Ray-theory amplitudes and synthetic seismograms in 2-D inhomogeneous isotropic layered structures. Program packages SEIS

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Summary

The compact disk SW3D-CD-6, included in this report, contains program package SEIS88, designed for the computation of seismic rays, travel times, ray amplitudes, ray synthetic seismograms and particle ground motions of seismic body waves, propagating in inhomogeneous isotropic layered 2-D structures. Program package SEIS88 represents a modification of previously designed and broadly distributed program packages SEIS81 and SEIS83. All these packages contain detailed documentations, which explain the input data and outputs. Notwithstanding, the exact meaning of computed ray amplitudes has not been strictly described in any generally available publication. This may lead to some confusion if, for example, program packages SEIS are used to study the reciprocity of ray-theory computations. This contribution should fill this gap and give a detailed explanation of ray-theory amplitudes computed by program packages SEIS.

Key words
Ray-theory amplitudes, 2-D structures, program packages SEIS.

1 Introduction

The main purpose of this article is to give general expressions for zero-order ray-theory amplitudes of high-frequency seismic body waves, propagating in inhomogeneous isotropic layered 2-D structures, computed in program packages SEIS. In the following, we call zero-order ray-theory amplitudes briefly ray-theory amplitudes. The first version of the package SEIS, called SEIS81, was designed before the year 1981. In that time, the equations for the ray-theory amplitudes were mostly taken from the book Červený, Molotkov

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and Pšenčík (1977). Although all the used equations are correct, they do not cover certain situations which play a very important role in recent seismological applications. For example, the above-mentioned book does not contain expressions for the ray-theory elastodynamic Green functions. It only contains expressions for omnidirectional (explosive) point sources. Moreover, the structure of the equations is not suitable for certain studies, as individual factors in the equations are not reciprocal (although they could be made reciprocal). Finally, there are some limitations in the computation of the KMAH index, in the computations of the line source waveforms, etc.

For these reasons, we first present very general more recent expressions for 2-D ray-theory amplitudes, more suitable and general than those used in older versions of packages SEIS. We take them either directly from the book of Červený (2001), or specify the general 3-D expressions presented there for the 2-D situation. We do not rederive these expressions here, we rely on the derivations presented in Červený (2001). We explain, however, in detail individual factors in these expressions. General 2-D expressions for ray-theory amplitudes are presented and discussed in Sections 2 and 3 of this contribution. In Section 4, we compare the general expressions with those presented in Červený, Molotkov and Pšenčík (1977), and explain the differences. Finally, in Section 5, we discuss the ray-theory amplitudes computed in program packages SEIS81, SEIS83 and SEIS88. The reason is that even the packages SEIS81 and SEIS83 are still being used at many seismological institutions all over the world. We believe that the following explanations and discussions will be valuable for all seismologists who have been using or intend to use program packages SEIS.

We consider an isotropic laterally varying perfectly elastic layered structure, a point source situated at S and a receiver situated at R. We denote by α the velocity of P waves, by β the velocity of S waves, and by ρ the density. Let us assume that the model is specified in Cartesian coordinates x_1, x_2, x_3 and that the medium parameters α, β, ρ and the structural interfaces do not depend on one Cartesian coordinate, say x_2, and that the source S and receiver R are situated in the plane $x_2 = 0$ (2-D case). The rays of an arbitrary multiply reflected (possibly converted) wave, connecting S and R, are then planar curves, situated completely in the plane $x_2 = 0$. The rays connecting S and R are computed by two-point ray tracing. At any point M of the ray, we determine the Cartesian components of the slowness vector p_1 and p_3 (with $p_2 = 0$), and the basis vectors of the ray-centered coordinate system $\vec{e}_1, \vec{e}_2, \vec{e}_3$. The unit vector $\vec{e}_3$ is tangent to the ray, and is related to $\vec{p}$ as follows:

$$\vec{e}_3(M) = V(M)\vec{p}(M) \; , \tag{1}$$

where $V(M)$ is the velocity of the wave under consideration (either α or β). The unit vectors $\vec{e}_1(M)$ and $\vec{e}_2(M)$ are perpendicular to the ray, and are taken as follows: $\vec{e}_2(M)$ is oriented along the Cartesian x_2-axis,

$$\vec{e}_2(M) = (0, 1, 0) \; , \tag{2}$$

and $\vec{e}_1(M)$ is given by the relation,

$$\vec{e}_1(M) = \vec{e}_2(M) \times \vec{e}_3(M) \; . \tag{3}$$

In the plane perpendicular to the ray (specified by $\vec{e}_1(M)$ and $\vec{e}_2(M)$), we can also introduce local Cartesian coordinates q_1, q_2, known as ray-centered coordinates. The relevant basis vectors are given by $\vec{e}_1(M)$, $\vec{e}_2(M)$. Note that the unit vectors $\vec{e}_1(M)$, $\vec{e}_2(M)$, $\vec{e}_3(M)$ represent also the polarization vectors of SV, SH and P waves, respectively.

