Properties of the zero-, first- and higher-order approximations of attributes of elastic waves in weakly anisotropic media

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Summary

Use of the perturbation theory in the study of attributes of elastic waves propagating in weakly anisotropic media leads to approximate but transparent and simple formulae, which have many applications in forward and inverse wave modeling. We present and study such formulae. We show that all studied attributes depend on elements of a matrix linearly dependent on parameters of a medium. We study this dependence with the goal to understand which parameters of the medium, and in which combinations, affect individual wave attributes. Alternative auxiliar vector bases, in which the matrix can be specified, are proposed and studied. The vector bases offer alternative specifications of polarization vectors of qS waves. One of the important observations is that the higher-order ($n \geq 2$) perturbation formulae for qS waves are obtained separately for $qS1$ and $qS2$ waves. We also study effects of the use of the perturbation theory on the accuracy of the determination of the acoustical axes in weakly anisotropic media. We show that longitudinal directions in the first-order approximation are identical with actual ones. In singular directions, however, the first-order formulae provide directions, which may deviate from the exact ones, or they may even indicate false singular directions. Again, the above-mentioned matrix depending linearly on the parameters of the medium plays a central role in this study.

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I. INTRODUCTION

Perturbation theory is a useful tool for approximate study of wave properties in weakly anisotropic media. Especially its first-order approximation yields simple and transparent formulae only slightly more complicated than those for isotropic media. Thus it allows approximate but very effective solution of forward and inverse problems like ray tracing, travel-time computations, travel-time tomography, local determination of parameters of media, etc. Perturbation theory also plays an important role in the coupling ray theory and quasi-isotropic approximation in particular, see Červený (2001), for example.

Farra (2001) extended standard techniques for the calculation of the first-order approximations of the phase velocity and polarization vectors (see, e.g., Jech and Pšenčík, 1989), and proposed a procedure how to calculate even higher-order approximations. In this contribution, we extend and apply results of Farra (2001). We extend it by a detailed description of the construction of perturbation series for qS waves, by introduction of new, useful, auxiliary vectorial bases, in which the perturbation formulae can be specified, and by the sensitivity study of the wave attributes. We then apply the formulae to study, both analytically and numerically, behaviour of first-order perturbation formulae in the directions of acoustical axes.

In perturbation formulae of any order, an important role is played by a matrix, B_{mn}, whose elements control various attributes of elastic waves. The matrix has several interesting and important properties. It is independent of the choice of a reference medium. It depends linearly on weak anisotropy (WA) parameters, which are, in turn, linearly related to elastic parameters. Explicit dependence of the matrix B_{mn} on various parameters of the medium or their combinations makes possible to study sensitivity of various attributes of elastic waves propagating in an anisotropic media to its parameters. This is important information for inversion of observed data into elastic moduli (Chu et al. 1994; Song et al. 2001; Zheng and Pšenčík, 2002). Attention is also paid to alternative ways of constructing approximate formulae for the polarization vectors of qS waves.

A special attention is devoted to the study of effects of the first-order perturbation formulae on the position of acoustical axes in weakly anisotropic media. Acoustical axes are a characteristic feature of anisotropic media. We can distinguish two kinds of acoustical axes (Schoenberg and Helbig, 1997). The first are the well-known *singular directions* in which the two qS waves propagating in anisotropic media propagate with the same phase velocity. These directions correspond to the places on the slowness surfaces at which the two qS-wave sheets are in a contact. Characteristic feature of singular directions and their vicinity are distortions of slowness surfaces and strong variations of qS-wave polarization vectors (Schoenberg and Helbig, 1997; Shuvalov and Every, 1997; Vavryčuk, 2003). The second kind of acoustical axes are *longitudinal directions*. Characteristic feature of longitudinal directions is propagation of purely longitudinal and purely transverse waves along them, i.e. propagation of qP waves with polarization parallel to the slowness vector (specifying the longitudinal direction) and propagation of qS waves with polarization perpendicular to the slowness vector. Helbig (1993) and Schoenberg and Helbig (1997) propose to use the acoustical axes for inversion of observed data into elastic parameters specifying anisotropy of the medium. It thus seems that importance of the acoustical axes might increase in near future. In this contribution, we study effects of frequently used approximations on the accuracy with which the acoustical axes can be determined.

In Sec.II, basic equations are given. Properties of the matrix B_{mn} and its sensitivity to parameters of a medium are studied. Alternative auxiliar vector bases, in which the matrix B_{mn} can be specified, are proposed and discussed. Then basic results of Farra (2001) are briefly reviewed and extended. It is shown that the higher-order ($n \geq 2$) perturbation formulae for qS waves are obtained separately for $qS1$ and $qS2$ waves. An alternative determination of two unit vectors specifying the qS-wave polarization, which leads to simplified polarization formulae, is proposed. In Sec.III, the conditions for the determination of acoustical axes in the first-order approximation of the perturbation theory are specified and discussed. The accuracy of the determination of acoustical axes determined by the perturbation formulae is studied. In Sec.IV, accuracy of perturbation formulae is illustrated on several numerical examples of media used in literature. The Appendix A contains explicit expressions for the elements for one of many possible specifications of the matrix B_{mn}.

In the following, component notation is used. All the Roman lowercase indices range over the values 1, 2 and 3 and the uppercase indices over the values 1 and 2. Einstein summation convention is used for the repeated subscripts. Where necessary, supperscripts are used. The supperscripts in parentheses indicate an order of the quantity. In the matrices $\hat{B}^{(j)}_{mn}$ and in the vectors $\hat{e}^{(j)m}_i$, the upper indices j in parentheses indicate that $\hat{B}^{(j)}_{mn}$ and $\hat{e}^{(j)m}_i$ are used in expressions for the quantities of the j-th or $(j+1)$-st order, see the text. The supperscripts in the curly brackets (for example, $g^{\{m\}}_i$) indicate the type of the wave, $m = 1, 2$ for the qS waves and $m = 3$ for the qP wave. Voigt notation $A_{\alpha\beta}$ for density-normalized elastic parameters, with α, β running from 1 to 6, is used in parallel with the tensor notation a_{ijkl}.

II. HIGHER-ORDER PERTURBATION FORMULAE

We first study properties of the matrix B_{mn} playing basic role in perturbation formulae for many wave attributes. Then we briefly review the formulae for the calculation of higher-order terms for the phase velocity and polarization of qP and qS waves derived by Farra (2001), and present their alternative form for qS waves.

A. Basic Equations

Let us start from the Christoffel matrix

$$\Gamma_{jk} = a_{ijkl} n_i n_l. \tag{1}$$

The symbol a_{ijkl} denotes a tensor of density-normalized elastic parameters and n_i is a unit wave vector. The Christoffel matrix Γ_{jk} is symmetric and positive definite, see, e.g., Červený (2001). Thus for any direction of the wave vector n_i, Γ_{jk} has three positive and real eigenvalues $G_m = G_m(n_j)$ and three corresponding eigenvectors $g^{\{m\}}_i(n_j)$. They satisfy the following system of equations

$$(\Gamma_{jk} - G_m \delta_{jk}) g^{\{m\}}_k = 0. \tag{2}$$

Two eigenvalues G_1 and G_2 with corresponding eigenvectors $g_i^{\{1\}}$ and $g_i^{\{2\}}$ correspond to the two qS waves, remaining eigenvalue G_3 with the eigenvector $g_i^{\{3\}}$ belongs to the qP wave. The eigenvalues are related to the squares of corresponding phase velocities $c_m(n_j)$,

$$G_m(n_j) = c_m^2(n_j). \tag{3}$$

The eigenvectors $g_i^{\{m\}}$ specify polarization vectors of the corresponding waves. The eigenvalues G_1 and G_2 are for most geological materials smaller than G_3 (Schoenberg and Helbig, 1997). Since the relation between G_1 and G_2 can be chosen arbitrarily, we choose it as follows: $G_2 \leq G_1 < G_3$.

The tensor a_{ijkl} can be expressed as follows:

$$a_{ijkl} = a_{ijkl}^{(0)} + \Delta a_{ijkl}. \tag{4}$$

Here $a_{ijkl}^{(0)}$ is a tensor of density-normalized elastic parameters in a reference isotropic medium:

$$a_{ijkl}^{(0)} = (\alpha^2 - 2\beta^2)\delta_{ij}\delta_{kl} + \beta^2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{5}$$

and Δa_{ijkl} is its perturbation. In (5), α and β denote P- and S-wave velocities of the reference isotropic medium.

B. Matrix B_{mn}

We introduce three mutually perpendicular unit vectors e_i^1, e_i^2 and e_i^3 so that $e_i^3 = n_i$. The vectors e_i^1 and e_i^2, situated in the plane perpendicular to e_i^3, can be chosen arbitrarily.

We use the vectors e_i^k to define a matrix B_{mn}

$$B_{mn} = \Gamma_{jk} e_j^m e_k^n. \tag{6}$$

The matrix B_{mn} is independent of the choice of the reference isotropic medium. It is related to the *weak anisotropy* (WA) matrix $\mathcal{B}_{mn} = \Delta a_{ijkl} n_i n_l e_j^m e_k^n$, which, in contrast to B_{mn}, depends on the choice of the reference medium. Eq.(13) of Pšenčík and Gajewski (1998) yields the relation between the two matrices:

$$B_{mn} = \mathcal{B}_{mn} + c_0^2 \delta_{mn}. \tag{7}$$

The symbol c_0 in Eq.(7) stands for the phase velocity of the reference isotropic medium. For $m = n = 1$ or 2, $c_0 = \beta$; for $m = n = 3$, $c_0 = \alpha$. From (7), we can see that the off-diagonal terms of the matrix B_{mn} are of the first order.

In the following we frequently use two specifications of the vectors e_i^K and, thus, of the matrix B_{mn}. In one we use a matrix $\hat{B}_{mn}^{(0)}$ defined by Eq.(6) with vectors e_i^1 and e_i^2 substituted by vectors $\hat{e}_i^{(0)1}$ and $\hat{e}_i^{(0)2}$ chosen so that

$$\hat{B}_{12}^{(0)} = 0, \qquad \hat{B}_{11}^{(0)} > \hat{B}_{22}^{(0)}. \tag{8}$$

The second condition in Eq.(8) excludes singular directions from our considerations. The singular directions are studied in Sec.III. The diagonal elements of the matrix $\hat{B}_{mn}^{(0)}$ specify the first-order approximations $G_m^{(1)}$ of the eigenvalues of the Christoffel matrix, see Farra (2001). Specifically,

$$G_1^{(1)} = \hat{B}_{11}^{(0)}, \qquad G_2^{(1)} = \hat{B}_{22}^{(0)}, \qquad G_3^{(1)} = \hat{B}_{33}^{(0)} = B_{33}. \tag{9}$$

Since the elements of the matrix B_{mn} are independent of the velocities α and β of the reference medium, $G_m^{(1)}$ in (9) are independent of them too.

