Slowness vectors of harmonic plane waves in viscoelastic anisotropic media

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Summary

Properties of inhomogeneous plane waves propagating in a viscoelastic anisotropic medium are investigated. The slowness vector $\mathbf{p}$ is described by the so-called mixed specification. In it, the vector $\mathbf{p}$ is expressed in terms of two given real-valued, mutually perpendicular vectors (one of them specifying the direction of propagation), and of a free complex-valued parameter σ. The parameter σ must be determined so that the slowness vector $\mathbf{p}$ satisfies a constraint relation following from an equation of motion for viscoelastic media. In this contribution, σ is determined by solving a complex-valued polynomial equation of the sixth degree. The used algorithm is quite general. It can be used for homogeneous as well as inhomogeneous plane waves propagating in elastic or viscoelastic, isotropic or anisotropic media. It is shown that the inhomogeneous plane waves propagating in anisotropic viscoelastic media exhibit certain phenomena, not known from elastic anisotropic or viscoelastic isotropic media. For example, the inhomogeneous plane qP wave may propagate with the same phase velocity as one of inhomogeneous plane qS wave. It is also shown that the attenuation angles of inhomogeneous plane waves can attain values greater than $\pi/2$ even for very weakly inhomogeneous plane waves.

Introduction

A harmonic plane wave in a viscoelastic medium can be written as

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{U} \exp\Big(-\mathrm{i}\omega(t - \mathbf{p}\cdot\mathbf{x})\Big). \tag{1}$$

Here $\mathbf{U}$ and $\mathbf{p}$ are complex-valued amplitude and slowness vectors, respectively. The slowness vector $\mathbf{p}$ can be expressed in terms of two real-valued vectors $\mathbf{P}$ and $\mathbf{A}$ (see e.g. Aki & Richards, 1980):

$$\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A}. \tag{2}$$

The vector $\mathbf{P}$ is called the *propagation vector*, the vector $\mathbf{A}$ the *attenuation vector*. The propagation vector $\mathbf{P}$ is perpendicular to the wavefront, i.e. it is parallel to the wave

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normal $\mathbf{N}$. It is oriented in the direction of the propagation of the phase front. The attenuation vector $\mathbf{A}$ is perpendicular to the plane of constant amplitudes. It is oriented in the direction of the exponential decay of amplitudes. If we denote by $\mathbf{M}$ a unit vector parallel to $\mathbf{A}$, we can introduce an *attenuation angle* γ ($0^0 \leq \gamma < 180^0$) as an angle made by $\mathbf{N}$ and $\mathbf{M}$:

$$\cos\gamma = \mathbf{N} \cdot \mathbf{M} \ . \tag{3}$$

The plane wave is called homogeneous if $\gamma = 0$, and inhomogeneous if $\gamma \neq 0$. The vectorial component of the attenuation vector $\mathbf{A}$ into the wavefront is the *wavefront attenuation vector* denoted by $\mathbf{d}$. The wavefront attenuation vector $\mathbf{d}$ is, by definition, perpendicular to $\mathbf{N}$. Its size d is the *inhomogeneity strength*. For d non-zero, the plane wave is inhomogeneous, for $d = 0$, the plane wave is homogeneous.

Eq.(2) for the complex-valued slowness vector $\mathbf{p}$ can be expressed in the following form:

$$\mathbf{p} = \mathcal{C}^{-1}(\mathbf{N} + \mathrm{i}\delta\mathbf{M}) \ . \tag{4}$$