The ray tracing also yields the ray-theory travel time $T(R,S)$ of the wave under consideration, along the ray between S and R.

2 General expressions for the 2-D ray-theory amplitudes for a point source

The Cartesian components of the ray-theory complex-valued displacement vector $u_i(R,t)$ of a selected elementary wave at the receiver point R are given by relations,

$$u_i(R,t) = U_i(R) F(t - T(R,S)) \ . \tag{4}$$

Here t is the running time, $T(R,S)$ the travel time from S to R, $F(\theta)$ analytical signal, and $U_i(R)$ the i-th Cartesian component of the ray-theory complex-valued vectorial amplitude function (briefly ray-theory amplitude). The analytical signal $F(\theta)$ is given by the relation

$$F(\theta) = x(\theta) + \mathrm{i} g(\theta) \ , \tag{5}$$

where $x(\theta)$ is the source-time function, which can be chosen as an arbitrary real-valued function of θ, and $g(\theta)$ the Hilbert transform of $x(\theta)$. For time-harmonic waves, $F(\theta) = \exp[-\mathrm{i}\omega\theta]$, where ω is the circular frequency, $\omega = 2\pi f$.

In this section, we present general expressions for 2-D ray-theory amplitudes $U_i(R)$, for a point source (or a line source) situated at S. For a detailed derivation, see Červený (2001). To shorten the text, we use a shorthand notation if we refer to equations of Červený (2001). For example, the reference (C-5.2.109) refers to Equation (5.2.109) of Červený (2001).

The Cartesian components of the complex-valued vectorial ray-theory amplitude $U_i(R)$ at the receiver point R of the elementary wave generated by a point source situated at S in a 2-D model are given by the following relation:

$$U_i(R) = \left[\frac{V(S)\rho(S)}{V(R)\rho(R)}\right]^{1/2} \frac{\exp[\mathrm{i}T^C(R,S)]}{\mathcal{L}(R,S)} \mathcal{D}_{ik}(R) \mathcal{R}^C(R,S) \mathcal{G}_j^{(q)}(S;\gamma_1,\gamma_2) \ , \tag{6}$$

see (C-5.2.109) and (C-5.2.113). The index $i = 2$ applies to SH waves (polarized along $\vec{e}_2$), and indices $i = 1$ or $i = 3$ to P-SV waves (with polarization vectors $\vec{e}_1, \vec{e}_3$ in the plane $x_2 = 0$). Thus, the SH-wave ray-theory amplitude is fully specified by one Cartesian component $U_2(R)$, but any of P- and SV-wave ray theory amplitudes is specified by two Cartesian components, $U_1(R)$ and $U_3(R)$.

For $i = 2$ (SH waves), also $k = 2$ and $j = 2$ in Eq.(6). For $i = 1, 3$ (P-SV waves), k and j may take only values 1 or 3. Indices k and j are fixed and are specified in the following way: j specifies the type of the wave generated by the source: $j = 1$ for SV wave, $j = 3$ for P waves. Similarly, k specifies the type of the wave arriving at the receiver R: $k = 1$ for SV wave, $k = 3$ for P wave. The equation (6) is valid for an arbitrary multiply reflected/transmitted unconverted or converted P-SV wave, specified by a suitable alphanumeric code. For $i = 2$, the wave cannot be converted; it is completely SH, along the whole ray.

The individual symbols in (6) have the following meaning:

1. $V(S)$ and $V(R)$ denote the velocities at S and R. V is either α or β, depending on the type of the wave at S and R.

2. $\mathcal{D}_{ik}(R)$ is the *conversion coefficient* at R. If the point R is situated in a smooth medium, the conversion coefficients reduce to the Cartesian components of the polarization vectors: $\mathcal{D}_{ik}(R) = e_{ki}(R)$, where $e_{ki}(R)$ is the i-th Cartesian component of the polarization vector $\vec{e}_k(R)$. For the receiver R situated at the Earth's surface, $\mathcal{D}_{ik}(R)$ are given by (C-5.3.33). For the receiver R situated at a structural interface, $\mathcal{D}_{ik}(R)$ are given by (C-5.3.34).

3. $\mathcal{R}^C(R,S)$ is the *complete energy-normalized R/T (reflection/transmission) coefficient* along the ray, between S and R. It equals the product of energy-normalized R/T coefficients, determined at all points of incidence of the ray on structural interfaces and/or the Earth's surface. The type of the R/T coefficient at any point of incidence must correspond to the type of the elementary wave under consideration. For SH waves ($i = 2$), only SH energy-normalized R/T coefficients are included in $\mathcal{R}^C$. For P-SV waves ($i = 1, 3$), P→P, P→SV, SV→P and SV→SV energy-normalized R/T coefficients may be involved in $\mathcal{R}^C$.