In the other specification, we substitute the vectors e_i^K by vectors $\bar{e}_i^K$ given by Eqs.(A1), (A2) of the Appendix A. We denote the corresponding matrix B_{mn} by a bar: $\bar{B}_{mn}$. Explicit expressions for the elements of the matrix $\bar{B}_{mn}$ can be found in Eqs.(A4) in the Appendix A. Elements of a matrix B_{mn} specified for arbitrarily chosen vectors e_i^J are related to $\bar{B}_{mn}$ by simple linear relations. If we denote by ϕ the acute angle between the vectors e_i^1 and $\bar{e}_i^1$, the elements of the matrix $\bar{B}_{mn}$ transform into B_{mn} in the following simple way, see, e.g., Pšenčík and Vavryčuk (2002):

$$\begin{aligned}
B_{11} &= \bar{B}_{11}\cos^2\phi + 2\bar{B}_{12}\cos\phi\sin\phi + \bar{B}_{22}\sin^2\phi, \\
B_{22} &= \bar{B}_{11}\sin^2\phi - 2\bar{B}_{12}\cos\phi\sin\phi + \bar{B}_{22}\cos^2\phi, \\
B_{12} &= (\bar{B}_{22} - \bar{B}_{11})\cos\phi\sin\phi + \bar{B}_{12}(\cos^2\phi - \sin^2\phi), \\
B_{13} &= \bar{B}_{13}\cos\phi + \bar{B}_{23}\sin\phi, \\
B_{23} &= -\bar{B}_{13}\sin\phi + \bar{B}_{23}\cos\phi, \qquad B_{33} = \bar{B}_{33}.
\end{aligned} \tag{10}$$

We can see that the elements $\bar{B}_{11}$, $\bar{B}_{12}$ and $\bar{B}_{22}$ transform into the elements of the matrix B_{mn} with the same subscripts. The same holds for the elements $\bar{B}_{13}$ and $\bar{B}_{23}$. The element $\bar{B}_{33}$ is unaffected by the rotation.

In what follows it is shown that in the first-order approximation the elements of the above matrices control phase velocities, polarizations, and orientations of acoustical axes of elastic waves propagating in weakly anisotropic media. The elements B_{11}, B_{12}, and B_{22} are related to the qS-wave phase velocities and control orientation of singular directions. The element B_{33} is related to the qP-wave phase velocity. The elements B_{13} and B_{23} are related to the polarization vectors of qP and qS waves and control orientation of the longitudinal directions. Since, for given e_i^K, the elements of the matrix B_{mn} are linear functions of the WA or elastic parameters, it is desirable to deduce which parameters can affect individual elements of the matrix B_{mn}, and which parameters and in which form (individually or in a combination with others) can be retrieved from individual elements of the matrix B_{mn}.

By a simple inspection of Eqs.(A4) in the Appendix A, we can find that the elements $\bar{B}_{11}$, $\bar{B}_{12}$ and $\bar{B}_{22}$ are controlled by 15 mutually independent coefficients, which depend linearly on all 21 WA parameters. The coefficients are: $\epsilon_{15} - \epsilon_{35}$, $\epsilon_{24} - \epsilon_{34}$, $\chi_x - \epsilon_{34}$, $\chi_y - \epsilon_{35}$, $\epsilon_x - \delta_x + \epsilon_z$, $\epsilon_y - \delta_y + \epsilon_z$, $\delta_z - \epsilon_x - \epsilon_y$, $\chi_z - \epsilon_{16}$, $\chi_z - \epsilon_{26}$, γ_x, γ_y, γ_z, ϵ_{45}, ϵ_{46} and ϵ_{56}. We can see that from the above elements of the matrix $\bar{B}_{mn}$ we can retrieve individually only 6 WA parameters. All remaining 15 WA parameters can be found only in combinations with others.

By a similar inspection of the element $\bar{B}_{33}$ we can find that the element is controlled by 15 different mutually independent coefficients, which depend on 15 qP-wave WA parameters. The coefficients are: ϵ_x, ϵ_y, ϵ_z, $\epsilon_x - \delta_x + \epsilon_z$, $\epsilon_y - \delta_y + \epsilon_z$, $\delta_z - \epsilon_x - \epsilon_y$, χ_x, χ_y, χ_z, ϵ_{15},

ϵ_{16}, ϵ_{24}, ϵ_{26}, ϵ_{34} and ϵ_{35}. We can see that in this case all 15 involved WA parameters can be determined from $\bar{B}_{33}$ individually.

The elements $\bar{B}_{13}$ and $\bar{B}_{23}$ are controlled by 14 mutually independent coefficients, which depend on 15 qP-wave WA parameters. The coefficients have the same form as for $\bar{B}_{33}$ except ϵ_x, ϵ_y, ϵ_z. Instead of these three coefficients we have their two combinations: $\epsilon_x - \epsilon_z$ and $\epsilon_y - \epsilon_z$. This means that from the elements $\bar{B}_{13}$ and $\bar{B}_{23}$ we can retrieve only 9 individual WA parameters. The remaining 6 can be determined only in combinations. This has been observed by Zheng and Pšenčík (2002) during their inversion of synthetic polarization data in a VTI model.

A similar analysis in terms of the elastic parameters $A_{\alpha\beta}$ yields the following conclusions, which can be simply deduced from Eqs.(A3). The elements $\bar{B}_{11}$, $\bar{B}_{12}$, and $\bar{B}_{22}$ depend on all 21 elastic parameters. Only 6 of them, A_{44}, A_{55}, A_{66}, A_{46}, A_{56} and A_{45} can be found individually. The remaining 15 elastic parameters can be found only in combinations with other parameters.

The element $\bar{B}_{33}$ depends also on all 21 elastic parameters. But only 9 of them, A_{11}, A_{22}, A_{33}, A_{15}, A_{16}, A_{24}, A_{26}, A_{34} and A_{35} can be found individually. The remaining 12 elastic parameters can be found only in combinations with others.

The elements $\bar{B}_{13}$ and $\bar{B}_{23}$ depend again on all 21 elastic parameters. But only 6 of them, A_{15}, A_{16}, A_{24}, A_{26}, A_{34} and A_{35} can be determined individually. The remaining 15 elastic parameters can be obtained only in combinations.

C. qP Wave

We follow Farra (2001) and express the polarization vector $g_i^{\{3\}}$, related to the qP wave, in the vector basis $(\hat{e}_i^{(0)1}, \hat{e}_i^{(0)2}, e_i^3)$. Inserting $g_i^{\{3\}}$ specified in this way into (2) and multiplying the resulting equation successively by $\hat{e}_i^{(0)1}$, $\hat{e}_i^{(0)2}$ and e_i^3, we get a set of three equations from which we can determine a vector parallel to the exact polarization vector $g_i^{\{3\}}$:

$$\frac{\hat{B}_{13}^{(0)}}{G_3 - G_1^{(1)}}\hat{e}_i^{(0)1} + \frac{\hat{B}_{23}^{(0)}}{G_3 - G_2^{(1)}}\hat{e}_i^{(0)2} + e_i^3. \tag{11}$$

Note that the vector (11) is generally not unit. Eq.(11) is derived under assumption that the expressions in the denominators are non-zero. As mentioned above, in most geological materials this is guaranteed.

The vector (11) specifies the exact direction of polarization vector of the qP wave. Since the determination of the vectors $\hat{e}_j^{(0)K}$ and e_j^3, of the elements of the matrix $\hat{B}_{mn}^{(0)}$ and of the first-order approximations of the eigenvalues is straightforward, only knowledge of the exact eigenvalue G_3 is necessary for the determination of the direction of the polarization vector $g_i^{\{3\}}$ from Eq.(11). Eq.(11) can be also used for the determination of the directions of the polarization vectors of different order of approximation, see Farra (2001). In the zero-order approximation, the polarization vector is e_i^3. In the first-order approximation we get

$$g_i^{(1)\{3\}} = \frac{\hat{B}_{13}^{(0)}}{G_3^{(0)} - G_1^{(0)}}\hat{e}_i^{(0)1} + \frac{\hat{B}_{23}^{(0)}}{G_3^{(0)} - G_2^{(0)}}\hat{e}_i^{(0)2} + e_i^3. \tag{12}$$

It is easy to show that in contrast to the general Eq.(11), Eq.(12) holds for arbitrarily chosen vectors e_i^K, not only for $\hat{e}_i^{(0)K}$. It can thus be rewritten in the form:

$$g_i^{(1)\{3\}} = \frac{B_{13}}{\alpha^2 - \beta^2}e_i^1 + \frac{B_{23}}{\alpha^2 - \beta^2}e_i^2 + e_i^3. \tag{13}$$

We took into account that $G_3^{(0)} = \alpha^2$ and $G_1^{(0)} = G_2^{(0)} = \beta^2$. Eq.(13) corresponds to the expression for the first-order perturbation of the qP-wave polarization vector of Jech and Pšenčík (1989). Since B_{13} and B_{23} are the first-order quantities, the vector $g_i^{(1)\{3\}}$ is a unit vector in the first-order approximation.

If we substitute $g_k^{\{m\}}$ in the Christoffel equation (2) by the vector (11), and multiply the resulting equation by e_j^3, we get

$$G_3 = G_3^{(1)} + \frac{(\hat{B}_{13}^{(0)})^2}{G_3 - G_1^{(1)}} + \frac{(\hat{B}_{23}^{(0)})^2}{G_3 - G_2^{(1)}}. \tag{14}$$

This is a basic equation for the iterative determination of the higher-order approximations of the eigenvalue G_3, see Eq.(30) of Farra (2001). We can proceed in the same way as Farra (2001) to obtain approximations of G_3 of different orders. Specifically, in the first-order approximation we obtain:

$$G_3^{(1)} = \hat{B}_{33}^{(0)} = B_{33} = \Gamma_{jk}e_j^3 e_k^3, \tag{15}$$

see Eq.(9). It is interesting to note that an odd-order approximation of the eigenvalue G_3 can be determined from a lower odd-order approximation. The same holds for even-order approximations, see Farra (2001). The odd-order approximations of the eigenvalue G_3 are independent of the choice of the reference medium while the even-order approximations depend on its choice.

Since the elements B_{13}, B_{23} and B_{33} appearing in Eqs.(13) and (15) are related only to the elements $\bar{B}_{13}$, $\bar{B}_{23}$ and $\bar{B}_{33}$, see Eq.(10), from the discussion in Sec.II.B follows that the first-order approximations of the polarization vector (13) and of the eigenvalue (15) (representing square of the phase velocity of the qP wave in the first-order approximation) depend on 15 WA parameters. From the phase velocity all 15 WA parameters can be determined. Nine individual WA parameters and remaining six in 5 combinations can be determined from the polarization vector. The polarization vector depends, in addition, on the difference of squared phase velocities in the reference isotropic medium: $\alpha^2 - \beta^2$.

