly anisotropic media. In deriving Eq.(18), we took into account Eqs.(13) and (14) specified for the reference isotropic medium, Eq.(5) for perturbation of the corresponding phase velocity Δc and an obvious identity $e_j \Delta g_j = 0$. Comparison of projections of the group velocity vector v_j given by Eqs.(16) and (17) onto the vector $e_j^{(I)}$ yields sought coefficients A_I:

$$A_I = c_0^{-2}[a_{ijkl}n_l e_i e_k e_j^{(I)} + (\alpha^2 - \beta^2)\Big((n_k e_k)(\Delta g_j e_j^{(I)}) - (n_k e_k)(e_j e_j^{(I)}) + (n_k \Delta g_k)(e_j e_j^{(I)})\Big)]. \tag{19}$$

Using Eqs.(15) and (19) we can write an approximate expression relating the ray vector N_J and the phase normal n_j

$$N_j = n_j + \Delta N_j, \tag{20}$$

where the term ΔN_j reads for qP waves

$$\Delta N_j^{(3)} = 2\alpha^{-2}(B_{13}e_j^{(1)} + B_{23}e_j^{(2)}) \tag{21a}$$

and for $qS1$ and $qS2$ waves

$$\Delta N_j^{(1)} = -n_j + \beta^{-2}(a_{ijkl}n_l e_i^{(1)} e_k^{(1)} - \bar{B}_{11}n_j - \bar{B}_{13}e_j^{(1)}),$$

$$\Delta N_j^{(2)} = -n_j + \beta^{-2}(a_{ijkl}n_l e_i^{(2)} e_k^{(2)} - \bar{B}_{22}n_j - \bar{B}_{23}e_j^{(2)}), \tag{21b}$$

Since A_I is a first-order quantity, Eq.(15) implies that everywhere, where a first-order quantity is multiplied by N_i or n_i, both vectors are interchangeable. This means that, within the first-order approximation, there is no difference if the perturbation ΔN_j is expressed with respect to n_j or N_j. This allows us to rewrite Eq.(20), within the same first-order approximation, into the form

$$n_j = N_j - \Delta N_j. \tag{22}$$

Eq.(22) should work without problems for a wave with convex slowness surface sections. It cannot work properly, however, when the slowness surface of the considered wave has a concave section (this indicates triplication of the group velocity surface and multivaluedness of the function $n_j = N_j(n_k)$). Eqs.(20) and (22) with (21) represent sought approximate formulae for the determination of the ray vector from phase normal and vice versa.

From Eqs.(16), (18), (20) and (21), we can also deduce final approximate expressions for the group velocity vectors in a weakly anisotropic medium. For qP waves, the group velocity vector has a form

$$v_j^{(3)} = (\alpha + \Delta c_3)n_j + \alpha \Delta N_j^{(3)}. \tag{23a}$$

For $qS1$ and $qS2$ waves, the group velocity vectors are

$$v_j^{(1)} = (\beta + \Delta c_1)n_j + \beta \Delta N_j^{(1)}, \qquad v_j^{(2)} = (\beta + \Delta c_2)n_j + \beta \Delta N_j^{(2)}. \tag{23b}$$

For Δc_k and $\Delta N_j^{(k)}$ see Eqs.(5) and (21), respectively.

Let us mention an interesting phenomenon. If we compare Eq.(6a) and Eq.(23a), we can see that the qP-wave polarization vector, qP-wave group velocity vector and the phase normal are coplanar in weakly anisotropic media. This is an extension of an observation made by Crampin (1981) for general anisotropy in symmetry planes. As Crampin (1981), we can also observe that the deviation of the group velocity and polarization vectors from the phase normal is in the same direction and the deviation of the group velocity vector exceeds the deviation of the polarization vector, at least for media close to Poisson solid ($\alpha^2 = 3\beta^2$). All three vectors become parallel when the elements B_{I3} vanish. Such directions are known as longitudinal directions, i.e., directions, in which pure modes propagate, Helbig (1994).

4 Numerical examples

We illustrate accuracy of the derived formulae on two models of anisotropic media. The first model is transversely isotropic with vertical axis of symmetry (VTI), see Shearer and Chapman (1989), the other is orthorhombic, see Farra (2001). Anisotropy (calculated as $2(c_{max} - c_{min})/(c_{max} + c_{min})\times$ 100%) of the VTI model is about 9% for qP wave, 29% and 30% for qS waves. We can see that in case of qS waves anisotropy is rather strong. Anisotropy of the orthorhombic model is 14% for qP waves and 7% and 5% for qS waves.

4.1 VTI symmetry

The medium is characterized by the density-normalized elastic parameters A_{ij}, in $(\mathrm{km/s})^2$, with values: $A_{11} = A_{22} = 20.16$, $A_{33} = 19.63$, $A_{12} = 7.40$, $A_{13} = A_{23} = 7.26$, $A_{44} = A_{55} = 3.48$, $A_{66} = 6.38$.

Figure 1 illustrates accuracy of approximate expressions for qP-wave (left) and qS-waves (right) phase velocities. Blue curve corresponds to the qSV wave, red curve to the

SH wave. The upper plots show comparison of exact phase velocity (solid curve) with approximate one (dotted curve) in a section containing the symmetry axis. The bottom plots show relative errors of the approximation. The angles along the horizontal axes specify phase normal. The angle 0^o corresponds to the phase normal along the symmetry axis, 90^o to the phase normal perpendicular to the symmetry axis. The approximate phase velocities are calculated as square roots of expressions (10) or (11). As pointed out in Sec.2, such an approximation is independent of the choice of the parameters of the reference medium. In their Appendix B, Pšenčík & Gajewski (1998) show that the approximate formula for the square of the qP-wave phase velocity yields always smaller or equal values than the exact formula. Equality occurs only in longitudinal directions (Helbig, 1994). All the mentioned phenomena can be clearly observed in Figure 1. We can see that in addition to longitudinal directions for 0^o and 90^o, there is an additional longitudinal direction for 43.6^o. The approximate formulae quite accurate; relative errors are less than 0.5 % for the qP wave and less than 2 % for the qSV wave. The approximate formula for squared phase velocity of the SH wave coincides with the exact one in the VTI case.

Figure 2 shows the group velocities as functions of phase normal direction. The display is the same as in Figure 1. The results of approximate formulae (dotted curves) are identical with results shown in Figure 1 because for a phase normal direction the ray velocities are equal to the phase velocities in the first-order approximation. In this case, the relative errors are larger. They reach nearly 2 % for the qP wave, 13 % for qSV wave and 4 % for SH wave.

Figure 2 illustrates accuracy of the approximate determination of the value of the group velocity. The group velocity vector is, however, also specified by its direction, the ray vector. Let us, therefore, test accuracy of approximate formulae (20) and (22) with (21) for the determination of the ray vector N_j from a given phase normal n_j and vice versa. The results are shown in Figure 3. The left column illustrates performance of the formula (20), the right column the formula (22). The upper plots show results for the qP wave, the bottom plots for qS waves. Solid curves correspond to exact and dotted curves to approximate results. The curves in the left column show the ray vector direction (vertical axes) as a function of the phase normal direction (horizontal axes), see Eq.(20). The curves in the right column show the phase normal direction as a function of the ray vector, see Eq.(22). For an isotropic medium the curves would be straight lines. Deviations from such lines indicate anisotropy. We can see that dotted curves obtained from Eq.(20) approximate solid curves quite well even in the case of the qSV wave (blue). In this case, we can see that some ray vector directions correspond to three different phase normal directions. This is an indication of the triplication of the corresponding wavefront (or, equivalently, of the group velocity surface). In the right column, we can see very good performance of formula (22) for the qP and SH waves. In the case of the qSV wave, however, the formula (22) yields, as expected, distorted result. It yields a single-valued and very inaccurate curve.