Any individual energy-normalized R/T coefficient $\mathcal{R}(Q)$ can be expressed in terms of a standard displacement R/T coefficient $R(Q)$ as follows, see (C-5.2.43),

$$\mathcal{R}(Q) = R(Q)\Big[\frac{V(\tilde{Q})\rho(\tilde{Q})\cos i(\tilde{Q})}{V(Q)\rho(Q)\cos i(Q)}\Big]^{1/2} . \tag{7}$$

Here Q is the point of incidence, and $\tilde{Q}$ is the appropriate R/T point, $\tilde{Q} \equiv Q$, $i(Q)$ is the acute angle of incidence, and $i(\tilde{Q})$ is the acute angle of R/T.

Using (7), we obtain

$$\mathcal{R}^C(R,S) = R^C(R,S)\prod_{i=1}^{N}\Big[\frac{V(\tilde{Q}_i)\rho(\tilde{Q}_i)\cos i(\tilde{Q}_i)}{V(Q_i)\rho(Q_i)\cos i(Q_i)}\Big]^{1/2} . \tag{8}$$

Here $R^C(R,S)$ is the complete displacement R/T coefficient, Q_i ($i = 1, 2, ..., N$) are points of incidence and $\tilde{Q}_i$ ($i = 1, 2, ..., N$) are the appropriate R/T points. It is assumed that the ray strikes N-times structural interfaces or the Earth's surface between S and R.

The advantage of the complete energy-normalized R/T coefficient $\mathcal{R}^C(R,S)$ over the complete displacement R/T coefficient $R^C(R,S)$ is that $\mathcal{R}^C(R,S)$ is reciprocal, while $R^C(R,S)$ is not,

$$\mathcal{R}^C(R,S) \;=\; \mathcal{R}^C(S,R) , \tag{9}$$

$$R^C(R,S) \;\; \neq \;\; R^C(S,R) \; . \tag{10}$$

The analytical expressions for individual R/T coefficients $R(Q)$ and $\mathcal{R}(Q)$ can be found in Section 5.3 of Červený (2001).

4) $\mathcal{G}_j^{(q)}(S;\gamma_1,\gamma_2)$ are *ray-centered components of the radiation function* at the point source S for the ray specified by ray parameters γ_1, γ_2. For a point source at S, the ray parameters γ_1, γ_2 usually represent the take-off angles of rays at S. In 2-D computations, one ray parameter (say, γ_2) is fixed. As before, $\mathcal{G}_1^{(q)}$ corresponds to the SV source, $\mathcal{G}_2^{(q)}$ to the SH source, and $\mathcal{G}_3^{(q)}$ to the P source. For point sources situated in a smooth isotropic medium, $\mathcal{G}_j^{(q)}(S;\gamma_1,\gamma_2)$ can be expressed in terms of the directivity patterns of the source at S, $\mathcal{F}_i^{(q)}(S;\gamma_1,\gamma_2)$, as follows:

$$\mathcal{G}_j^{(q)}(S;\gamma_1,\gamma_2) = V(S)\mathcal{F}_j^{(q)}(S;\gamma_1,\gamma_2) \; , \tag{11}$$

see (C-5.2.24). From seismological point of view, $\mathcal{F}_j^{(q)}(S;\gamma_1,\gamma_2)$ represent the ray-centered components of the complex-valued amplitudes generated by a point source along a unit sphere, with its center at S. For an omnidirectional point source, $\mathcal{F}_j^{(q)}(S;\gamma_1,\gamma_2)$ do not depend on the ray parameters γ_1, γ_2. Consequently, $\mathcal{F}_j^{(q)}(S;\gamma_1,\gamma_2) = a_j$, where a_j are constants. For example, the directivity pattern $\mathcal{F}_3^{(q)}(S;\gamma_1,\gamma_2)$ of P waves generated by a center of dilatation is omnidirectional, see (C-5.2.35).

An important role is played by a single force point source. In this case, the radiation function $\mathcal{G}_j^{(q)}(S;\gamma_1,\gamma_2)$ is given by the relation, see (C-5.2.78),

$$\mathcal{G}_j^{(q)}(S;\gamma_1,\gamma_2) = \frac{1}{4\pi\rho(S)V(S)}\mathcal{D}_{lj}(S)f_{0l}^{(x)}(S) \; . \tag{12}$$

($l = 1,2,3$, and Einstein summation convention over l is considered.) Here $f_{0l}^{(x)}(S)$ is the l-th Cartesian component of the single force applied at S. Equation (12) remains valid even for single force situated on a structural interface or on the Earth's surface. $\mathcal{D}_{lj}(S)$ again denotes the conversion coefficient. For a source S situated in a smooth medium, $\mathcal{D}_{lj}(S)$ may be replaced by $e_{jl}(S)$. For moment-tensor point sources, see (C-5.2.34).