D. qS Wave

We consider the two qS waves propagating in anisotropic media. We denote the faster wave $qS1$ and associate it with the index $K = 1$. The slower $qS2$ wave is associated with the index $K = 2$. For the qS waves we must use a procedure designed for the degenerate cases. In this section, we assume that the considered direction is not a singular direction.

Singular directions are considered in Sec.III. We again follow Farra (2001) and express the polarization vector of the K-th qS wave, $g_i^{\{K\}}$, in the basis (e_i^1, e_i^2, e_i^3). Note that arbitrarily chosen vectors e_i^J are considered. Inserting $g_i^{\{K\}}$ specified in this way into (2) and multiplying the resulting equation successively by e_i^1, e_i^2 and e_i^3, we get, after some algebra, the following set of three equations for the projections of the vector $g_i^{\{K\}}$ into e_i^1, e_i^2 and e_i^3

$$(M_{11}^{\{K\}} - G_K)g_m^{\{K\}}e_m^1 + M_{12}^{\{K\}}g_m^{\{K\}}e_m^2 = 0, \tag{16a}$$

$$M_{12}^{\{K\}}g_m^{\{K\}}e_m^1 + (M_{22}^{\{K\}} - G_K)g_m^{\{K\}}e_m^2 = 0, \tag{16b}$$

$$B_{13}g_m^{\{K\}}e_m^1 + B_{23}g_m^{\{K\}}e_m^2 + (G_3^{(1)} - G_K)g_m^{\{K\}}e_m^3 = 0. \tag{16c}$$

Here we use the notation of Farra (2001):

$$M_{ij}^{\{K\}} = B_{ij} + \frac{B_{i3}B_{j3}}{G_K - G_3^{(1)}}. \tag{17}$$

Note that $M_{ij}^{\{K\}}$ in Eq.(17) depends, through the index K, on the type of the considered qS wave. From the condition of solvability of Eqs.(16a,b), we get expressions for the eigenvalues G_K of the two qS waves:

$$G_1 = \frac{1}{2}[M_{11}^{\{1\}} + M_{22}^{\{1\}} + \sqrt{D_1}] \tag{18a}$$

and

$$G_2 = \frac{1}{2}[M_{11}^{\{2\}} + M_{22}^{\{2\}} - \sqrt{D_2}], \tag{18b}$$

where

$$D_K = (M_{11}^{\{K\}} - M_{22}^{\{K\}})^2 + 4(M_{12}^{\{K\}})^2. \tag{19}$$

From Eqs.(16) and (17), we can determine vectors parallel to the exact polarization vectors $g_i^{\{1\}}$ and $g_i^{\{2\}}$. They can be written as follows

$$P_K e_i^1 + Q_K e_i^2 + R_K e_i^3. \tag{20}$$

The symbols P_K and Q_K in (20) have the following meaning:

$$P_1 = \sqrt{\frac{1}{2}(1 + \frac{M_{11}^{\{1\}} - M_{22}^{\{1\}}}{\sqrt{D_1}})},$$

$$Q_1 = sgn(M_{12}^{\{1\}})\sqrt{\frac{1}{2}(1 - \frac{M_{11}^{\{1\}} - M_{22}^{\{1\}}}{\sqrt{D_1}})} \tag{21a}$$

and

$$P_2 = -sgn(M_{12}^{\{2\}})\sqrt{\frac{1}{2}(1 - \frac{M_{11}^{\{2\}} - M_{22}^{\{2\}}}{\sqrt{D_2}})},$$

$$Q_2 = \sqrt{\frac{1}{2}(1 + \frac{M_{11}^{\{2\}} - M_{22}^{\{2\}}}{\sqrt{D_2}})}. \tag{21b}$$

The symbol R_K in (20) reads

$$R_K = \frac{B_{13}P_K + B_{23}Q_K}{G_K - G_3^{(1)}}. \tag{21c}$$

As in the case of qP waves, the vectors (20) are generally not unit.

Eqs.(18) and (20), which hold for arbitrarily chosen vectors e_i^1 and e_i^2, are basic equations for the iterative determination of the higher-order approximations of the eigenvalues and eigenvectors of the two qS waves propagating in anisotropic media, see Farra (2001). It is only necessary to find higher-order approximations $M_{ij}^{(n)\{K\}}$ of the matrix $M_{ij}^{\{K\}}$ and insert them into (18) and (20). From (17), the first-order approximation $M_{ij}^{(1)\{K\}}$ is obviously $M_{ij}^{(1)\{K\}} = B_{ij}$, and it is thus the same for both qS waves. The nth-order approximation $M_{ij}^{(n)\{K\}}$ of the matrix $M_{ij}^{\{K\}}$, for $n \geq 2$, is defined by Eq.(17) with G_K substituted by $G_K^{(n-2)}$, see Farra (2001). Let us consider, for example, the $qS1$ wave. Inserting $M_{ij}^{(n)\{1\}}$ into (18a) yields nth-order approximation $G_1^{(n)}$ of the eigenvalue G_1. Inserting $M_{ij}^{(n)\{1\}}$ into (20), specified for $K = 1$, yields $(n-1)$st-order approximation $g_i^{(n-1)\{1\}}$ of the polarization vector $g_i^{\{1\}}$. For the $qS2$ wave, the procedure is analogous. As in the case of qP wave, the odd-order approximations of the eigenvalues (18) are independent of the choice of the reference medium while the even-order approximations depend on it.

Let us again consider the $qS1$ wave, for example, and let us introduce vectors $\hat{e}_i^{(n-1)1}$, $\hat{e}_i^{(n-1)2}$ chosen in an analogous way to the vectors $\hat{e}_i^{(0)1}$, $\hat{e}_i^{(0)2}$ in Eq.(8). Specifically, let us choose them so that they define the nth-order approximation $\hat{M}_{ij}^{(n)\{1\}}$ of the matrix $M_{ij}^{\{1\}}$, which satisfies

$$\hat{M}_{12}^{(n)\{1\}} = 0, \qquad \hat{M}_{11}^{(n)\{1\}} > \hat{M}_{22}^{(n)\{1\}}. \tag{22}$$

We can see that for $n = 1$, Eq.(22) reduces to Eq.(8) since $\hat{M}_{ij}^{(1)\{K\}} = \hat{B}_{ij}^{(0)}$. This is consequence of the equation $M_{ij}^{(1)\{K\}} = B_{ij}$, which holds for arbitrarily chosen vectors e_i^J. Using vectors $\hat{e}_i^{(n-1)1}$ and $\hat{e}_i^{(n-1)2}$ chosen in the above described way, we get from (18a) and (19) the expression for the nth approximation $G_1^{(n)}$ of the eigenvalue G_1:

$$G_1^{(n)} = \hat{M}_{11}^{(n)\{1\}}. \tag{23}$$

In the first-order approximation ($n = 1$), Eq.(23) yields the first equation of (9):

$$G_1^{(1)} = \hat{M}_{11}^{(1)\{1\}} = \hat{B}_{11}^{(0)}. \tag{24}$$

The direction of the vector $g_i^{(n)\{1\}}$, which represents the nth-order approximation ($n \geq 1$) of the polarization vector $g_i^{\{1\}}$ of the considered $qS1$ wave is specified by the vector

$$\hat{e}_i^{(n)1} + \frac{\hat{B}_{13}^{(n)}}{G_1^{(n-1)} - G_3^{(1)}} e_i^3. \tag{25}$$

Note again that the vector (25) is not unit. Eq.(25) follows from Eq.(20) specified for $K = 1$. For $n = 0$, Eq.(20) specified for $K = 1$ yields the vector $\hat{e}_i^{(0)1}$ as the zero-order approximation of the vector $g_i^{\{1\}}$. The symbol $\hat{B}_{kl}^{(n)}$ denotes a matrix defined by Eq.(6), in which the vectors e_i^J are substituted by the vectors $\hat{e}_i^{(n)J}$. For $n = 0$, the matrix $\hat{B}_{kl}^{(n)}$ reduces to $\hat{B}_{kl}^{(0)}$ satisfying (8). In the first-order approximation ($n = 1$), Eq.(25) yields the direction of the vector $g_i^{(1)\{1\}}$

$$\hat{e}_i^{(1)1} + \frac{\hat{B}_{13}^{(1)}}{\beta^2 - \alpha^2} e_i^3. \tag{26}$$

Note that when using the vectors $\hat{e}_i^{(n)J}$, the nth-order approximation of the polarization vector $g_i^{\{1\}}$ has zero projection into the vector $\hat{e}_i^{(n)2}$. Since the vector $\hat{e}_i^{(1)1}$ used in (26) differs from $\hat{e}_i^{(0)1}$, Eq.(26) differs from a similar equation for the first-order perturbation of the $qS1$-wave polarization vector given by Jech and Pšenčík (1989).

In an analogous way, we can derive corresponding approximations of the eigenvalue G_2 and of the direction of the polarization vector $g_i^{\{2\}}$ of the $qS2$ wave. In Eq.(22), we must substitute $\hat{M}_{ij}^{(n)\{1\}}$ by $\hat{M}_{ij}^{(n)\{2\}}$. This defines a new set of vectors $\hat{e}_i^{(n-1)1}$ and $\hat{e}_i^{(n-1)2}$, which are generally different from the set of vectors $\hat{e}_i^{(n-1)1}$ and $\hat{e}_i^{(n-1)2}$ related to the $qS1$ wave. Then, Eq.(23) becomes $G_2^{(n)} = \hat{M}_{22}^{(n)\{2\}}$ and in Eq.(25), $\hat{e}_i^{(n)1}$, $\hat{B}_{13}^{(n)}$ and $G_1^{(n-1)}$ must be substituted by $\hat{e}_i^{(n)2}$, $\hat{B}_{23}^{(n)}$ and $G_2^{(n-1)}$, respectively.