Plots in the upper part of Figure 4 show angular deviations of ray vectors of qP wave (left) and of qS waves (right) from the phase normal. Plots in the bottom part of Figure 4 show the errors of the approximate formulae (20), (21): deviations of the approximate ray vector from the exact one in degrees. The angles along horizontal axes specify the phase normal. In the upper plots, we can see that the deviation of the ray vector from the phase normal is always to one side and can reach 10^o for the qP wave, 30^o for the qSV wave and 20^o for the SH wave. The maximum errors of the approximate formulae are about 10 % of the maximum deviations of ray vectors and phase normals. They are slightly higher in the case of SH waves.

Figure 5 shows sections of the group velocity surface (wavefront at time t=1). The errors are, of course, the same as in Figure 1, but their distribution, because of a different parameterization of the horizontal axes (ray vector), is slightly different. An interesting phenomenon can be observed on the curve corresponding to the qSV wave. It shows that even the approximate formula can describe, very roughly, triplication of the group velocity surface.

4.1 Orthorhombic symmetry

The medium is characterized by the density-normalized elastic parameters A_{ij}, in $(\text{km/s})^2$, with values: $A_{11} = 10.8$, $A_{22} = 11.3$, $A_{33} = 8.5$, $A_{12} = 2.2$, $A_{13} = 1.9$, $A_{23} = 1.7$, $A_{44} = 3.6$, $A_{55} = 3.9$, $A_{66} = 4.3$.

Figure 6 shows plots of one half of the equal area plots of exact qP-, $qS1$- (faster) and $qS2$-wave phase velocities (top, middle and bottom). Figure 7 shows, in the same display, the plot of qS-wave phase velocity differences (top left), the plot of relative qS-wave phase velocity differences (bottom left) and relative errors of approximate formulae (11) for $qS1$- (top right) and $qS2$-wave phase velocities (bottom right). The phase velocities are calculated in the same way as in Figure 1. It can be clearly seen that the approximate formula for faster ($qS1$) wave works better than for the slower wave. We can also see that the relative errors are considerably smaller than the relative differences of qS waves.

The same phenomena can be observed in Figure 8, which shows relative errors of the approximate formulae for the group velocity of the qP wave (top), $qS1$ wave (middle) and $qS2$ wave (bottom). Similarly as in the case of phase velocities, we can see that the approximate formulae work relatively well close to horizontal and vertical directions. Zero deviations in the horizontal and vertical directions correspond to longitudinal directions, for which the approximate formulae give exact results.

Figure 8 illustrates accuracy of the determination of values of the group velocity. Figures 9-11 show accuracy of the approximate formulae (20), (21): deviations of the approximate ray vector from the exact one. Figure 9 shows qP-wave angular differences between the phase normal and the ray vector (top) and the deviations (in degrees) of the approximate ray vector determined from Eqs.(20) and (21a) from the exact one (bottom). As in Figure 4, the maximum deviations represent approximately 10% of the difference between the phase normal and the ray vector.

Figures 10 and 11 show the same as Figure 9 but for for $qS1$ (Figure 10) and $qS2$ (Figure 11) waves. Here maximum deviations are comparable with maximum differences between the phase normal and the ray vector. It is, however, necessary to emphasize that the maximum deviations are concentrated to very narrow regions around 45^o inclination of the phase normal. In the rest, the deviations are less than 1^o. The deviations are slightly larger for the slower wave.

5 Conclusions

An approximate formula relating the ray vector and the phase normal was presented. It can be used for the determination of each vector from knowledge of the other one for qP waves generally and for qS waves outside singular regions and regions of triplication of the group velocity surface.

The accuracy of the approximate formula was tested on two examples of models of anisotropic media. The results show that the approximate formula works suprisingly well even for a rather strong anisotropy. Generally less accurate results are obtained for qS waves. With formula (20), we can even approximately describe triplication. On the contrary, the reverse formula (22) fails in the case of triplication.

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Appendix A: Expressions for the elements of the weak anisotropy matrix

The elements of the symmetric matrix B_{mn} given in Eq.(4) can be expressed in the following way:

$$\begin{aligned}
B_{11} = {} & 2\alpha^2 n_3^3 D^{-2}[(\chi_x n_1 + \chi_y n_2)n_1 n_2 + 2(\epsilon_{15} n_1^3 + \epsilon_{24} n_2^3)] \\
& +2\alpha^2 n_3^2 \Big((\epsilon_z - \delta_x)n_1^2 + (\epsilon_z - \delta_y)n_2^2 + 2\chi_z n_1 n_2 \\
& +D^{-2}[\epsilon_x n_1^4 + \epsilon_y n_2^4 + \delta_z n_1^2 n_2^2 + 2(\epsilon_{16} n_1^2 + \epsilon_{26} n_2^2)n_1 n_2]\Big) \\
& -2\alpha^2 n_3[(\chi_x n_1 + \chi_y n_2)n_1 n_2 + \epsilon_{34} n_2 + \epsilon_{35} n_1 + D^{-1}(\epsilon_{15} n_1^3 + \epsilon_{24} n_2^3)] \\
& +2\beta^2 D^{-2}(\gamma_x n_1^2 + \gamma_y n_2^2 - 2\epsilon_{45} n_1 n_2),
\end{aligned}$$

$$\begin{aligned}
B_{12} = {} & \alpha^2 n_3^2\Big((\epsilon_{35} n_2 - \epsilon_{34} n_1) + D^{-2}[\chi_x n_1^3 - \chi_y n_2^3)n_1 n_2 + (\chi_y - 2\epsilon_{15})n_1^2 n_2 - (\chi_x - 2\epsilon_{24})n_1 n_2^2]\Big) \\
& +\alpha^2 n_3\Big((\delta_x - \delta_y)n_1 n_2 + \chi_z n_2^2 - \chi_z n_1^2 \\
& +D^{-2}[(\delta_z - 2\epsilon_x)n_1^3 n_2 - (\delta_z - 2\epsilon_y)n_1 n_2^3 + 3(\epsilon_{26} - \epsilon_{16})n_1^2 n_2^2 + \epsilon_{16} n_1^4 - \epsilon_{26} n_2^4]\Big) \\
& +\beta^2 D^{-2} n_3[2(\gamma_y - \gamma_x)n_1 n_2 + \epsilon_{45} n_1^2 - \epsilon_{45} n_2^2] \\
& +\alpha^{-2}[(\chi_x - \epsilon_{24})n_1 n_2^2 - (\chi_y - \epsilon_{15})n_1^2 n_2 + \epsilon_{56} n_1 + \epsilon_{46} n_2],
\end{aligned}$$

$$\begin{aligned}
B_{22} = {} & 2\beta^2 n_3^2[D^{-2}(\gamma_x n_2^2 + \gamma_y n_1^2 - \epsilon_{45} n_1 n_2) - \gamma_z] \\
& +2\alpha^2 n_3\Big(\epsilon_{56} n_2 + \epsilon_{46} n_1) - D^{-2}[(\chi_x - \epsilon_{24})n_1^2 n_2 + (\chi_y - \epsilon_{15})n_1 n_2^2]\Big) \\
& +2\alpha^2 D^{-2}[\epsilon_x + \epsilon_y - \delta_z)n_1^2 n_2^2 + (\epsilon_{26} - \epsilon_{16})n_1^3 n_2 - (\epsilon_{26} - \epsilon_{16})n_1 n_2^3] + 2\beta^2 \gamma_z,
\end{aligned}$$