5) $\mathcal{L}(R,S)$ is the *relative geometrical spreading.* In 2-D, the relative geometrical spreading may be factorized, see (C-4.13.52),

$$\mathcal{L}(R,S) = \mathcal{L}^{\parallel}(R,S)\mathcal{L}^{\perp}(R,S) \; . \tag{13}$$

Here $\mathcal{L}^{\parallel}(R,S)$ is the in-plane relative geometrical spreading, and $\mathcal{L}^{\perp}(R,S)$ the transverse relative geometrical spreading. Computation of the transverse relative geometrical spreading for a point source at S is easy, see (C-4.13.54),

$$\mathcal{L}^{\perp}(R,S) = \int_S^R V \mathrm{d}s \; . \tag{14}$$

The integration is performed along the ray. The variable s represents the arclength along the ray. The computation of the in-plane relative geometrical spreading $\mathcal{L}^{\parallel}(R,S)$ requires dynamic ray tracing along the ray. By dynamic ray tracing, we understand the solution of the dynamic ray tracing equations. In 2-D computations, the dynamic ray tracing system consists of two linear ordinary differential equations for $Q^{\|} = \partial q_1/\partial\gamma_1$ and $P^{\|} = \partial p_1^{(q)}/\partial\gamma_1$, where $p_1^{(q)}$ is the ray-centered component of the slowness vector along q_1. They read

$$\frac{\mathrm{d}Q^{\|}}{\mathrm{d}s} = VP^{\|} , \qquad \frac{\mathrm{d}P^{\|}}{\mathrm{d}s} = -V^{-2}V_{11}Q^{\|} , \tag{15}$$

see (C-4.13.11) and (C-4.13.13). Here V_{11} is the second derivative of the velocity V with respect to the ray-centered coordinate q_1 (perpendicular to the ray, in the plane $x_2 = 0$). For a point source at S, the initial conditions for (15) are as follows

$$Q^{\|}(S) = 0 , \qquad P^{\|}(S) = 1 . \tag{16}$$

The in-plane relative geometrical spreading $\mathcal{L}^{\|}(R,S)$ is then given by the relation, see (C-4.13.53),

$$\mathcal{L}^{\|}(R,S) = |Q^{\|}(R,S)|^{1/2} . \tag{17}$$

At points Q_i at which the ray strikes a structural interface (or the Earth's surface), transformation equations for $Q^{\|}$ and $P^{\|}$ must be applied. They are described in Section 4.13.2 of Červený (2001).

Note that the relative geometrical spreading quantities are reciprocal,

$$\mathcal{L}(R,S) = \mathcal{L}(S,R) , \quad \mathcal{L}^{\|}(R,S) = \mathcal{L}^{\|}(S,R) , \quad \mathcal{L}^{\perp}(R,S) = \mathcal{L}^{\perp}(S,R) . \tag{18}$$

This is a great advantage of the *relative* geometrical spreading quantities, in comparison with geometrical spreading quantities introduced in different ways.

6) $T^C(R,S)$ is the *phase-shift due to caustics.* It can be expressed in terms of the so-called KMAH index $k(R,S)$ as follows, see (C-4.12.1),

$$T^C(R,S) = -\tfrac{1}{2}\pi k(R,S) . \tag{19}$$

In 2-D computations, the KMAH index $k(R,S)$ represents the number of zeros of $Q^{\|}(s)$ along the ray, between S and R. The zeros of $Q^{\|}(s)$ represent caustic points. KMAH index can be calculated as a byproduct of dynamic ray tracing (15) along the ray, from S to R. Note that the phase shift due to caustics $T^C(R,S)$ is reciprocal,

$$T^C(R,S) = T^C(S,R) . \tag{20}$$

3 Several comments to general equations

The equations given in Section 2 are very general. They can be used to compute elementary synthetic seismograms, Green functions etc. The 2-D isotropic structure may contain curved structural interfaces and curved Earth's surface. Inside individual layers and blocks, the velocities and density may vary laterally. The elementary wave under consideration may be multiply reflected/transmitted and converted or unconverted. Here we give several comments and/or extensions of the above equations.