The vectors $\hat{e}_i^{(n)J}$ corresponding to one of the considered qS waves, let us say $qS1$ wave, can be expressed in terms of arbitrarily chosen, mutually perpendicular unit vectors e_i^1 and e_i^2 in the following way

$$\begin{aligned} \hat{e}_i^{(n)1} &= e_i^1 \cos\phi^{(n)} + e_i^2 \sin\phi^{(n)}, \\ \hat{e}_i^{(n)2} &= -e_i^1 \sin\phi^{(n)} + e_i^2 \cos\phi^{(n)}. \end{aligned} \tag{27}$$

The angle $\phi^{(n)}$ can be determined from the equation

$$\tan 2\phi^{(n)} = \frac{2M_{12}^{(n+1)\{1\}}}{M_{11}^{(n+1)\{1\}} - M_{22}^{(n+1)\{1\}}}. \tag{28}$$

Eq.(28) follows from the first condition in Eq.(22) if we take into account that the matrix $M_{ij}^{(n+1)\{1\}}$ satisfies the same relations (10) as the matrix B_{ij}. The second condition in (22) guarantees unique determination of the angle $\phi^{(n)}$ from Eq.(28). For $n = 0$, Eq.(28) yields

$$\tan 2\phi^{(0)} = \frac{2B_{12}}{B_{11} - B_{22}}. \tag{29}$$

For $qS2$ wave, we can proceed in the same way. We obtain, of course, generally different vectors $\hat{e}_i^{(n)1}$ and $\hat{e}_i^{(n)2}$ and different values of the angles $\phi^{(n)}$. Only in the zero-order (see Eq.(29)) and the first-order approximation, $\phi^{(0)}$ and $\phi^{(1)}$ have universal meaning for both qS waves since $M_{ij}^{(1)\{1\}} = M_{ij}^{(1)\{2\}}$ and $M_{ij}^{(2)\{1\}} = M_{ij}^{(2)\{2\}}$.

Let us mention that the procedure of selection of vectors $\hat{e}_i^{(n-1)J}$ satisfying Eq.(22) can be extended to the limiting case. For Kth qS wave we can specify vectors $\hat{e}_i^J$ so that the elements of the matrix $\hat{M}_{ij}^{\{K\}}$ (the matrix $\hat{M}_{ij}^{\{K\}}$ is the exact matrix $M_{ij}^{\{K\}}$, see (17), specified for the vectors $\hat{e}_i^J$) satisfy similar relations to those in Eq.(22):

$$\hat{M}_{12}^{\{K\}} = 0, \qquad \hat{M}_{11}^{\{K\}} > \hat{M}_{22}^{\{K\}}. \tag{30}$$

For such a specification, we get for the $qS1$ wave from (18a) and (19) specified for $K = 1$

$$G_1 = \hat{M}_{11}^{\{1\}}, \tag{31}$$

and for the direction of the corresponding exact polarization vector:

$$\hat{e}_i^1 + \frac{\hat{B}_{13}}{G_1 - G_3^{(1)}} e_i^3. \tag{32}$$

The matrix $\hat{B}_{mn}$ is specified by the vectors $\hat{e}_i^J$. Again, analogous expressions for the eigenvalue G_2 and for the direction of the polarization vector $g_i^{\{2\}}$ can be written for the $qS2$ wave. Let us again emphasize that the corresponding vectors $\hat{e}_i^1$ and $\hat{e}_i^2$ are generally different from the vectors $\hat{e}_i^J$ related to the $qS1$ wave.

The first-order approximation of the qS-wave polarization vectors (see (26) for $qS1$ wave, for example) depends on the elements $\bar{B}_{13}$ and $\bar{B}_{23}$ of the matrix $\bar{B}_{mn}$ (the elements $\hat{B}_{13}^{(1)}$ and $\hat{B}_{23}^{(1)}$ are functions of $\bar{B}_{13}$ and $\bar{B}_{23}$ only, see Eq.(10)) and, through the angle $\phi^{(1)}$, which controls specification of the vectors $\hat{e}_i^{(1)J}$ (see (28) specified for $n = 1$), also on elements $\bar{B}_{11}$, $\bar{B}_{12}$ and $\bar{B}_{22}$. The first-order approximation of the eigenvalues (24) depends on elements $\bar{B}_{11}$, $\bar{B}_{12}$ and $\bar{B}_{22}$ (since $\hat{B}_{11}^{(0)}$ and $\hat{B}_{22}^{(0)}$ depend on $\bar{B}_{11}$, $\bar{B}_{12}$ and $\bar{B}_{22}$ only, see Eq.(10)). From the discussion in Sec.II.B, it follows that the first-order approximations of the eigenvalues (representing squares of the phase velocities in the first-order approximations) and of the polarization vectors of qS waves depend on the complete set of 21 WA parameters. Six individual WA parameters and remaining fifteen in 9 combinations can be determined from the phase velocity. Fifteen individual WA parameters and remaining six in 5 combinations can be determined from the polarization vectors. The polarization vectors are, in addition, also controlled by the difference of squared velocities of the reference isotropic medium: $\alpha^2 - \beta^2$.

III. ACOUSTICAL AXES

Characteristic feature of any anisotropic medium are so-called *acoustical axes*. According to Schoenberg and Helbig (1997) the acoustical axes specify two kinds of directions: longitudinal and singular.

A. Longitudinal directions

A longitudinal direction is a direction in which the qP wave is purely longitudinal and the qS waves are purely transverse. The polarization vector $g_i^{\{3\}}$ of the qP wave is parallel to the corresponding wave normal n_i, the polarization vectors $g_i^{\{K\}}$ of the qS waves are perpendicular to n_i. By taking this into account in Eqs.(11) and in the expression for R_K in (21c), we can see that in the longitudinal direction

$$\hat{B}_{13}^{(0)} = \hat{B}_{23}^{(0)} = B_{13} = B_{23} = 0. \tag{33}$$

We can see that Eq.(33) holds for arbitrary choice of vectors e_i^K. From Eqs.(12) or (13), we can see that if Eq.(33) holds, also the first-order approximation of the qP-wave polarization vector is parallel to the wave vector n_i. From Eq.(26), we can see that the first-order approximation of the $qS1$-wave polarization vector is perpendicular to the wave vector n_i if Eq.(33) holds. Thus the longitudinal directions determined from the first-order perturbation formulae are identical with exact longitudinal directions.

Let us mention that the condition (33) is automatically satisfied in isotropic media. This means that every direction in an isotropic medium is a longitudinal direction.

From Eq.(14) we can derive another interesting property of the first-order perturbation of the qP-wave eigenvalue. For $G_3 > G_1^{(1)}$ and $G_3 > G_2^{(1)}$, Eq.(14) indicates that $G_3 \geq G_3^{(1)}$, i.e., the qP-wave first-order formula yields value which is less or at most equal to exact eigenvalue. An alternative proof of this inequality can be found in the Appendix of Pšenčík and Gajewski (1998). We can see from Eq.(14) that the equality $G_3 = G_3^{(1)}$ occurs when (33) holds, i.e., in longitudinal directions.

Similar property as above holds also for the qS waves. The proof is only slightly more complicated. Let us consider vectors $\hat{e}_i^{(0)J}$ chosen so that conditions (8) are satisfied and let us rewrite Eqs.(18) for this specification. We get

$$G_1 = \frac{1}{2}\Big(\hat{B}_{11}^{(0)} + \hat{B}_{22}^{(0)} + \frac{(\hat{B}_{13}^{(0)})^2 + (\hat{B}_{23}^{(0)})^2}{G_1 - G_3^{(1)}} + \sqrt{D_1}\Big)$$

$$G_2 = \frac{1}{2}\Big(\hat{B}_{11}^{(0)} + \hat{B}_{22}^{(0)} + \frac{(\hat{B}_{13}^{(0)})^2 + (\hat{B}_{23}^{(0)})^2}{G_2 - G_3^{(1)}} - \sqrt{D_2}\Big). \tag{34}$$

Let us also rewrite Eq.(19) for the same specification:

$$D_K = (\hat{B}_{11}^{(0)} - \hat{B}_{22}^{(0)})^2 + \Big(\frac{(\hat{B}_{13}^{(0)})^2 + (\hat{B}_{23}^{(0)})^2}{G_K - G_3^{(1)}}\Big)^2$$

$$+2(\hat{B}_{11}^{(0)} - \hat{B}_{22}^{(0)})\frac{(\hat{B}_{13}^{(0)})^2 - (\hat{B}_{23}^{(0)})^2}{G_K - G_3^{(1)}}. \tag{35}$$

The quantity D_K satisfies the following inequalities

$$\Big(\hat{B}_{11}^{(0)} - \hat{B}_{22}^{(0)} + \frac{(\hat{B}_{13}^{(0)})^2 + (\hat{B}_{23}^{(0)})^2}{G_K - G_3^{(1)}}\Big)^2 \leq D_K, \tag{36a}$$

$$D_K \leq \Big(\hat{B}_{11}^{(0)} - \hat{B}_{22}^{(0)} - \frac{(\hat{B}_{13}^{(0)})^2 + (\hat{B}_{23}^{(0)})^2}{G_K - G_3^{(1)}}\Big)^2. \tag{36b}$$

The inequalities in (36) follow from

$$-[(\hat{B}_{13}^{(0)})^2 + (\hat{B}_{23}^{(0)})^2] \leq (\hat{B}_{13}^{(0)})^2 - (\hat{B}_{23}^{(0)})^2 \leq (\hat{B}_{13}^{(0)})^2 + (\hat{B}_{23}^{(0)})^2 \tag{37}$$

and from $G_K < G_3^{(1)}$. Inserting (36b) to the expression for G_1 in (34) and (36a) to G_2, we get two important results

$$G_1 \leq \hat{B}_{11}^{(0)} = G_1^{(1)}, \qquad G_2 \leq \hat{B}_{22}^{(0)} = G_2^{(1)}. \tag{38}$$

We can see that the first-order formulae yield values which are greater or at least equal to exact eigenvalues of the qS waves. As in the case of qP wave, we can see from Eqs.(17)-(19) that the equality $G_1 = G_1^{(1)}$ and $G_2 = G_2^{(1)}$ occurs when (33) holds, i.e., in longitudinal directions.

Let us mention that the equality of the first-order approximation of only one of the qS-wave eigenvalues to its exact counterpart does not imply a longitudinal direction. The SH wave in a TI medium can be taken as an example.

Eqs.(33) can be thus used for the determination of longitudinal directions if the WA parameters are known or, vice versa, for the determination (or constraint) of the WA parameters if the longitudinal directions are known from observations. In the (x, z) plane of an orthorhombic medium, the equation $\bar{B}_{13} = 0$ yields two "bound" directions (Helbig, 1993) along coordinate axes x and z and an equation

$$2\epsilon_x n_1^2 - 2\epsilon_z n_3^2 + \delta_x(n_3^2 - n_1^2) = 0. \tag{39}$$

Equation $\bar{B}_{23} = 0$ is satisfied automatically. For $\tan^2\theta = n_1^2/n_3^2$, where θ is measured from the z-axis, and under one of the conditions

$$\delta_x < \min(\epsilon_x, \epsilon_z), \qquad \delta_x > \max(\epsilon_x, \epsilon_z), \tag{40}$$

Eq.(39) yields

$$\tan^2\theta = \frac{2\epsilon_z - \delta_x}{2\epsilon_x - \delta_x}. \tag{41}$$

Eq.(41) and conditions (40) are equivalent to those given for the same situation by Schoenberg and Helbig (1997). In a similar way we get for the (x, y) plane of an orthorhombic medium from Eqs.(33) two "bound" directions along coordinate axes x and y and the direction

$$\tan^2\theta = \frac{2\epsilon_y - \delta_z}{2\epsilon_x - \delta_z}. \tag{42}$$

Here θ is measured from the y-axis. The longitudinal directions can exist if one of the conditions

$$\delta_z < \min(\epsilon_x, \epsilon_y), \qquad \delta_z > \max(\epsilon_x, \epsilon_y) \tag{43}$$

is satisfied.