Here $\mathcal{C}$ and δ are two real-valued and non-negative quantities, to be determined. $\mathcal{C}$ is the *phase velocity*. Quantity δ represents the ratio of the lengths of the attenuation and propagation vectors, $\delta = |A|/|P|$. It is called the *attenuation amplitude ratio*. The phase velocity and the attenuation amplitude ratio can be determined from the complex-valued equation, which the slowness vector $\mathbf{p}$ in (4) must satisfy:

$$\det[a_{ijkl}p_j p_l - \delta_{ik}] = 0 \ . \tag{5}$$

Eq.(5) is the condition of solvability of the equation

$$a_{ijkl}p_j p_l U_k = U_i \ , \qquad i = 1, 2, 3 \ , \tag{6}$$

which results from an equation of motion for viscoelastic media. In (5) and (6), a_{ijkl} are complex-valued density-normalized viscoelastic parameters, which are generally frequency dependent. p_i and U_i are components of the slowness and amplitude vectors of the considered wave, respectively. Inserting Eq.(4) into Eq.(5), we can determine $\mathcal{C}$ and δ. Unfortunately, we obtain a system of two coupled equations for $\mathcal{C}$ and δ, which cannot be decoupled into two individual equations, one for $\mathcal{C}$, the other for δ. Consequently, it is, in general, very complicated to solve the system. Moreover, for certain values of $\mathbf{N}$ and $\mathbf{M}$, the system may yield non-physical solutions, for example the negative values of the square of the phase velocity $\mathcal{C}$. In such a case we speak about forbidden directions $\mathbf{N}, \mathbf{M}$ of the slowness vector, see Krebes and Le (1994) or Carcione and Cavallini (1995).

Červený (2003) proposed an alternative specification of the complex-valued slowness vector, called *mixed specification*. In the mixed specification, the slowness vector $\mathbf{p}$ is expressed in terms of two given real-valued mutually perpendicular vectors $\mathbf{n}$ and $\mathbf{d}$ and one complex-valued parameter σ as follows:

$$\mathbf{p} = \sigma\mathbf{n} + \mathrm{i}\mathbf{d} \ . \tag{7}$$

In (7), $\mathbf{n}$ is a unit vector, $\mathbf{N} = \mathbf{n}$ if the real part of the quantity σ is positive and $\mathbf{N} = -\mathbf{n}$ if it is negative. The vector $\mathbf{d}$ is the wavefront attenuation vector. The quantity σ can be determined by inserting Eq.(7) into Eq.(5). This yields:

$$\det[a_{ijkl}(\mathrm{i}d_j + \sigma n_j)(\mathrm{i}d_l + \sigma n_l) - \delta_{ik}] = 0 \ . \tag{8}$$

Eq.(8) is a polynomial equation of the sixth degree with complex-valued coefficients. For the coefficients, see Fedorov (1968) (note missprints in the coefficients with powers 5 and 2). Eq.(8) has six roots corresponding to $qS1$, $qS2$ and qP plane waves propagating in the directions of $\pm\mathbf{n}$. Various methods can be used to solve Eq.(8), see Červený (2003). Here we shall solve Eq.(8) directly, using the Laguerre's method, see Press et al.(1986).

For a found σ, the complex-valued slowness vector can be determined from (7). Here, more than in the slowness vector, we are interested in the values of the phase velocity $\mathcal{C}$, of the cosine of the attenuation angle γ and attenuation amplitude ratio δ. From comparison of Eqs.(3) and (4) with (7), we get for these quantities

$$\mathcal{C} = 1/|\mathrm{Re}\sigma| \ , \qquad \delta = \mathcal{C}[(\mathrm{Im}\sigma)^2 + \mathbf{d}\cdot\mathbf{d}]^{1/2} \ , \qquad \cos\gamma = \epsilon\, \mathrm{Im}(\sigma)/[(\mathrm{Im}\sigma)^2 + \mathbf{d}\cdot\mathbf{d}]^{1/2} \ . \quad (9)$$

Here

$$\epsilon = sgn(\mathrm{Re}\sigma) \ . \quad (10)$$

The proposed algorithm of the computation of the slowness vector, based on Eqs.(8)-(10), is simple and quite general. It can be applied both to homogeneous and inhomogeneous plane waves, propagating in isotropic and anisotropic, perfectly elastic or viscoelastic, media. The algorithm does not yield non-physical solutions and removes problems of forbidden directions.