1) **Elementary synthetic seismograms**. Equation (4) gives an expression for the complex-valued elementary seismogram. We present now equations for real-valued elementary seismograms $\mathrm{Re}(u_i(R,t))$. We use the notation

$$U_k(R) = |U_k(R)| \exp[\mathrm{i}\chi_k(R)] \; . \tag{21}$$

Then (4) yields

$$\begin{aligned} \mathrm{Re}(u_i(R,t)) &= |U_i(R)|\{x(t - T(R,S)) \cos\chi_i(R) \\ &- g(t - T(R,S)) \sin\chi_i(R)\} \; , \end{aligned} \tag{22}$$

(no summation over i), see (C-6.1.10).

Expression (22) for an elementary seismogram can be simplified if analytic expressions for the Hilbert transform $g(\theta)$ are known. For harmonic signals modulated by a broad envelope, the Hilbert transform is approximately known analytically, see (C-A.2.14). As an example, let us consider the popular Gabor signal,

$$x(t) = \exp[-(2\pi f_M t/\gamma)^2] \cos(2\pi f_M t + \nu) \; , \tag{23}$$

where f_M, γ and ν are three parameters of the signal. If γ is sufficiently large (i.e., if the signal is sufficiently broad), the Hilbert transform of $x(t)$ is given by an approximate relation, see (C-A.2.15),

$$g(t) \sim -\exp[-(2\pi f_M t/\gamma)^2] \sin(2\pi f_M t + \nu) \; . \tag{24}$$

See also Fig. 6.1 in Červený (2001). The approximate expression for the elementary seismogram (22) then reads

$$\begin{aligned} \mathrm{Re}(u_i(R,t)) &= |U_i(R)| \exp[-(2\pi f_M (t - T(R,S))/\gamma)^2] \\ &\times \cos[2\pi f_M (t - T(R,S)) + \nu - \chi_i(R)] \; . \end{aligned} \tag{25}$$

The approximate expression (25) gives sufficiently accurate results for large γ, say $\gamma \geq 4$.

The approximate expression (25) for an elementary synthetic seismogram can be also modified for weakly dissipative media, see (C-6.3.6).

2) **Ray synthetic seismograms**. Ray synthetic seismograms $\mathrm{Re}(\bar{u}_n(R,t))$ are obtained by a superposition of elementary ray synthetic seismograms, corresponding to individual elementary seismic body waves arriving at the receiver R within a specified time window:

$$\mathrm{Re}(\bar{u}_n(R,t)) = \mathrm{Re}\; \Sigma_{(\Omega)}\, u_n(R,t) \; . \tag{26}$$

Here the summation runs over all (or some selected) rays Ω from S to R.

Thus, to compute approximately ray synthetic seismograms for the Gabor source-time function (23) in the time domain, we can use (25) for each elementary wave, and perform summation of all (or some selected) elementary synthetic seismograms (26).

Alternatively, we can compute the ray synthetic seismograms in the **frequency domain**. For any elementary wave, we compute the **elementary frequency response**, see (4),

$$u_n(R,t) = U_n(R) \exp[-\mathrm{i}\omega(t - T(R,S))] \; . \tag{27}$$

To compute the elementary frequency response (27) for real-valued $T(R,S)$ in the frequency domain, we can use effectively the fast frequency response algorithm (see Section 6.1.7 of Červený, 2001). After this, we can perform the summation over all (or selected) rays from S to R, similar to (26), multiply the elementary frequency response by the Fourier spectrum of the source-time function, and use the inverse Fourier transform to obtain the ray synthetic seismogram in the time domain.

The computations in the frequency domain are considerably more general than those based on (26). They allow us to consider arbitrary source-time functions, weakly dissipative media, and various frequency-dependent effects (frequency-dependent site effects, R/T coefficients for transition layers, 2-D frequency filters, frequency filters of recording instruments, frequency-dependent radiation functions, etc.).

3) **Line source**. Let us consider a line source parallel to the x_2-axis, intersecting the plane $x_2 = 0$ at the point S. In the frequency domain, the i-th Cartesian component of the displacement vector $u_i(R,\omega)$ is given by the equation, see (C-5.2.121),

$$u_i(R,t) = U_i(R) F^{2D}(\omega) \exp[-\mathrm{i}\omega(t - T(R,S))] \; . \tag{28}$$

Here $F^{2D}(\omega)$ is the 2-D frequency filter, given by the relation

$$F^{2D}(\omega) = (2\pi/|\omega|)^{1/2} \exp[\mathrm{i}\tfrac{\pi}{4}\mathrm{sgn}\,\omega] \; , \tag{29}$$

see (C-2.6.29). The ray-theory amplitude $U_i(R)$ is again given by (6), only $\mathcal{L}(R,S)$ is replaced by $\mathcal{L}^{\|}(R,S)$. This expresses the fact that the transverse relative geometrical spreading $\mathcal{L}^{\perp}(R,S)$ equals unity for a line source. For a single force line source, the radiation function is again given by the relation (12). See (C-5.2.123) with (C-5.2.120).