B. Singular directions

The second type of acoustical axes is connected with singular directions. Singular directions exist only for qS waves and occur when the phase velocities (or the eigenvalues G_M) of the two qS waves are equal. This is so-called degenerate situation in perturbation analysis. In such a situation, it is not possible to specify uniquely the polarization vectors of the qS waves. It is only possible to find the plane in which the polarization vectors are situated, i.e., the plane perpendicular to the third polarization vector. This fact reflects the indeterminacy of corresponding formulae for the polarization vectors. The expression (17) is independent of the type of the considered wave in the singular direction because $G_1 = G_2$. Inspection of Eqs.(18) reveals that the equality $G_1 = G_2$ occurs if $D_1 = D_2 = 0$.

Because D_K is given, see (19), by a sum of two quadratic terms, each of these terms must be zero. This yields

$$M_{11}^{\{K\}} = M_{22}^{\{K\}}, \quad M_{12}^{\{K\}} = 0. \tag{44}$$

Eq.(44) holds simultaneously for $K = 1$ and $K = 2$, and for an arbitrary choice of the vectors e_i^K. In the first-order approximation, the singular direction is specified by

$$B_{11} = B_{22}, \quad B_{12} = 0. \tag{45}$$

In the (x, z) plane of an orthorhombic medium, the equation $B_{12} = 0$ is automatically satisfied and equation $B_{11} = B_{22}$ has an explicit form

$$\alpha^2(\epsilon_x - \delta_x + \epsilon_z)\cos^4\theta + [\beta^2(\gamma_y - \gamma_z) - \alpha^2(\epsilon_x - \delta_x + \epsilon_z)]\cos^2\theta + \beta^2(\gamma_z - \gamma_x) = 0. \tag{46}$$

In the plane (y, z), equation $B_{11} = B_{22}$ has the form

$$\alpha^2(\epsilon_y - \delta_y + \epsilon_z)\cos^4\theta + [\beta^2(\gamma_x - \gamma_z) - \alpha^2(\epsilon_y - \delta_y + \epsilon_z)]\cos^2\theta + \beta^2(\gamma_z - \gamma_y) = 0. \tag{47}$$

If a singular direction differs from a longitudinal direction, the equality $G_1 = G_2$ does not imply generally $G_1^{(1)} = G_2^{(1)}$. This means that the singular direction in the first-order approximation, specified by (45), differs generally from the actual singular direction specified by (44). This also holds for approximations of higher orders. In the limit, the singular direction should converge to the actual one. If Eqs.(44) and (45) are satisfied for the same direction, this automatically implies that that direction is longitudinal. Let us mention that the conditions (44), (45) are automatically satisfied in isotropic media. This means that every direction in an isotropic medium is a singular direction.

IV. NUMERICAL EXAMPLES

In applications, the most frequently used formulae are formulae based on the first-order approximation. In the following, we shall compare results of the first- and second-order perturbation formulae for qS waves with exact results for several types of anisotropic media. This will enable us to estimate accuracy of the first-order formulae in a vicinity of singular directions. Let us emphasize that the results of this section were obtained by using Eqs.(18) and (20). In order to calculate the n-th order approximation of the square of the phase velocity from Eq.(18), the n-th order approximation of the matrix $M_{ij}^{\{K\}}$ was used. The velocity was calculated as a square root from the corresponding eigenvalue. For the determination of the n-th order approximation of polarization vectors from (20), the $(n+1)$-th order approximation of $M_{ij}^{\{K\}}$ was used.

For illustration of effects of a singularity on the first-order approximation formulae, let us consider a simple transversely isotropic medium with vertical axis of symmetry. We use the model of thin water-filled cracks of Shearer and Chapman (1989). The model is specified by the density-normalized elastic parameters A_{ij}, in $(\mathrm{km/s})^2$, with values: $A_{11} = A_{22} = 20.16$, $A_{33} = 19.63$, $A_{12} = 7.40$, $A_{13} = A_{23} = 7.26$, $A_{44} = A_{55} = 3.48$, $A_{66} = 6.38$. Anisotropy of this model is rather strong, about 29% and 30% for qS waves. Due to the axial symmetry of the medium, it is sufficient to investigate just a quadrant of a vertical plane containing the axis of symmetry. Figure 1 shows variations of phase velocities of the two qS waves in such a section. The angle θ specifies the direction of the wave vector, $\theta = 0^0$ corresponds to the direction along the axis of symmetry, $\theta = 90^0$ corresponds to the direction perpendicular to it. The two distinct curves correspond to the SH and qSV waves. Polarization of the SH wave is perpendicular to the plane of propagation and thus to the wave vector (therefore we omit the letter q in front of SH), the qSV wave is polarized in the plane of propagation, its polarization is generally not perpendicular to the wave vector. The first-order phase velocity formula for the SH wave yields exact values of the phase velocity (because the zero-order approximation of the SH-wave polarization vector coincides with its exact counterpart). For the qSV wave the values generally differ, the approximate values being always greater or equal to the exact ones, see the analytical proof above. Results of the approximate formula coincide with exact for $\theta = 0^0$ (kiss singularity), for approximately 45^0 (longitudinal direction) and for 90^0 (another longitudinal direction). For the goals of this paper, the most interesting is the intersection of the SH- and qSV-wave phase velocities at 58.59^0, which corresponds to an intersection singularity. We can see that the singularity estimated by the first-order formula is shifted from the direction of the true singularity to the direction $\theta = 59.7^0$ determined from (46). The first-order perturbation formula thus yields distorted result. It is of interest that there is no shift in case of the kiss singularity. The kiss singularity direction coincides with the longitudinal direction, which is determined exactly by the first-order formulae.

Let us now consider a more complicated situation, an orthorhombic model used by Farra (2001). The model is specified by the density-normalized elastic parameters A_{ij}, in $(\mathrm{km/s})^2$, with values: $A_{11} = 10.8$, $A_{22} = 11.3$, $A_{33} = 8.5$, $A_{12} = 2.2$, $A_{13} = 1.9$, $A_{23} = 1.7$, $A_{44} = 3.6$, $A_{55} = 3.9$ and $A_{66} = 4.3$. Squares of the P- and S-wave velocities of the reference isotropic medium are chosen 10.04 and 4.01 $(\mathrm{km/s})^2$, respectively. Figure 2 shows exact (top), first-order (middle) and second-order (bottom) maps of relative differences (in %) of $qS1$ and $qS2$ phase velocities as functions of the polar angle θ and the azimuth φ specifying the wave vector. The relative differences are calculated as differences of the qS-wave velocities normalized by their average. There is a point singularity at $\varphi = 0^0$ and $\theta = 34.4^0$ and the region of small differences extends to larger values of φ in the map of exact differences. Another region of small differences can be observed on the right-hand side of the exact map for $\varphi = 90^0$ and $\theta \sim 35^0$. Let us now compare the exact map with its first-order approximation in the middle frame. We can see several interesting phenomena. The most significant is the shift of the region of minimum differences to higher values of θ, from 34.4^0 to 38.91^0 following from (46). Another important feature is appearence of two minima seemingly indicating existence of singularities for $\varphi = 90^0$, $\theta = 33^0$ and $\theta = 45^0$. The bottom of the valley crossing the plots for $\theta \sim 30 - 40^0$ is deeper in the first-order map. For relative differences of $qS1$ and $qS2$ phase velocities greater than, approximately 3%, the differences between the top and middle frames are minimum. From inspection of the frame of the second-order approximation, we can see that it gives effectively the same results as the exact formula. Slight exception is the region of $\varphi = 90^0$ and $\theta \sim 35^0$.

In the following, we show effects of the first-order perturbations on the model of orthorhombic medium proposed and studied in detail by Schoenberg and Helbig (1997). The model has been obtained by combining finely layered (VTI) model with vertical fractures.

The model is specified by the density-normalized elastic parameters A_{ij}, in $(\mathrm{km/s})^2$, with values: $A_{11} = 9.00$, $A_{22} = 9.84$, $A_{33} = 5.94$, $A_{12} = 3.60$, $A_{13} = 2.25$, $A_{23} = 2.40$, $A_{44} = 2.00$, $A_{55} = 1.60$ and $A_{66} = 2.18$. Squares of the P- and S-wave velocities of the reference isotropic medium are chosen 7.60 and 2.26 $(\mathrm{km/s})^2$, respectively. Figure 3 shows again exact (top), first-order (middle) and second-order (bottom) maps of relative differences of $qS1$ and $qS2$ phase velocities. This model has four singularities in the shown map: two singularities (conical points) for $\varphi = 0^0$ and $\theta = 20.1^0$ and $\theta = 59.8^0$, one conical point for $\varphi = 90^0$ and $\theta = 72.5^0$, and another one for $\varphi = 44.89^0$ and $\theta = 46.53^0$, see Schoenberg and Helbig (1997). Comparison of the exact and first-order maps yields several interesting phenomena. We can see that, as in Fig.2, the singularity $\varphi = 0^0$, $\theta = 59.8^0$ is shifted within the plane of symmetry $\varphi = 0^0$ to $\theta = 66.23^0$ following from (46). The other singularity in the same plane, $\varphi = 0^0$, $\theta = 20.1^0$ remains, however, at nearly the same position, specifically at $\theta = 20.09^0$. This is result of closeness of the longitudinal direction specified by $\varphi = 0^0$ and $\theta = 20.3^0$. As mentioned above, the first-order formulae yield exact results for the longitudinal directions. Thus the differences between the first-order approximations of phase velocities of the qS waves in a vicinity of a longitudinal direction are also close to exact ones. The singularity $\varphi = 90^0$, $\theta = 72.5^0$ is shifted in the plane $\varphi = 90^0$ to $\theta = 76.74^0$ following from (47). Interesting is the shift of the point singularity $\varphi = 44.89^0$, $\theta = 46.53^0$ situated off the symmetry planes. The shift of the singularity in the first-order approximation is in both directions. It is about 2^0 in φ and about 4^0 in the polar angle θ. As in Fig.2, the results of the second-order formula practically coincide with exact results. The only exception is the singularity $\varphi = 90^0$, $\theta = 72.5^0$ which remains shifted by about 2^0. The use of the third-order formula would shift the singularity to the proper position.