Numerical tests

We consider the matrix of complex-valued parameters used by Gajewski and Pšenčík (1992). The parameters specify a hexagonally symmetric medium with the horizontal axis of symmetry. The anisotropy is due to aligned, partially liquid saturated cracks in an isotropic host rock. Kinematic viscosity of the fluid is 0.04 St (1 Stokes= 10^{-4} m^2 s^{-1}), the degree of the saturation is 70%:

$$\begin{pmatrix} (5.28,-0.127) & (1.76,-0.043) & (1.76,-0.043) & (0.,0.) & (0.,0.) & (0.,0.) \\ & (8.59,-0.014) & (2.59,-0.014) & (0.,0.) & (0.,0.) & (0.,0.) \\ & & (8.59,-0.014) & (0.,0.) & (0.,0.) & (0.,0.) \\ & & & (3.,0.) & (0.,0.) & (0.,0.) \\ & & & & (2.39,0.) & (0.,0.) \\ & & & & & (2.39,0.) \end{pmatrix} .$$

We can see that the parameters A_{44}, A_{55} and A_{66} describing the qS-wave propagation have zero imaginary parts. For the above set of parameters, we calculated phase velocity $\mathcal{C}$, cosine of the attenuation angle γ and the attenuation amplitude ratio δ of one qP and two qS waves as a function of the vector $\mathbf{n}$ situated in the vertical plane containing the axis of symmetry, i.e. in the symmetry plane. We made calculations for inhomogeneity strength d varying from zero to infinity. The vector $\mathbf{d}$ is chosen so that it is also situated in the symmetry plane and points to the left from $\mathbf{n}$. One of the qS waves is polarized perpendicularly to the plane of symmetry and we call it the SH wave. The other, polarized in the symmetry plane is called the qSV wave. The SH plane wave is fully controlled by parameters A_{44} and A_{66}. As the imaginary parts of these parameters are zero, the SH

wave represents, in fact, an inhomogeneous plane wave propagating in a perfectly elastic medium. Notwithstanding, we treat the SH wave in the same way as inhomogeneous plane qP and qSV waves. This illustrates that the proposed algorithm works safely even for inhomogeneous plane waves propagating in perfectly elastic anisotropic media.

In Figure 1, we can see phase velocity sections for 6 selected values of the inhomogeneity strength d, d=0., 0.3, 0.4, 0.44, 0.5 and 1. Red colour corresponds to the fastest phase velocity sheet, blue colour to the slowest. The sheet with intermediate velocity is black. Due to mutual intersections, the colours do not always represent physical wave sheets. For d varying between 0. and 0.1, the phase velocity changes are negligible. The two qS-wave velocity sections coincide along the horizontal axis. The qP-wave velocity sheet is separated from them. For d=0.3, $\mathcal{C}$ is significantly reduced, qP-wave phase velocity being more affected. We can observe non-symmetric behaviour of phase velocity sections caused by non-zero d. For $d = 0.4$, the qP-wave velocity section is still separated but clear deformations of velocity sections can be observed. They become points of contact of qP- and qS-wave velocity sections for $d = 0.44$. In the direction of the points of contact the qP and qS waves propagate with the same phase velocity. For d greater than 0.5, all three velocity sheets intersect each other, the velocities of all waves decrease. For d greater than 100., all velocities are effectively zero. Let us note that in viscoelastic *isotropic* media, the relevant curves for both qP and qS waves, would be circular.