$$B_{13} = \alpha^2 D^{-1}\{2\epsilon_z n_3^5 + n_3^4(\epsilon_{34}n_2 + \epsilon_{35}n_1) + n_3^3(\delta_x n_1^2 + \delta_y n_2^2 + 2\chi_z n_1 n_2 - 2\epsilon_z)$$
$$+n_3^2[(4\chi_x - 3\epsilon_{34})n_1^2 n_2 + (4\chi_y - 3\epsilon_{35})n_1 n_2^2 + (4\epsilon_{15} - 3\epsilon_{35})n_1^3 + (4\epsilon_{24} - 3\epsilon_{34})n_2^3]$$
$$+n_3[(2\delta_z - \delta_x - \delta_y)n_1^2 n_2^2 + 2(2\epsilon_{16} - \chi_z)n_1^3 n_2 + 2(2\epsilon_{26} - \chi_z)n_1 n_2^3 + (2\epsilon_x - \delta_x)n_1^4 + (2\epsilon_y - \delta_y)n_2^4]$$
$$-\chi_x n_1^2 n_2 - \chi_y n_1 n_2^2 - \epsilon_{15}n_1^3 - \epsilon_{24}n_2^3\},$$

$$B_{23} = \alpha^2 D^{-1}\{n_3^3(\epsilon_{34}n_1 - \epsilon_{35}n_2) + n_3^2[(\delta_y - \delta_x)n_1 n_2 + \chi_z n_1^2 - \chi_z n_2^2]$$
$$+n_3[(2\chi_y - 3\epsilon_{15})n_1^2 n_2 - (2\chi_x - 3\epsilon_{24})n_1 n_2^2 + \chi_x n_1^3 - \chi_y n_2^3]$$
$$+(\delta_z - 2\epsilon_x)n_1^3 n_2 + (2\epsilon_y - \delta_z)n_1 n_2^3 + 3(\epsilon_{26} - \epsilon_{16})n_1^2 n_2^2 + \epsilon_{16}n_1^4 - \epsilon_{26}n_2^4\},$$

$$B_{33} = 2\alpha^2[\epsilon_z n_3^4 + 2n_3^3(\epsilon_{34}n_2 + \epsilon_{35}n_1) + n_3^2(\delta_x n_1^2 + \delta_y n_2^2 + 2\chi_z n_1 n_2)$$
$$+2n_3(\chi_x n_1^2 n_2 + \chi_y n_1 n_2^2 + \epsilon_{15}n_1^3 + \epsilon_{24}n_2^3) + \epsilon_x n_1^4 + \delta_z n_1^2 n_2^2 + \epsilon_y n_2^4 + 2\epsilon_{16}n_1^3 n_2 + 2\epsilon_{26}n_1 n_2^3]. \quad (A1)$$

The symbol D is defined in Eq.(3). The coefficients of the directional cosines n_i in (A1) are the 21 WA parameters and they read

$$\delta_x = \frac{A_{13} + 2A_{55} - \alpha^2}{\alpha^2}, \quad \delta_y = \frac{A_{23} + 2A_{44} - \alpha^2}{\alpha^2}, \quad \delta_z = \frac{A_{12} + 2A_{66} - \alpha^2}{\alpha^2},$$

$$\chi_x = \frac{A_{14} + 2A_{56}}{\alpha^2}, \quad \chi_y = \frac{A_{25} + 2A_{46}}{\alpha^2}, \quad \chi_z = \frac{A_{36} + 2A_{45}}{\alpha^2},$$

$$\epsilon_{15} = \frac{A_{15}}{\alpha^2}, \quad \epsilon_{16} = \frac{A_{16}}{\alpha^2}, \quad \epsilon_{24} = \frac{A_{24}}{\alpha^2}, \quad \epsilon_{26} = \frac{A_{26}}{\alpha^2},$$

$$\epsilon_{34} = \frac{A_{34}}{\alpha^2}, \quad \epsilon_{35} = \frac{A_{35}}{\alpha^2}, \quad \epsilon_{46} = \frac{A_{46}}{\alpha^2}, \quad \epsilon_{56} = \frac{A_{56}}{\alpha^2}, \quad \epsilon_{45} = \frac{A_{45}}{\beta^2},$$

$$\epsilon_x = \frac{A_{11} - \alpha^2}{2\alpha^2}, \quad \epsilon_y = \frac{A_{22} - \alpha^2}{2\alpha^2}, \quad \epsilon_z = \frac{A_{33} - \alpha^2}{2\alpha^2},$$

$$\gamma_x = \frac{A_{55} - \beta^2}{2\beta^2}, \quad \gamma_y = \frac{A_{44} - \beta^2}{2\beta^2}, \quad \gamma_z = \frac{A_{66} - \beta^2}{2\beta^2}. \quad (A2)$$

Here α and β are P and S wave velocities of the reference isotropic medium, which can be chosen arbitrarily (for example so that one or two of the WA parameters are made zero).