4) **Ray-theory Green functions**. Ray-theory Green functions are easily obtained from previous general equations, if we insert the radiation function $\mathcal{F}_i^{(q)}(S;\gamma_1,\gamma_2)$ corresponding to a unit single force oriented along some Cartesian axis. For a unit single force oriented along the n-th Cartesian axis, we use (12) with $f_{0l}^{(x)}(S) = \delta_{ln}$. Further, we have to consider the analytical signal $F(\theta)$ corresponding to the analytical delta function, $F(\theta) = \delta^{(A)}(\theta) = \delta(\theta) - \mathrm{i}/\pi\theta$, see (C-A.3.6). Consequently, we use (4), (6), and (12) to obtain

$$\begin{aligned} G_{in}(R,t;S,t_0) \;\; &= \;\; \frac{\exp[\mathrm{i}T^C(R,S)]}{4\pi[\rho(S)\rho(R)V(S)V(R)]^{1/2}} \, \frac{1}{\mathcal{L}(R,S)} \mathcal{R}^C(R,S) \\ &\times \;\; \mathcal{D}_{ik}(R)\mathcal{D}_{nj}(S)\delta^{(A)}(t - t_0 - T(R,S)) \; , \end{aligned} \tag{30}$$

see (C-5.2.111) and (C-5.2.115). The ray-theory Green function (30) corresponds to any selected elementary wave; the complete ray-theory Green function would be obtained by summation of elementary Green functions over all rays arriving at R. Equation (30) is very general. It can be used even for the source S and the receiver R situated at structural interfaces or at the Earth's surface. Eq. (30) is, of course, complex-valued.

As we can see from (30), the ray-theory Green function $G_{in}(R,t;S,t_0)$ is reciprocal in the following sense:

$$G_{in}(R,t;S,t_0) = G_{ni}(S,t;R,t_0) \; . \tag{31}$$

This is very important property of ray-theory Green functions. This property is valid even for 3-D isotropic or anisotropic structures. We emphasize that the property is satisfied even for sources and receivers situated on structural interfaces and/or on the Earth's surface.

For a line source, we can also introduce the ray-theory line-source Green function $G^L_{in}(R,t;S,t_0)$. It is given by the relation

$$\begin{aligned} G^L_{in}(R,t;S,t_0) &= \frac{\exp[\mathrm{i}T^C(R,S)]}{4\pi[\rho(S)\rho(R)V(S)V(R)]^{1/2}}\,\frac{1}{\mathcal{L}^{\Vert}(R,S)}\,\mathcal{R}^C(R,S) \\ &\times\ \mathcal{D}_{ik}(R)\mathcal{D}_{nj}(S)U(t-t_0-T(R,S))\ , \end{aligned} \tag{32}$$

where

$$\begin{aligned} U(\theta) &= [\sqrt{2}H(\theta)\theta^{-1/2}]^{(A)} \\ &= \sqrt{2}H(\theta)\theta^{-1/2}+\mathrm{i}\sqrt{2}H(-\theta)(-\theta)^{-1/2}\ . \end{aligned} \tag{33}$$

Here $H(\theta)$ denotes Heaviside function, see (C-A.3.9). In the frequency domain, we put $t_0=0$ and replace $U(\theta)$ by $F^{2D}(\omega)\exp[\mathrm{i}\omega T(R,S)]$, see (C-5.2.128). Here $F^{2D}(\omega)$ is the 2-D frequency filter, given by (29).

As we can see from (32), the ray-theory line-source Green function $G^L_{in}(R,t;S,t_0)$ is again reciprocal, in the sense of (31).

4 Alternative expressions for 2-D ray-theory amplitudes for a point source

Many alternative expressions for 2-D ray-theory amplitudes for a point and line sources can be found in seismological literature. We believe, however, that the expressions presented in Sections 2 and 3 are most general and transparent, and that they exhibit many useful properties (for example, the reciprocity of individual factors, reciprocity of elementary Green functions, etc.).

Often, standard geometrical spreading $\mathcal{L}(R)$ has been used instead of the relative geometrical spreading $\mathcal{L}(R,S)$. For a point source and for ray parameters γ_1,γ_2 corresponding to the radiation take-off angles, the relation between $\mathcal{L}(R,S)$ and $\mathcal{L}(R)$ is as follows:

$$\mathcal{L}(R,S)=V(S)\mathcal{L}(R)/|\sin i(S)|^{1/2}\ , \tag{34}$$

where $i(S)$ is the take-off angle of the ray at S, see (C-4.10.11) and (C-4.10.15). The geometrical spreading $\mathcal{L}(R)$ has been often expressed in terms of the ray Jacobian $J(R)=D(x_1,x_2,x_3)/D(\gamma_1,\gamma_2,s)$. For a point source, the relation reads