The effect of singularities on polarization vectors is much more pronounced. In singular directions the formulae for polarization vectors are indetermined. In their vicinity, polarization vectors change fast. This can be seen in Fig.4. Fig.4 shows deviations of the exact polarization vector of the faster $qS1$ wave from the zero-order (top), first-order (middle) and (second-order) approximation of the same polarization vector. From the upper plot we can see that the zero-order approximation yields deviation less than 1^0 in only small regions. Generally, the deviation of the zero-order approximation is large, it reaches values of 90^0! The points, in which the isolines concentrate, represent actual and shifted positions of singularities. The actual positions are related to the exact polarization vector, which varies significantly in their vicinity. The shifted positions are related to the zero-order approximation of the polarization vector, which varies significantly in the vicinity of the shifted positions, see the top plot in Figure 3. There is only very slightly increased variation of deviations in the vicinity of the singularity $\varphi = 0^0$, $\theta = 20.1^0$. The first-order approximation (middle plot) of the polarization vector reduces the deviations significantly. The maximum deviations are again upto 90^0 but they are concentrated to the narrow region connecting three singularities. Except the singularity $\varphi = 90^0$, $\theta = 72.5^0$, the concentrations of isolines are close to actual positions of singularities. This picture further improves when second-order approximation of the polarization vector is used, see the bottom plot in Fig.4. The maximum deviation does not exceed 1^0 in most of the studied region, all non-negligible deviations are concentrated in the narrow belt connecting the three actual singularities whose positions are indicated by concentration of isolines.

Fig.5 shows detailed maps of the deviations of the exact polarization vector of the faster

$qS1$ wave from the zero-order (top), first-order (middle) and (second-order) approximation of the same polarization vector. We concentrate there on the vicinity of the singularity which can be seen in Fig.4 in the region of φ and θ between 40^0 and 50^0. The shift of the singularity with varying order of the approximation is clearly visible.

V. CONCLUSIONS

A detailed analysis of higher-order formulae of Farra (2001) for the eigenvalues and directions of the polarization vectors of qP and qS waves in weakly anisotropic media was performed. Possible ways of constructing separate higher-order perturbation for $qS1$ and $qS2$ waves were shown. The formulae depend on various elements of the matrix B_{mn}, which, in turn, depend linearly on the WA or elastic parameters of a medium. Thus the sensitivity analysis of attributes of elastic waves to the parameters of the medium reduces to the sensitivity analysis of the elements of the matrix B_{mn}.

Analysis of the first-order perturbation formulae for phase velocities and polarization vectors yielded the following results. From inversion of qP-wave phase velocity, 15 independent WA parameters can be found. If information about qP-wave polarization and/or orientation of longitudinal axes is available in addition, the same 15 WA parameters can be found, no more. From inversion of qS-wave velocities, only 6 independent WA parameters can be found. Addition of information about singular directions cannot change this situation. If, however, information about qS-wave polarization and/or of orientation of the longitudinal axes is available, the number of determinable independent WA parameters increases to 15. In order to determine all 21 WA parameters, observations of both qP- and qS-wave phase velocities are necessary. From observations of qP- and qS-wave polarizations alone, only 9 individual WA parameters can be found. Additional use of the acoustical axes, specifically of the singular directions adds further 6 independent WA parameters. Using (A3), a similar analysis can be made for elastic parameters.

Another goal of this contribution was to show the effect of the first-order perturbation formulae on the determination of acoustical axes. For this purpose, explicit analytical conditions for the determination of the acoustical axis were given. We have found, both analytically and in numerical examples, that the approximations do not affect positions of the longitudinal directions. They may, however, affect quite significantly positions of singularities. The effects can be removed by using higher-order approximations.

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APPENDIX A: EXPRESSIONS FOR ELEMENTS OF THE MATRIX $\bar{B}_{mn}$

We introduce three mutually perpendicular unit vectors $\bar{e}_i^1$, $\bar{e}_i^2$ and $\bar{e}_i^3$ so that $\bar{e}_i^3 = n_i$, where n_i is a unit wave vector. The vectors $\bar{e}_i^1$ and $\bar{e}_i^2$ are situated in the plane perpendicular to $\bar{e}_i^3$ and they are selected so that

$$\bar{e}_i^1 \equiv D^{-1}(n_1 n_3, n_2 n_3, n_3^2 - 1), \quad \bar{e}_i^2 \equiv D^{-1}(-n_2, n_1, 0), \quad \bar{e}_i^3 = n_i \equiv (n_1, n_2, n_3), \tag{A1}$$

where

$$D = (n_1^2 + n_2^2)^{1/2}, \quad n_1^2 + n_2^2 + n_3^2 = 1. \tag{A2}$$

In the following, a reference isotropic medium with the P-wave velocity α and S-wave velocity β is considered. We specify the symmetric matrix B_{mn} given in Eq.(6) for vectors (A1) and denote it $\bar{B}_{mn}$. The matrix $\bar{B}_{mn}$ can be expressed in terms of 21 WA parameters

$$\epsilon_x = \frac{A_{11} - \alpha^2}{2\alpha^2}, \quad \epsilon_y = \frac{A_{22} - \alpha^2}{2\alpha^2}, \quad \epsilon_z = \frac{A_{33} - \alpha^2}{2\alpha^2},$$

$$\delta_x = \frac{A_{13} + 2A_{55} - \alpha^2}{\alpha^2}, \quad \delta_y = \frac{A_{23} + 2A_{44} - \alpha^2}{\alpha^2}, \quad \delta_z = \frac{A_{12} + 2A_{66} - \alpha^2}{\alpha^2},$$

$$\chi_x = \frac{A_{14} + 2A_{56}}{\alpha^2}, \quad \chi_y = \frac{A_{25} + 2A_{46}}{\alpha^2}, \quad \chi_z = \frac{A_{36} + 2A_{45}}{\alpha^2},$$

$$\epsilon_{15} = \frac{A_{15}}{\alpha^2}, \quad \epsilon_{16} = \frac{A_{16}}{\alpha^2}, \quad \epsilon_{24} = \frac{A_{24}}{\alpha^2}, \quad \epsilon_{26} = \frac{A_{26}}{\alpha^2}, \quad \epsilon_{34} = \frac{A_{34}}{\alpha^2}, \quad \epsilon_{35} = \frac{A_{35}}{\alpha^2},$$

$$\epsilon_{46} = \frac{A_{46}}{\alpha^2}, \quad \epsilon_{56} = \frac{A_{56}}{\alpha^2}, \quad \epsilon_{45} = \frac{A_{45}}{\beta^2},$$

$$\gamma_x = \frac{A_{55} - \beta^2}{2\beta^2}, \quad \gamma_y = \frac{A_{44} - \beta^2}{2\beta^2}, \quad \gamma_z = \frac{A_{66} - \beta^2}{2\beta^2}. \tag{A3}$$

Although the WA parameters depend on velocities α and β of the reference isotropic medium, the matrix $\bar{B}_{mn}$ is *independent* of these velocities. Its elements have the following form:

$$\bar{B}_{11} = \beta^2 + 2\alpha^2 D^{-2}\Big(n_3^3[(\epsilon_{15} - \epsilon_{35})n_1^3 + (\epsilon_{24} - \epsilon_{34})n_2^3 + (\chi_x - \epsilon_{34})n_1^2 n_2 + (\chi_y - \epsilon_{35})n_1 n_2^2]$$
$$+n_3^2[(\epsilon_x - \delta_x + \epsilon_z)n_1^4 + (\epsilon_y - \delta_y + \epsilon_z)n_2^4 + (\delta_z - \delta_x - \delta_y + 2\epsilon_z)n_1^2 n_2^2$$
$$+2(\epsilon_{16} - \chi_z)n_1^3 n_2 + 2(\epsilon_{26} - \chi_z)n_1 n_2^3]$$
$$+n_3[(\epsilon_{35} - \epsilon_{15})n_1^5 + (\epsilon_{34} - \epsilon_{24})n_2^5 + (\epsilon_{34} - \chi_x)n_1^4 n_2 + (\epsilon_{35} - \chi_y)n_1 n_2^4$$
$$+(2\epsilon_{35} - \chi_y - \epsilon_{15})n_1^3 n_2^2 + (2\epsilon_{34} - \chi_x - \epsilon_{24})n_1^2 n_2^3]\Big)$$

$$+2\beta^2 D^{-2}(\gamma_x n_1^2 + \gamma_y n_2^2 + \epsilon_{45} n_1 n_2),$$

$$\bar{B}_{12} = \alpha^2 D^{-2}\Big(n_3^2[(\chi_x - \epsilon_{34})n_1^3 - (\chi_y - \epsilon_{35})n_2^3 + (\epsilon_{35} + \chi_y - 2\epsilon_{15})n_1^2 n_2 + (2\epsilon_{24} - \chi_x - \epsilon_{34})n_1 n_2^2]$$
$$+n_3[(\epsilon_{16}-\chi_z)n_1^4-(\epsilon_{26}-\chi_z)n_2^4+(\delta_x-\delta_y-2\epsilon_x+\delta_z)n_1^3 n_2+(\delta_x-\delta_y+2\epsilon_y-\delta_z)n_1 n_2^3+3(\epsilon_{26}-\epsilon_{16})n_1^2 n_2^2]$$
$$+[(\epsilon_{15}-\chi_y)n_1^4 n_2+(\chi_x-\epsilon_{24})n_1 n_2^4+(\epsilon_{15}-\chi_y)n_1^2 n_2^3+(\chi_x-\epsilon_{24})n_1^3 n_2^2+\epsilon_{46}n_2^3-\epsilon_{56}n_1^3+\epsilon_{46}n_1^2 n_2-\epsilon_{56}n_1 n_2^2]\Big)$$
$$+\beta^2 D^{-2} n_3[2(\gamma_y - \gamma_x)n_1 n_2 + \epsilon_{45}n_1^2 - \epsilon_{45}n_2^2],$$

$$\bar{B}_{22} = \beta^2 + 2\beta^2 D^{-2} n_3^2(\gamma_x n_2^2 + \gamma_y n_1^2 - \epsilon_{45} n_1 n_2)$$
$$+2\alpha^2 D^{-2}\Big(n_3[\epsilon_{46}n_1^3 + \epsilon_{56}n_2^3) + (\epsilon_{56} - \chi_x + \epsilon_{24})n_1^2 n_2 + (\epsilon_{46} - \chi_y + \epsilon_{15})n_1 n_2^2]$$
$$+(\epsilon_x + \epsilon_y - \delta_z)n_1^2 n_2^2 + (\epsilon_{26} - \epsilon_{16})n_1^3 n_2 - (\epsilon_{26} - \epsilon_{16})n_1 n_2^3]\Big) + 2\beta^2\gamma_z(n_1^2 + n_2^2),$$