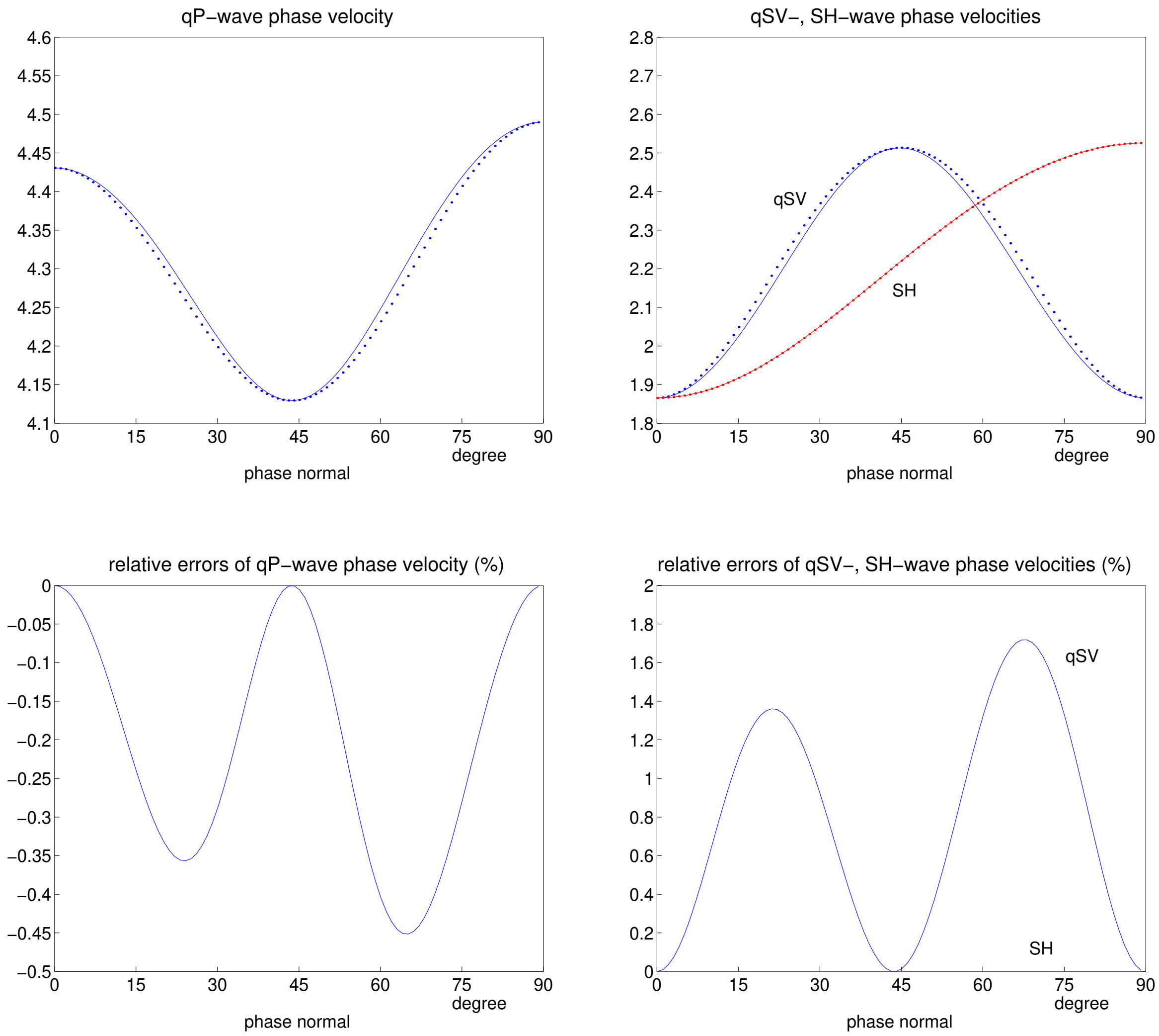

qP−wave phase velocity
qSV−, SH−wave phase velocities
qSV
SH
relative errors of qP−wave phase velocity (%)
relative errors of qSV−, SH−wave phase velocities (%)
qSV
SH
phase normal
degree

Figure 1

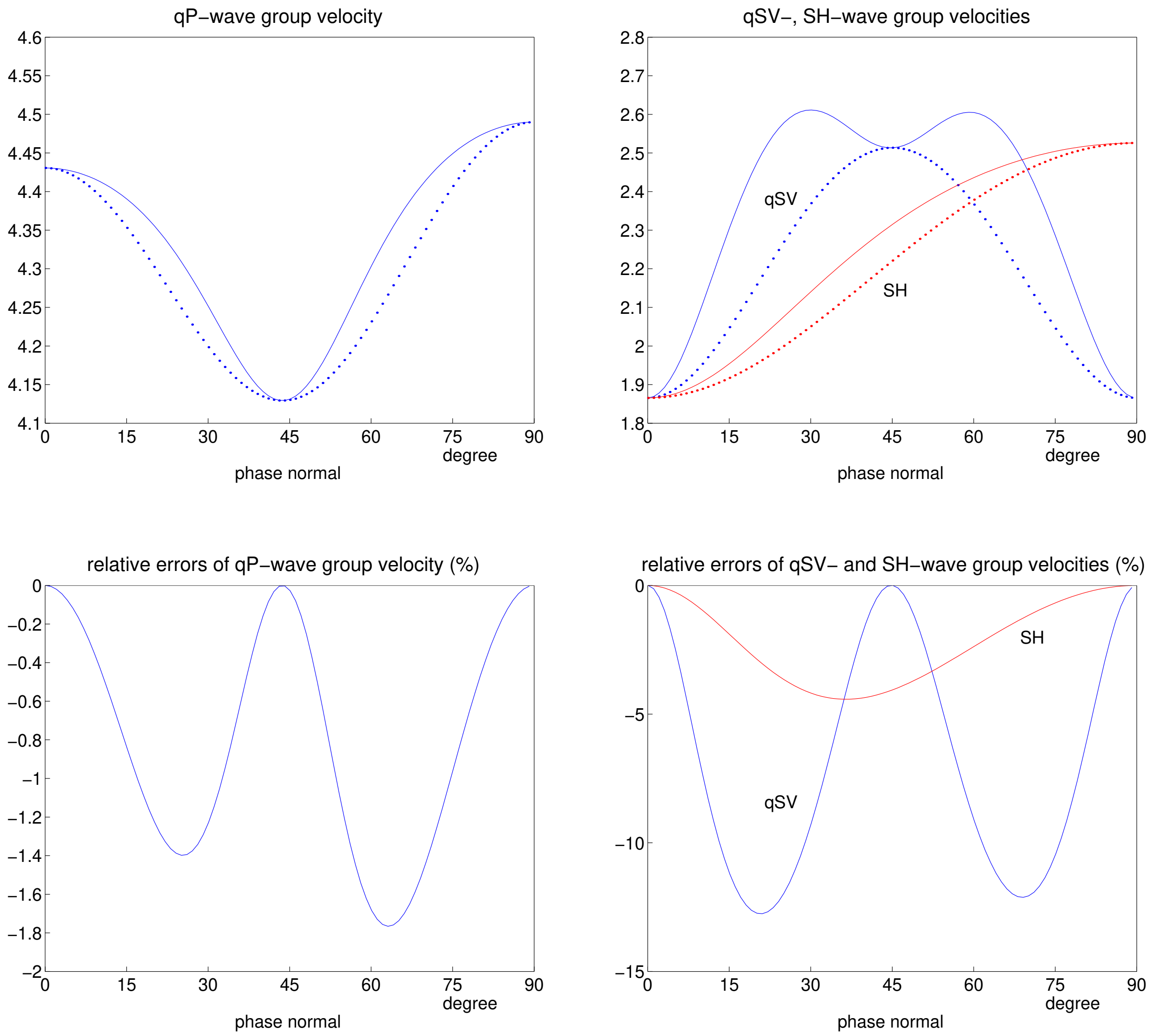

qP–wave group velocity
qSV–, SH–wave group velocities
qSV
SH
relative errors of qP–wave group velocity (%)
relative errors of qSV– and SH–wave group velocities (%)
phase normal
degree