$$\mathcal{L}(R)=\sqrt{|J(R)|}\ , \tag{35}$$

see (C-4.10.1). Consequently, the relative geometrical spreading $\mathcal{L}(R,S)$ is expressed for a point source at S in terms of $J(R)$ as follows:

$$\mathcal{L}(R,S)=V(S)|J(R)/\sin i(S)|^{1/2}\ . \tag{36}$$

Note that $\mathcal{L}(R)$ and $J(R)$ are not reciprocal. Using (6), (8) and (36), we can also express $U_i(R)$ as follows

$$U_i(R) = \frac{VR^C(R,S)\mathcal{D}_{ik}(R)\mathcal{F}_j^{(q)}(S)}{|L|}\exp[\mathrm{i}T^C(R,S)] \ , \tag{37}$$

where

$$\begin{aligned} V &= \left[\frac{V(S)\rho(S)}{V(R)\rho(R)}\right]^{1/2}\prod_{j=1}^{N}\left[\frac{V(\tilde{Q}_j)\rho(\tilde{Q}_j)}{V(Q_j)\rho(Q_j)}\right]^{1/2} \ , \\ L &= \left[\frac{J(R)}{\sin i(S)}\right]^{1/2}\prod_{j=1}^{N}\left(\frac{\cos i(Q_j)}{\cos i(\tilde{Q}_j)}\right)^{1/2} \ . \end{aligned} \tag{38}$$

The phase shift due to caustics $T^C(R,S)$ is directly related to the number of zeros of L along the ray. If a specific rule how to determine the relevant sign of $L(R)$ at R is given, (37) can be also expressed in the following form

$$U_i(R) = \frac{VR^C(R,S)\mathcal{D}_{ik}(R)\mathcal{F}_j^{(q)}(S)}{L} \ . \tag{39}$$

This corresponds to Equations (2.56), (2.58) and (2.64), used in Červený, Molotkov and Pšenčík (1977). These equations were used in the program package SEIS81. The notation used in the above reference is slightly different than here, but it is simply understandable. The symbols q_x and q_z denote the conversion coefficients $\mathcal{D}_{ik}$, g denotes the directivity patterns of the source $\mathcal{F}_j^{(q)}(S)$, etc.

Expression (39) has certain disadvantages:
a. Individual factors in (39), particularly $R^C(R,S)$ and L, are not reciprocal.
b. The phase shift due to caustics $T^C(R,S)$ is not strictly specified.
c. The procedure how to compute $J(R)$ is not specified.

5 Program packages SEIS

In this section, we give brief explanations and comments to the computations of 2-D ray-theory amplitudes and synthetic seismograms in program packages SEIS81, SEIS83 and SEIS88. Certain limitations of these program packages are also discussed.

5.1 Program package SEIS81

Program package SEIS81 was designed before the year 1981. In that time, the equations for the ray-theory amplitudes were mostly taken from Červený, Molotkov and Pšenčík (1977), see (39). Program package SEIS81 was distributed mainly by both authors, and by the Stanford Exploration Project, Stanford University, U.S.A.

The most important explanations and comments to SEIS81 are as follows:

1) Standard P-SV displacement R/T coefficients $R(Q_i)$ are computed in the routine COEF8. The same routine also computes the conversion coefficients at the Earth's surface. The conversion coefficients at structural interfaces are not included. The complete displacement R/T coefficient $R^C(R, S)$ is computed in the routine AMPL, which calls COEF8. SH R/T coefficients are not computed in COEF8. The SH waves are not considered in SEIS81 at all.

2) The receivers R may be situated only on the Earth's surface. The conversion coefficients are applied if MREG=0. If MREG=1, the conversion coefficients at R are not applied, and computations are performed as if the receiver R were situated in a smooth medium.

3) Point source S may be situated at any point of the model (with the exception of coinciding interfaces). In SEIS81, however, only a small attention was devoted to the directivity pattern of the source. The routine SOURCE, formally included in SEIS81 for this purpose, is practically empty. It only includes an omnidirectional directivity pattern, and a possibility to introduce an arbitrary directivity pattern in a numerical form, by a table. However, the tabular specification of the directivity pattern has not been used in our computations at all, and has not been sufficiently tested. Thus, practically only the omnidirectional directivity pattern is available in SEIS81. The omnidirectional directivity pattern is used not only for P waves (explosive source), but also for SV waves (where it has more or less a formal meaning). Directivity patterns of various important point sources (like single-force point source, moment-tensor point source) are not included.