$$\bar{B}_{13} = \alpha^2 D^{-1}\{n_3^4(\epsilon_{34}n_2 + \epsilon_{35}n_1) + n_3^3[(\delta_x - \epsilon_x - \epsilon_z)n_1^2 + (\delta_y - \epsilon_y - \epsilon_z)n_2^2 + 2\chi_z n_1 n_2]$$
$$+n_3^2[(4\chi_x - 3\epsilon_{34})n_1^2 n_2 + (4\chi_y - 3\epsilon_{35})n_1 n_2^2 + (4\epsilon_{15} - 3\epsilon_{35})n_1^3 + (4\epsilon_{24} - 3\epsilon_{34})n_2^3]$$
$$+n_3[(2\delta_z - \delta_x - \delta_y - \epsilon_x - \epsilon_y + 2\epsilon_z)n_1^2 n_2^2 + 2(2\epsilon_{16} - \chi_z)n_1^3 n_2 + 2(2\epsilon_{26} - \chi_z)n_1 n_2^3$$
$$+(\epsilon_x + \epsilon_z - \delta_x)n_1^4 + (\epsilon_y + \epsilon_z - \delta_y)n_2^4 + (\epsilon_x - \epsilon_z)n_1^2 + (\epsilon_y - \epsilon_z)n_2^2]$$
$$-\chi_x n_1^2 n_2 - \chi_y n_1 n_2^2 - \epsilon_{15}n_1^3 - \epsilon_{24}n_2^3\},$$

$$\bar{B}_{23} = \alpha^2 D^{-1}\{n_3^3(\epsilon_{34}n_1 - \epsilon_{35}n_2) + n_3^2[(\delta_y - \delta_x - \epsilon_y + \epsilon_x)n_1 n_2 + \chi_z n_1^2 - \chi_z n_2^2]$$
$$+n_3[(2\chi_y - 3\epsilon_{15})n_1^2 n_2 - (2\chi_x - 3\epsilon_{24})n_1 n_2^2 + \chi_x n_1^3 - \chi_y n_2^3]$$
$$+(\delta_z - \epsilon_x - \epsilon_y)n_1^3 n_2 - (\delta_z - \epsilon_x - \epsilon_y)n_1 n_2^3 + 3(\epsilon_{26} - \epsilon_{16})n_1^2 n_2^2 + \epsilon_{16}n_1^4 - \epsilon_{26}n_2^4 + (\epsilon_y - \epsilon_x)n_1 n_2\},$$

$$\bar{B}_{33} = \alpha^2 + 2\alpha^2[2n_3^3(\epsilon_{34}n_2 + \epsilon_{35}n_1) + n_3^2((\delta_x - \epsilon_x - \epsilon_z)n_1^2 + (\delta_y - \epsilon_y - \epsilon_z)n_2^2 + 2\chi_z n_1 n_2 + \epsilon_z)$$
$$+2n_3(\chi_x n_1^2 n_2 + \chi_y n_1 n_2^2 + \epsilon_{15}n_1^3 + \epsilon_{24}n_2^3) + \epsilon_x n_1^2 + \epsilon_y n_2^2 + (\delta_z - \epsilon_x - \epsilon_y)n_1^2 n_2^2 + 2\epsilon_{16}n_1^3 n_2 + 2\epsilon_{26}n_1 n_2^3]. \tag{A4}$$

Červený, V. (**2001**), *Seismic Ray Theory* (Cambridge University Press, Cambridge).

Chu, Y.C., Degtuar, A.D. and Rokhlin, S.I. (**1994**), "On determination of orthotropic material moduli from ultrasonic velocity data in nonsymmetry planes," J. Acoust. Soc. Am. **95**, 3191-3203.

Farra, V. (**2001**), "High order expressions of the phase velocity and polarization of qP and qS waves in anisotropic media," Geophys. J. Int. **147**, 93-105.

Helbig, K. (**1993**), "Longitudinal directions in media of arbitrary anisotropy," Geophysics **58**, 680-691.

Jech, J. and Pšenčík, I. (**1989**), "First-order perturbation method for anisotropic media," Geophys. J. Int. **99**, 369-376.

Pšenčík, I. and Gajewski, D. (**1998**), "Polarization, phase velocity and NMO velocity of qP waves in arbitrary weakly anisotropic media," Geophysics **63**, 1754-1766.

Pšenčík, I. and Vavryčuk, V. (**2002**), "Approximate relation between the ray vector and the wave normal in weakly anisotropic media," Studia geophys. geod. **46**, 793-807.

Shearer, P.M. and Chapman, C.H. (**1989**), "Ray tracing in azimuthally anisotropic media - I. Results for models of aligned cracks in the upper crust," Geophys. J. Int. **96**, 51-64.

Schoenberg, M. and Helbig, K. (**1997**), "Orthorhombic media: Modeling elastic wave behavior in a vertically fractured earth," Geophysics **62**, 1954-1974.

Shuvalov, A.L. and Every, A.G. (**1997**), "Shape of the acoustic slowness surface of anisotropic solids near points of conical degeneracy," J. Acoust. Soc. Am. **101**, 2381-2383.

Song, L.-P., Every, A.G. and Wright, C. (**2001**), "Tomography of joint P-wave traveltime and polarization data: A simple approach for media with low to moderate velocity gradients," J. Acoust. Soc. Am. **109**, 1796-1807.

Vavryčuk, V. (**2003**), "Parabolic lines and caustics in homogeneous weakly anisotropic solids," Geophys. J. Int., **152**, 318-334.

Zheng, X. and Pšenčík, I. (**2002**), "Local determination of weak anisotropy parameters from qP-wave slowness and particle motion measurements," Pure and Appl. Geophys. **159**, 1881-1905.

FIGURE CAPTIONS

Figure 1. qSV- and SH-wave phase velocity sections for the VTI medium specified in the text. Comparison of exact and first-order values.

Figure 2. Exact, first-order and second-order maps of relative differences of $qS1$- and $qS2$-wave phase velocities as functions of azimuth ϕ and polar angle θ for an orthorhombic medium of Farra (2001).

Figure 3. Exact, first-order and second-order maps of relative differences of $qS1$- and $qS2$-wave phase velocities as functions of azimuth ϕ and polar angle θ for an orthorhombic medium of Schoenberg and Helbig (1997).

Figure 4. Deviations of the exact $qS1$-wave polarization vector from its zero-, first- and second-order approximations as functions of azimuth ϕ and polar angle θ for an orthorhombic medium of Schoenberg and Helbig (1997).

Figure 5. The same as in Figure 4 but for the azimuths and polar angles in the interval $(30^0 - 60^0)$.

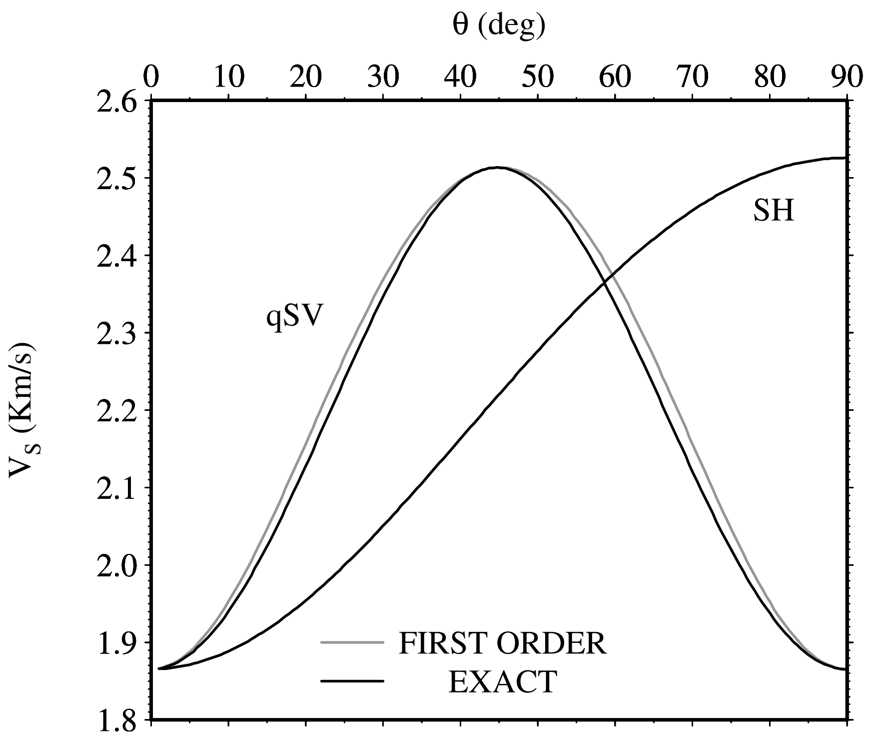

θ (deg)
0
10
20
30
40
50
60
70
80
90
2.6
2.5
2.4
2.3
2.2
2.1
2.0
1.9
1.8
V_S (Km/s)
qSV
SH
FIRST ORDER
EXACT