Figure 2

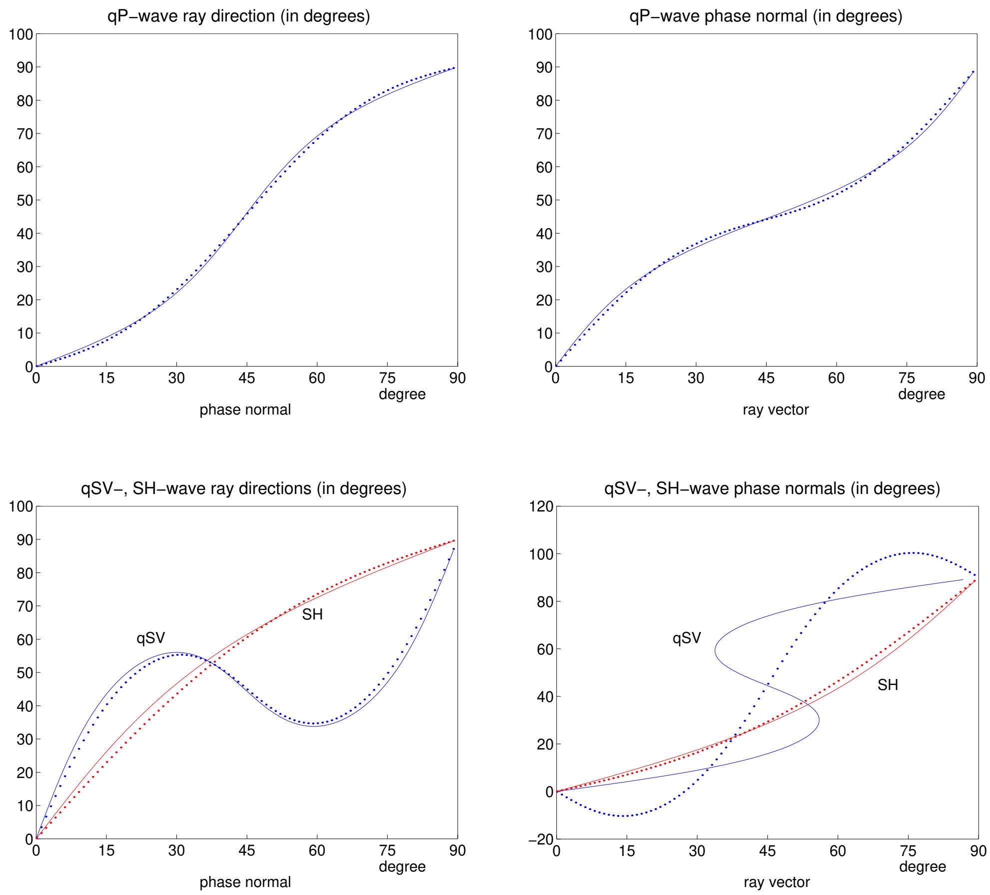

qP−wave ray direction (in degrees)
phase normal
degree
qP−wave phase normal (in degrees)
ray vector
degree
qSV−, SH−wave ray directions (in degrees)
qSV
SH
phase normal
degree
qSV−, SH−wave phase normals (in degrees)
qSV
SH
ray vector
degree

Figure 3

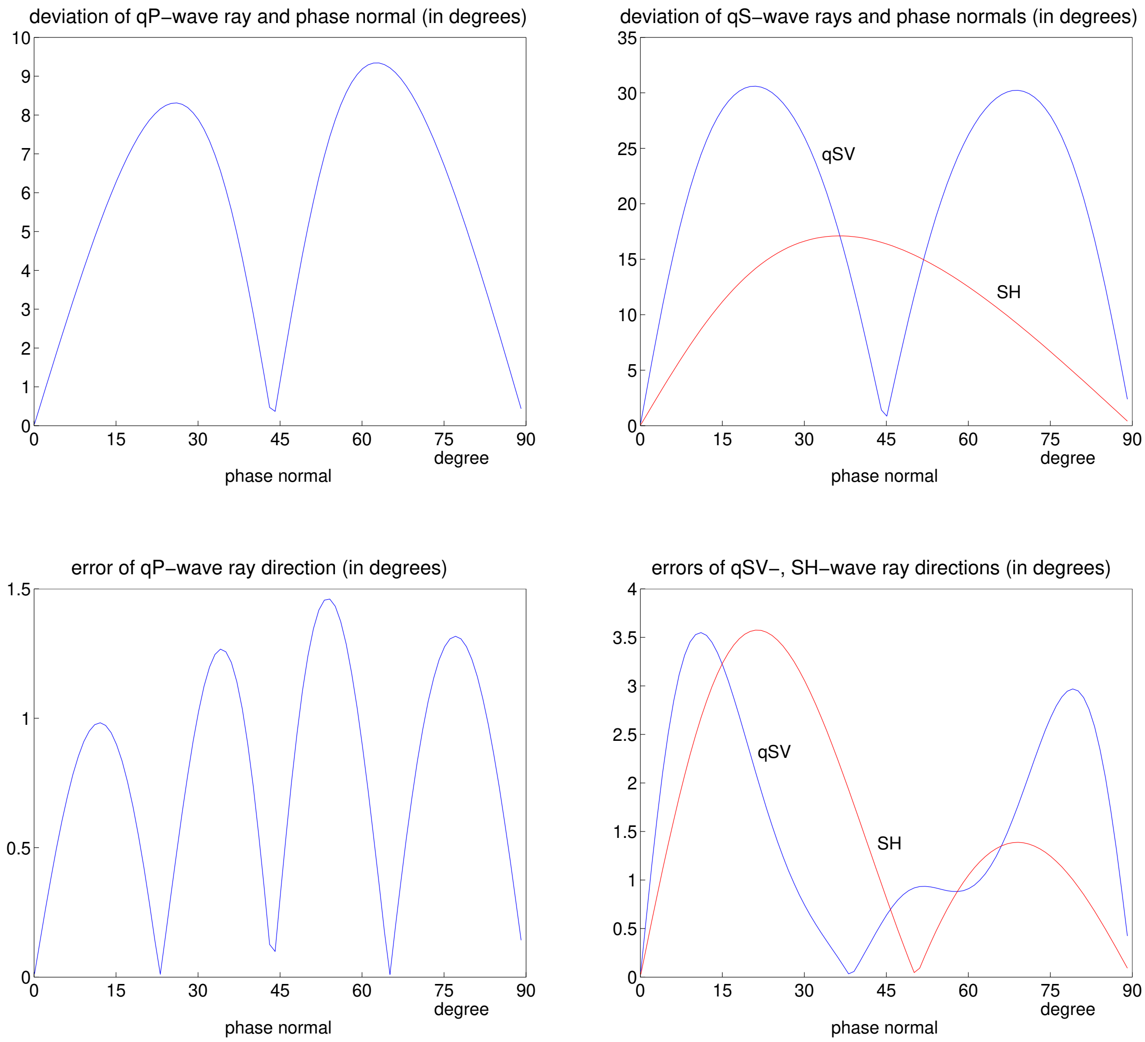
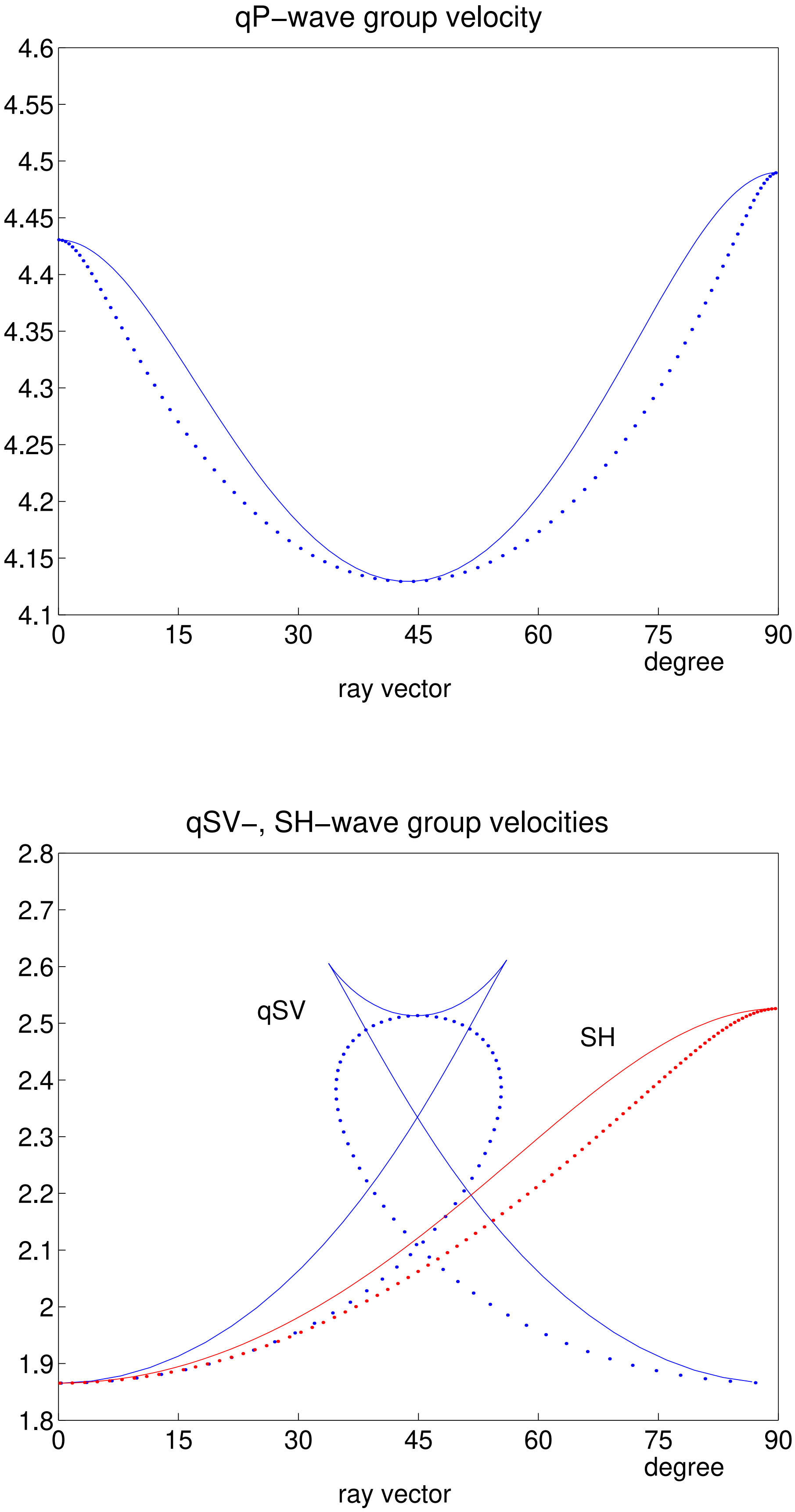