Note that the ray-theory amplitudes of seismic body waves, computed for an omnidirectional point source, are not reciprocal (even if P wave source is used). Consequently, SEIS81 does not yield reciprocal results. This is not a problem of the program package, but property of the omnidirectional point source. To obtain reciprocal results, it would be necessary to consider a single-force point source (ray-theory Green function). Alternatively, for P waves generated by the omnidirectional point source it would be necessary to record pressure or dilatation at R, not the displacement.

4) The ray Jacobian $J(R)$ is computed using dynamic ray tracing, by the routine JPAR(Q0, P0, Q, P, APRINT) with $Q = Q^{\parallel}$ and $P = P^{\parallel}$, see (15). The quantities Q0 and P0 specify the initial conditions for the dynamic ray tracing system (15) at the point source S. Instead of standard initial conditions (16), however, different initial conditions are used: $Q0 = 0$, $P0 = 1/V(S)$. This difference is adjusted in the routine AMPL.

Note that the dynamic ray tracing system along smooth parts of the ray is solved by the modified Euler method in the routine RK. Routine RK is called from JPAR. Routine JPAR contains also transformation formulae at structural interfaces.

5) Phase shift due to caustics $T^C(R, S)$. In SEIS81 the function L of (38) is expressed as follows:

$$L = |L| \cdot \exp[\tfrac{i}{2}\pi k] \ ,$$

where $k = 0$ for L positive, and $k = 1$ for L negative. Thus, the phase shift due to caustics is given correctly in SEIS81 if no or one caustic point exists along the ray, between S and

R. For more caustic points along the ray between S and R, the final complex-valued amplitudes computed by SEIS81 may be incorrectly phase shifted, by a factor $\exp[\pm i\frac{\pi}{2}n]$.

6) Ray amplitudes are computed in the routine AMPL(FJ, AX, AZ, PHX, PHZ). Here FJ represents the function L from (38), and it is the input quantity. AX and PHX are the amplitude and phase of the radial component, AZ and PHZ the amplitude and phase of the vertical component of the displacement vector, both taken at point R. Equations (37)–(39) are used in the computation.

7) Elementary synthetic seismograms are computed by the program SYNTPL. The source-time function corresponds to the Gabor signal (23). The approximate relation for the Hilbert transform is used. This approximate relation is sufficiently accurate only for higher γ, say $\gamma \geq 4$. No other source-time functions are included in SEIS81. The particle ground motions are not computed.

8) Line source computations: The 2-D frequency filter $F^{2D}(\omega)$, given by (29), is not considered in SEIS81. Actually, this simplification does not matter very much in most of practical applications; it only means that a different source-time function is considered. The simplification, however, may play an important role if we wish to compare the line-source solutions computed by SEIS81 with the line-source solutions computed by a different method (for example, by the method of finite differences). Thus, program package SEIS81 cannot be used for such comparisons.

9) Ray-theory Green functions are not computed in SEIS81.

5.2 Program package SEIS83

Program package SEIS83 represents a revised version of the package SEIS81. Most of the comments to SEIS81, presented in Section 5.1, remain valid even for SEIS83. Program package SEIS83 was distributed by both authors and by the World Data Center A for Solid Earth Geophysics, Boulder, U.S.A. See Červený and Pšenčík (1984).

5.3 Program package SEIS88

Program package SEIS88 is an updated version of the package SEIS83, in which several limitations listed in Sec. 5.1 are removed. The modifications are listed below in the order of comments in Sec. 5.1. Program package was distributed by authors.

1) SH waves are considered in SEIS88.

2) The receivers may be situated at the Earth's surface, at actual or artificial internal interfaces, and along vertical profiles (for VSP modeling). The conversion coefficients can be applied only at the Earth's surface. The receivers can record only displacement vectors.

3) Several types of point sources, for example explosive, single-force, moment-tensor sources, can be considered in SEIS88. For the single-force point source reciprocal results

are obtained. Directivity patterns corresponding to point sources situated in a smooth medium are applied even to point sources situated at structural interfaces or at the Earth's surface.

4) The initial conditions for the dynamic ray tracing system (15) in the routine JPAR have the form of Eq. (16). Fundamental solution of the dynamic ray tracing system is calculated.

5) The phase shift due to caustics is calculated correctly even if more caustic points exist along the considered ray.

6) In addition to radial and vertical, also transverse component of the displacement vector can be calculated.

7) Ray synthetic seismograms can be computed in SEIS88 in two modes. They can be computed in the time-domain mode used previously in SEIS81 and SEIS83, see Sec. 5.1. They can also be computed in the frequency-domain mode. In the latter mode several different source-time functions can be considered. It is also possible to consider a weak attenuation. The results can be displayed in the form of seismograms as well as of particle ground motions.

8) This point is unchanged. The 2-D frequency filter is not calculated. Thus the program package SEIS88 cannot be used for direct comparison with FD computations.

9) Ray theory Green functions can be computed in SEIS88.

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