Figure 1

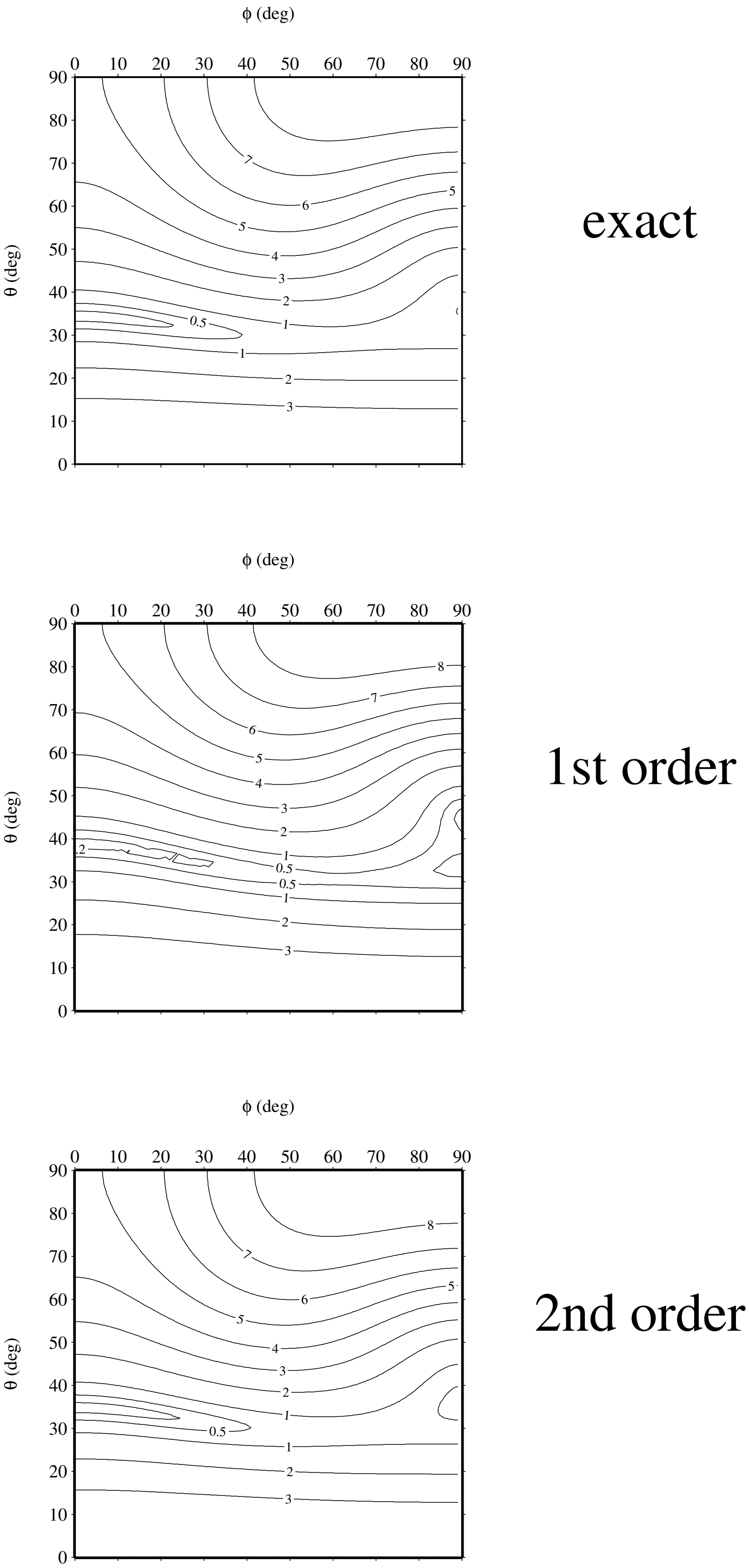


Figure 2

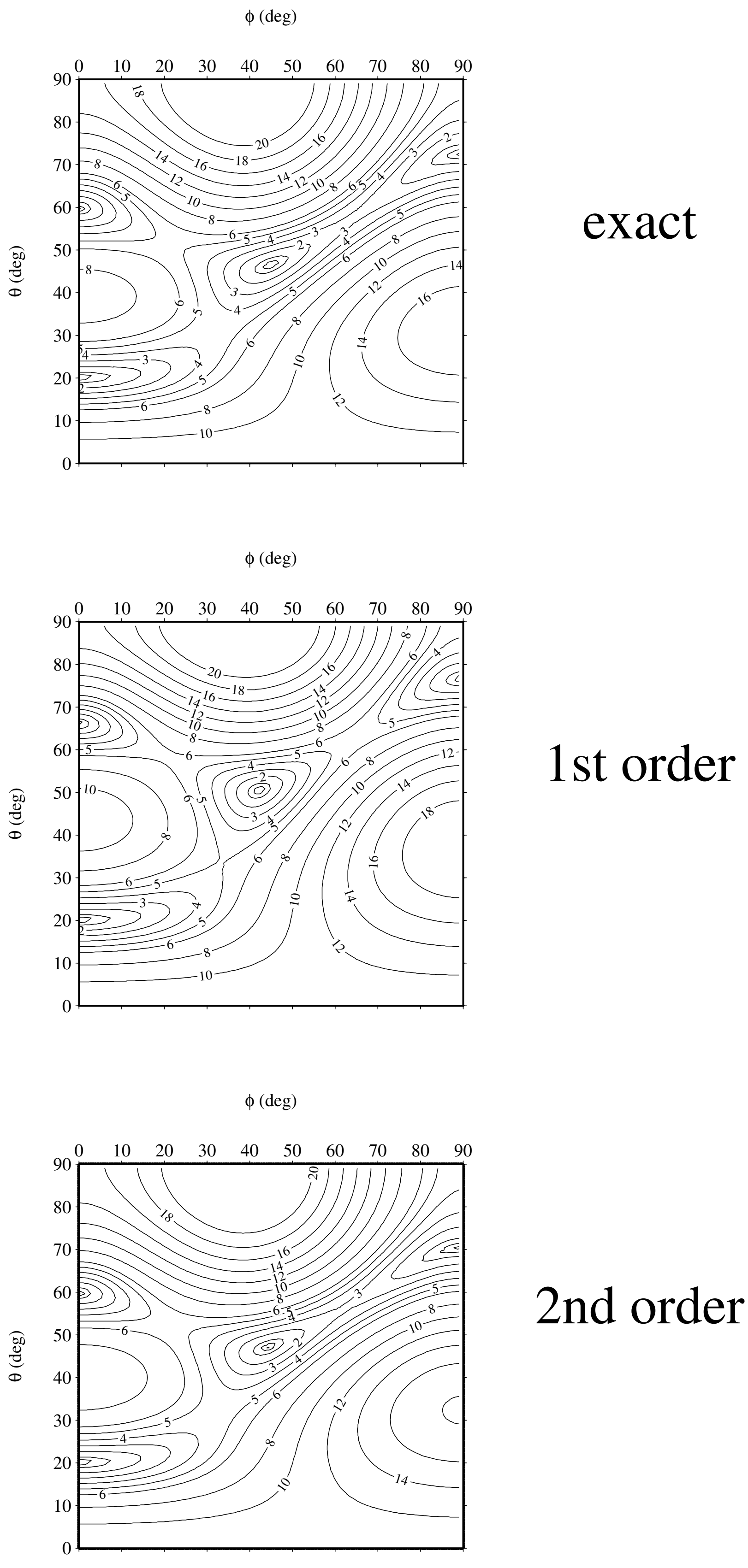
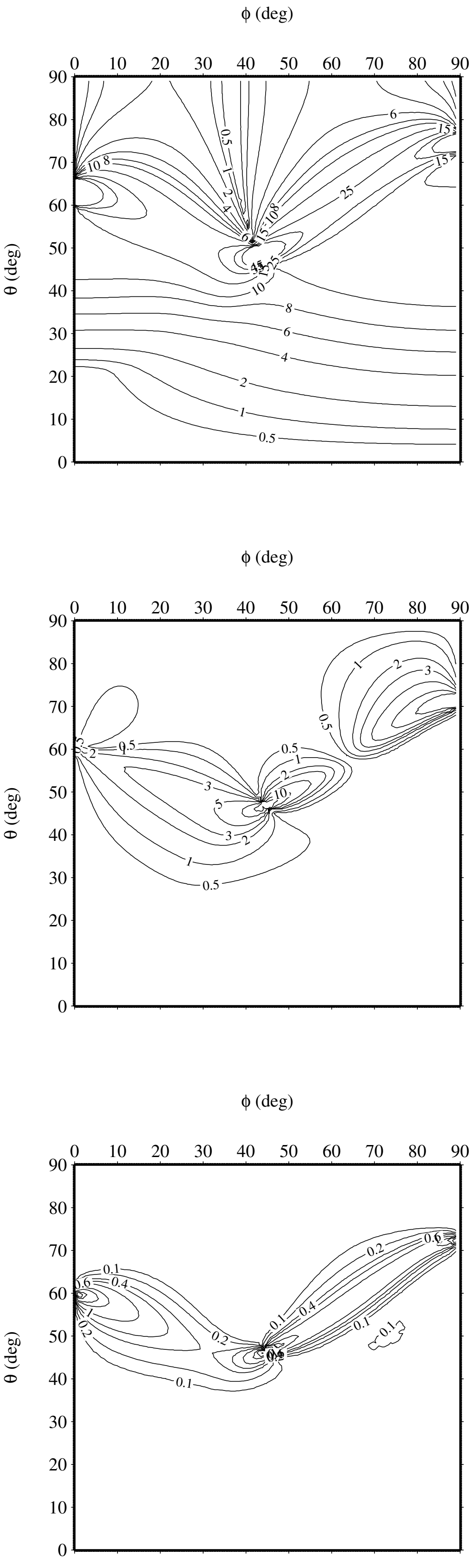


Figure 3

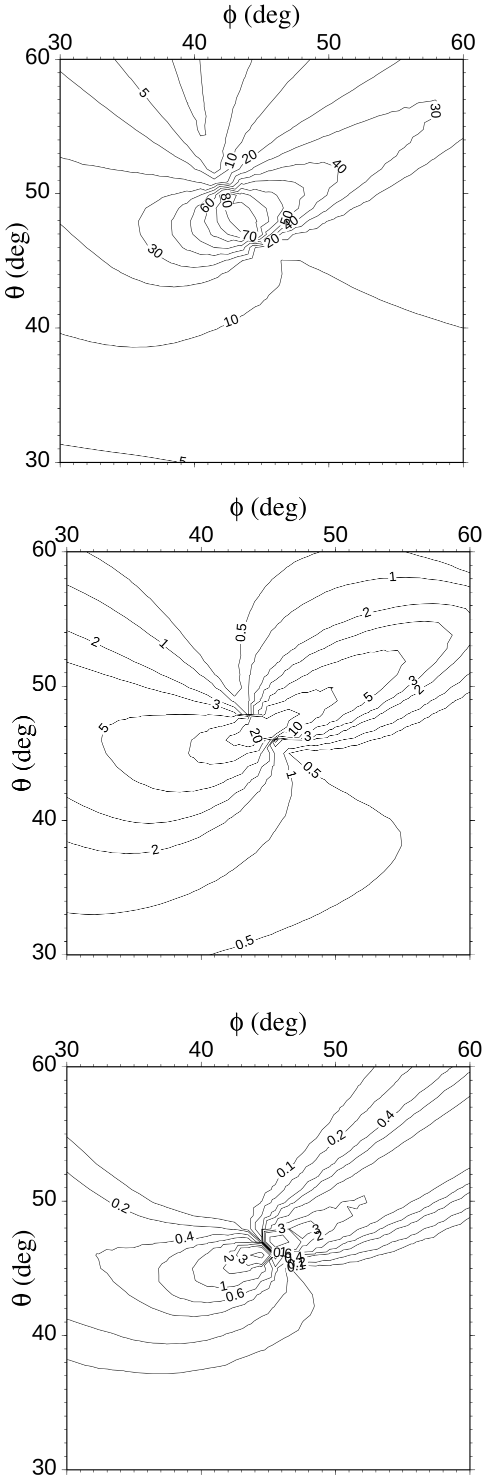


0th order

1st order

2nd order

Figure 4



0th order

1st order

2nd order

Figure 5