deviation of qP−wave ray and phase normal (in degrees)
deviation of qS−wave rays and phase normals (in degrees)
qSV
SH
error of qP−wave ray direction (in degrees)
errors of qSV−, SH−wave ray directions (in degrees)
qSV
SH
degree
phase normal

Figure 4


qP–wave group velocity
4.6
4.55
4.5
4.45
4.4
4.35
4.3
4.25
4.2
4.15
4.1
0
15
30
45
60
75
90
degree
ray vector
qSV–, SH–wave group velocities
2.8
2.7
2.6
2.5
2.4
2.3
2.2
2.1
2
1.9
1.8
qSV
SH
0
15
30
45
60
75
90
degree
ray vector
Figure 5

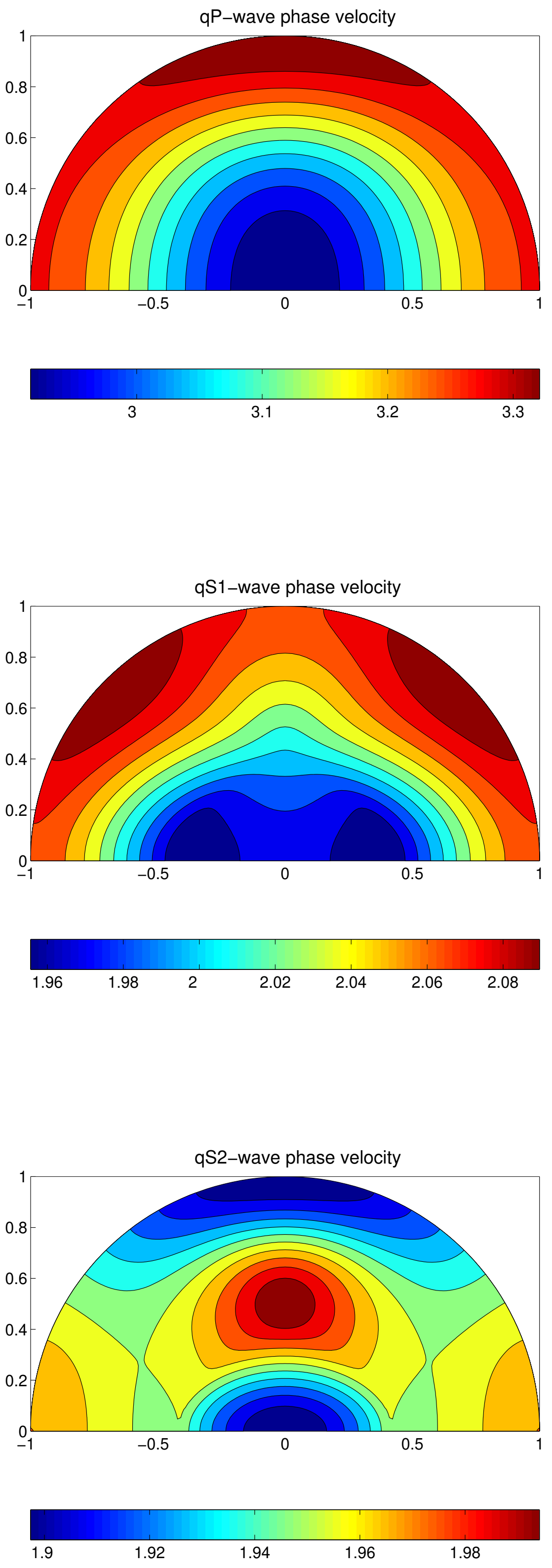

qP−wave phase velocity
1
0.8
0.6
0.4
0.2
0
−1
−0.5
0
0.5
1
3
3.1
3.2
3.3
qS1−wave phase velocity
1
0.8
0.6
0.4
0.2
0
−1
−0.5
0
0.5
1
1.96
1.98
2
2.02
2.04
2.06
2.08
qS2−wave phase velocity
1
0.8
0.6
0.4
0.2
0
−1
−0.5
0
0.5
1
1.9
1.92
1.94
1.96
1.98

Figure 6

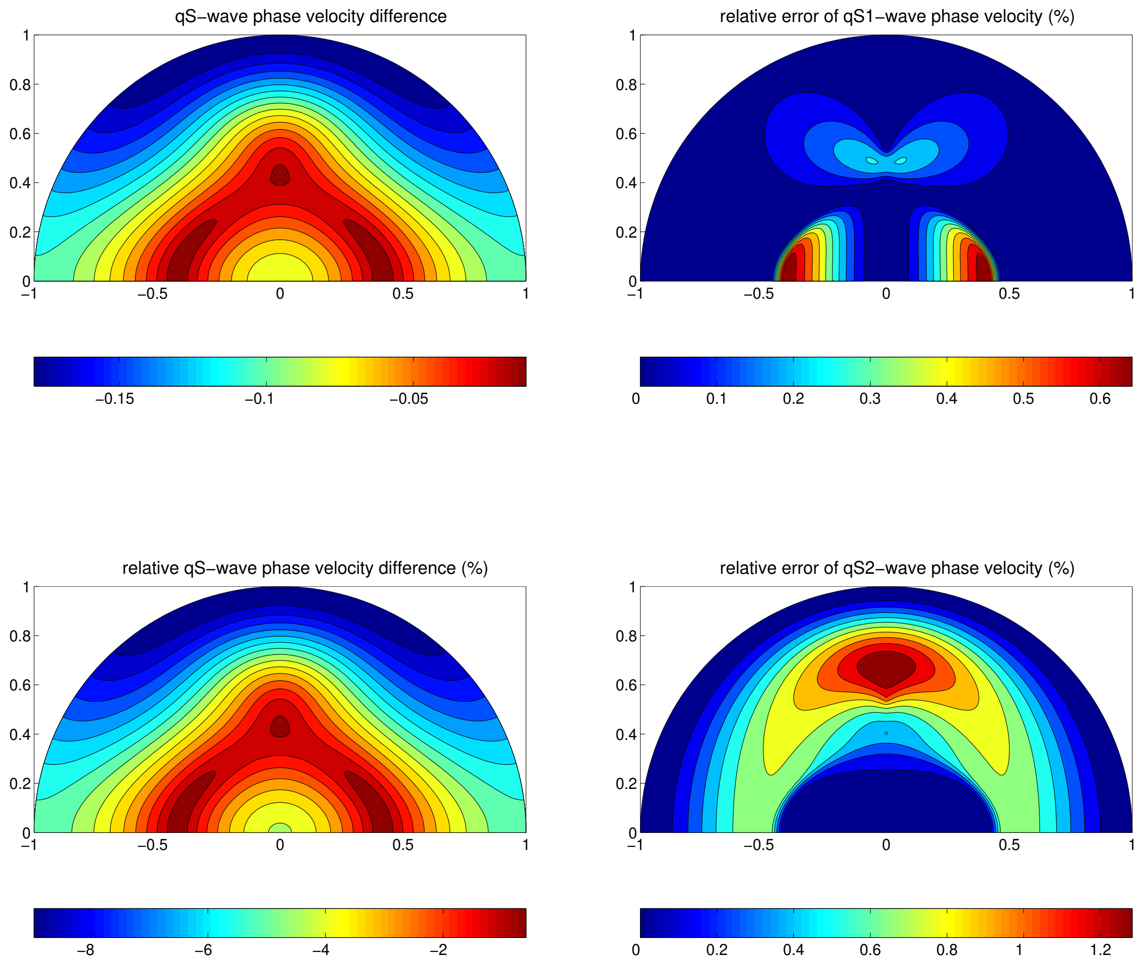

qS−wave phase velocity difference
relative error of qS1−wave phase velocity (%)
relative qS−wave phase velocity difference (%)
relative error of qS2−wave phase velocity (%)

Figure 7

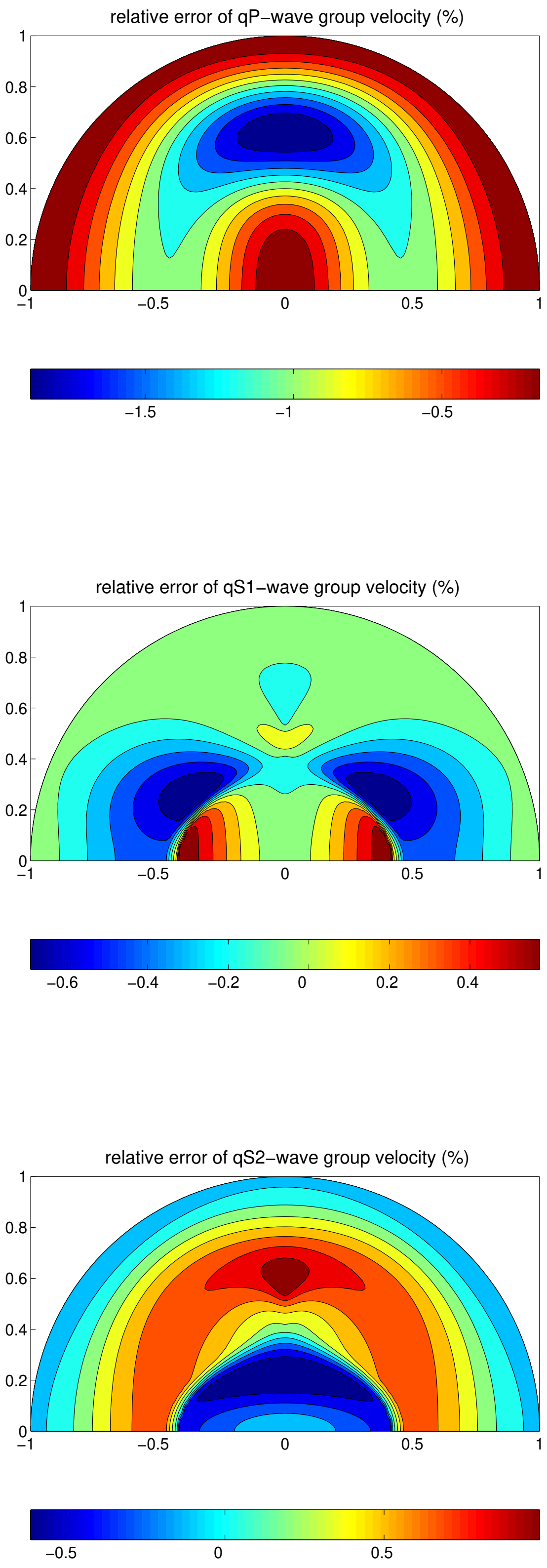
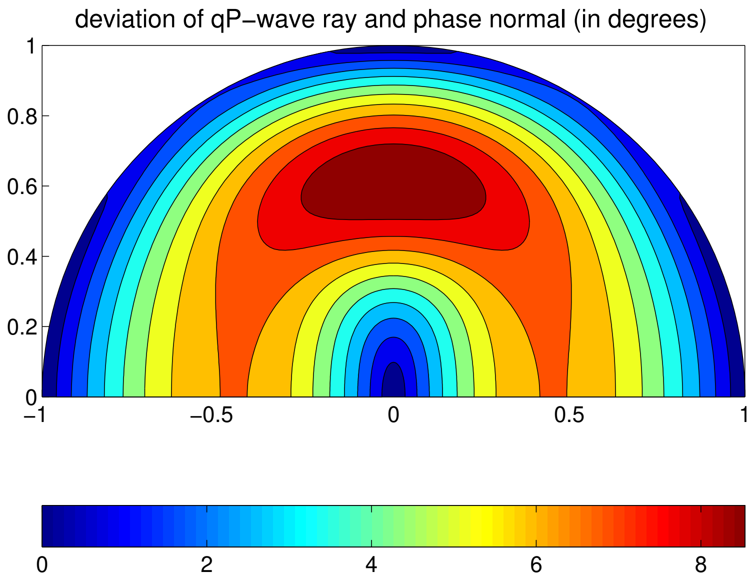

relative error of qP−wave group velocity (%)
relative error of qS1−wave group velocity (%)
relative error of qS2−wave group velocity (%)

Figure 8


deviation of qP−wave ray and phase normal (in degrees)
1
0.8
0.6
0.4
0.2
0
−1
−0.5
0
0.5
1
0
2
4
6
8

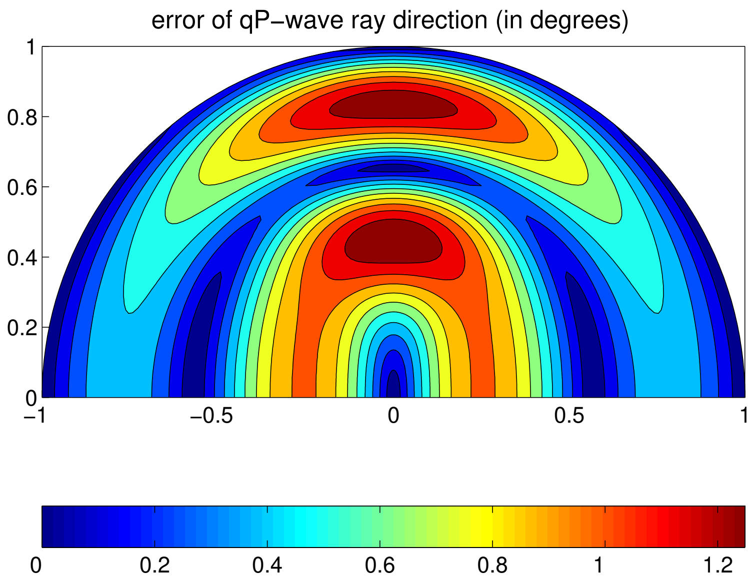

error of qP−wave ray direction (in degrees)
1
0.8
0.6
0.4
0.2
0
−1
−0.5
0
0.5
1
0
0.2
0.4
0.6
0.8
1
1.2

Figure 9

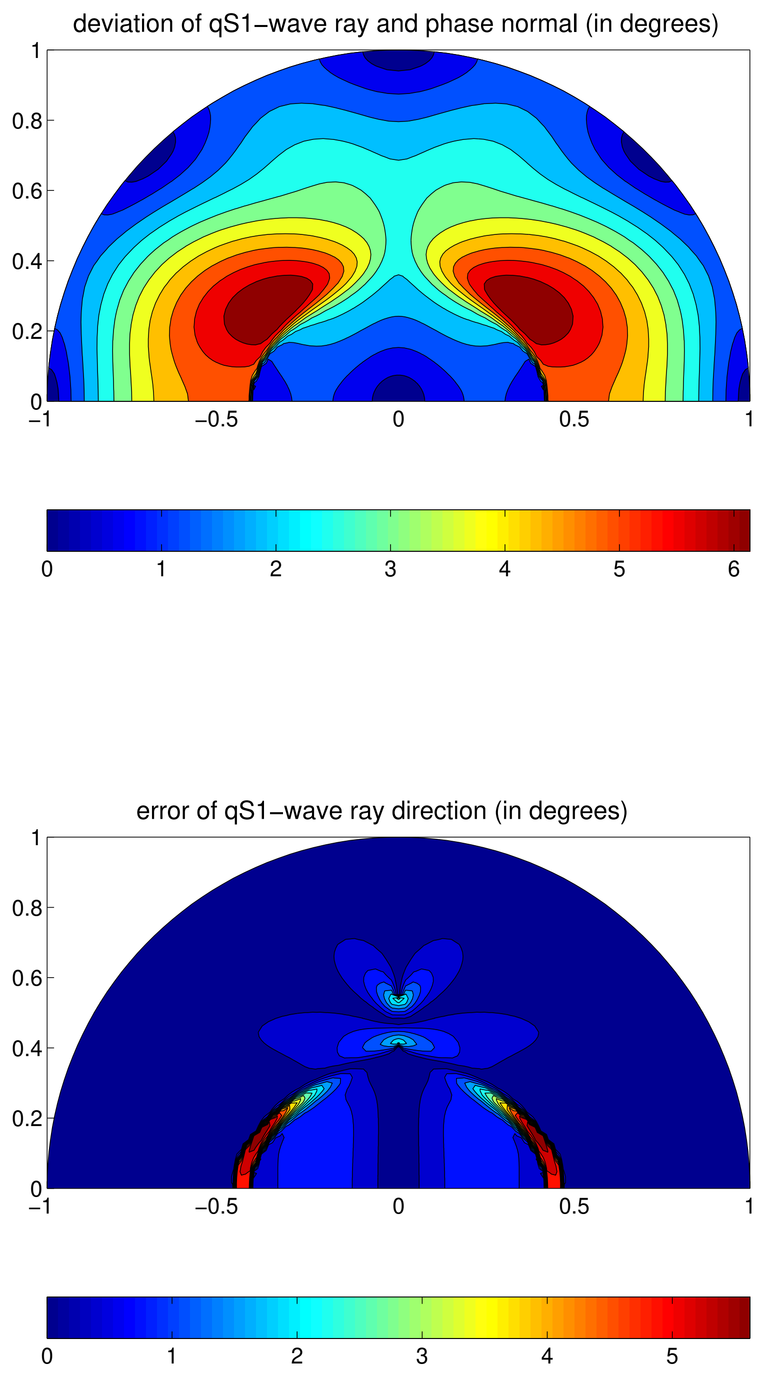

deviation of qS1−wave ray and phase normal (in degrees)
error of qS1−wave ray direction (in degrees)

Figure 10

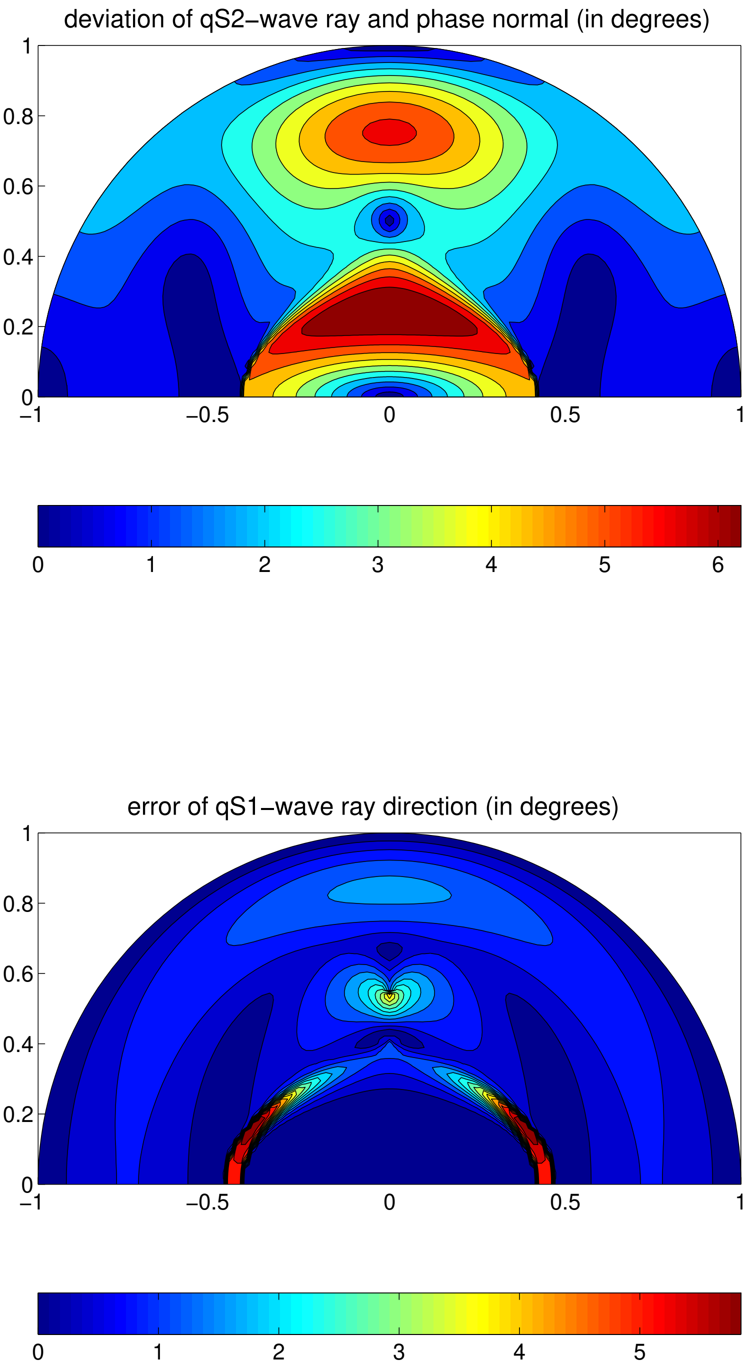

deviation of qS2−wave ray and phase normal (in degrees)
error of qS1−wave ray direction (in degrees)

Figure 11