In Figure 2 we can see how the cosine of the attenuation angle depends on d. Red colour is now used to denote positive values of cosines, i.e. values of γ less or equal to $\pi/2$. Blue colour denotes negative values of cosines, i.e., γ greater than $\pi/2$. For d=0., cosines of the attenuation angles γ of qP and qS waves are unit, i.e., $\gamma = 0$, which indicates homogeneous waves. For all nonzero values of d, it is easy to identify the SH wave, which exhibits properties of an inhomogeneous plane wave propagating in a perfectly elastic medium (as A_{44} and A_{66} are real-valued). The cosine of the SH wave has four lobes, two red and two blue, which, effectively, do not vary with varying d. The cosine is very close to zero, which means that the attenuation angle is close to $\pi/2$. It may be smaller or greater than $\pi/2$ depending on the direction of propagation. In contrast to it, the attenuation angle γ of inhomogeneous plane waves propagating in *perfectly elastic isotropic* media always equals $\pi/2$ and does not depend on the direction of propagation.

Let us now discuss the attenuation angles γ of qP and qSV waves. For d=0.0001, cosine corresponding to qP wave remains nearly unit. It is only slightly reduced in the direction perpendicular to the axis of symmetry. Cosine of the qSV wave is also unit but only outside the direction of the axis of symmetry and the direction perpendicular to it. In these directions, the cosine becomes nearly zero. The cosine of the qSV wave thus has four lobes. For $d = 0.001$, the cosine of the qP wave is unit only along the axis of symmetry. It has minimum values in the direction close to the normal to the axis of symmetry. Non-symmetry of the picture starts to be observable. The lobes of the qSV wave are reduced and narrower. For $d = 0.01$, cosine of the qP wave has a form of inclined infinity symbol with two small blue lobes perpendicular to it. Thus for $d = 0.01$, even qP wave attenuation angle becomes, in certain directions, larger than $\pi/2$. The qSV wave has still four-lobe character. Starting with $d = 0.1$, cosines of all waves become insensitive to the variation of d. They remain such even for the limiting case of infinite d. This means that the plots for d greater or equal 0.1 show limiting values of the attenuation angle. The red lobes indicate how close γ can get to $\pi/2$. The blue lobes indicate how far behind $\pi/2$ the attenuation angle can get. The limiting values of γ corresponding to red lobes specify *forbidden directions* discussed by Krebes and Le (1994) and Carcione and Cavallini (1995). We can see that the forbidden directions have greatest extent in the directions close to 45^o from the axis of symmetry. Their extent is negligible along the axis of symmetry or perpendicularly to it. Let us note again that in viscoelastic *isotropic media*, all curves for qP and qS waves would be circular and red, as the attenuation angle γ is always less then $\pi/2$ there.

Figure 3 shows behaviour of the attenuation amplitude ratio δ as d varies. The colours have the same meaning as in Fig.1. The quantity δ is rather small and it practically does not vary between d=0. and 0.0001. δ corresponding to the qP wave has a form of the infinity symbol. In the directions perpendicular to the axis of symmetry δ is close to zero. The value of δ corresponding to SH wave is effectively zero in every direction. δ corresponding to the qSV wave has a four-lobe form. For d = 0.001, δ of the SH wave becomes nonzero and it is independent of the direction. For d=0.01 and 0.1, δ increases. δ of the qP wave is represented by the outer curve, the inner curves correspond to the qS waves. From d=1. to infinity the picture remains the same. The curves corresponding to all three waves intersect each other.

Conclusions

Although we concentrated only on a symmetry plane of an viscoelastic medium of hexagonal symmetry, we could observe some phenomena unknown from elastic anisotropic or viscoelastic isotropic media. We found that in certain situations, viscoelastic anisotropic media can contain directions, in which an inhomogeneous plane qP wave propagates with the same phase velocity as one of the inhomogeneous plane qS waves. We also found that even for very small values of the inhomogeneity strength d, the attenuation angle γ can attain values greater than $\pi/2$.

Since we concentrated on the plane of symmetry in the present study, we were able to identify individual waves even if they were interconnected. The identification will be difficult outside symmetry planes. An important planned step which will make the identification possible will be computation of the complex-valued polarization vectors of the studied waves.

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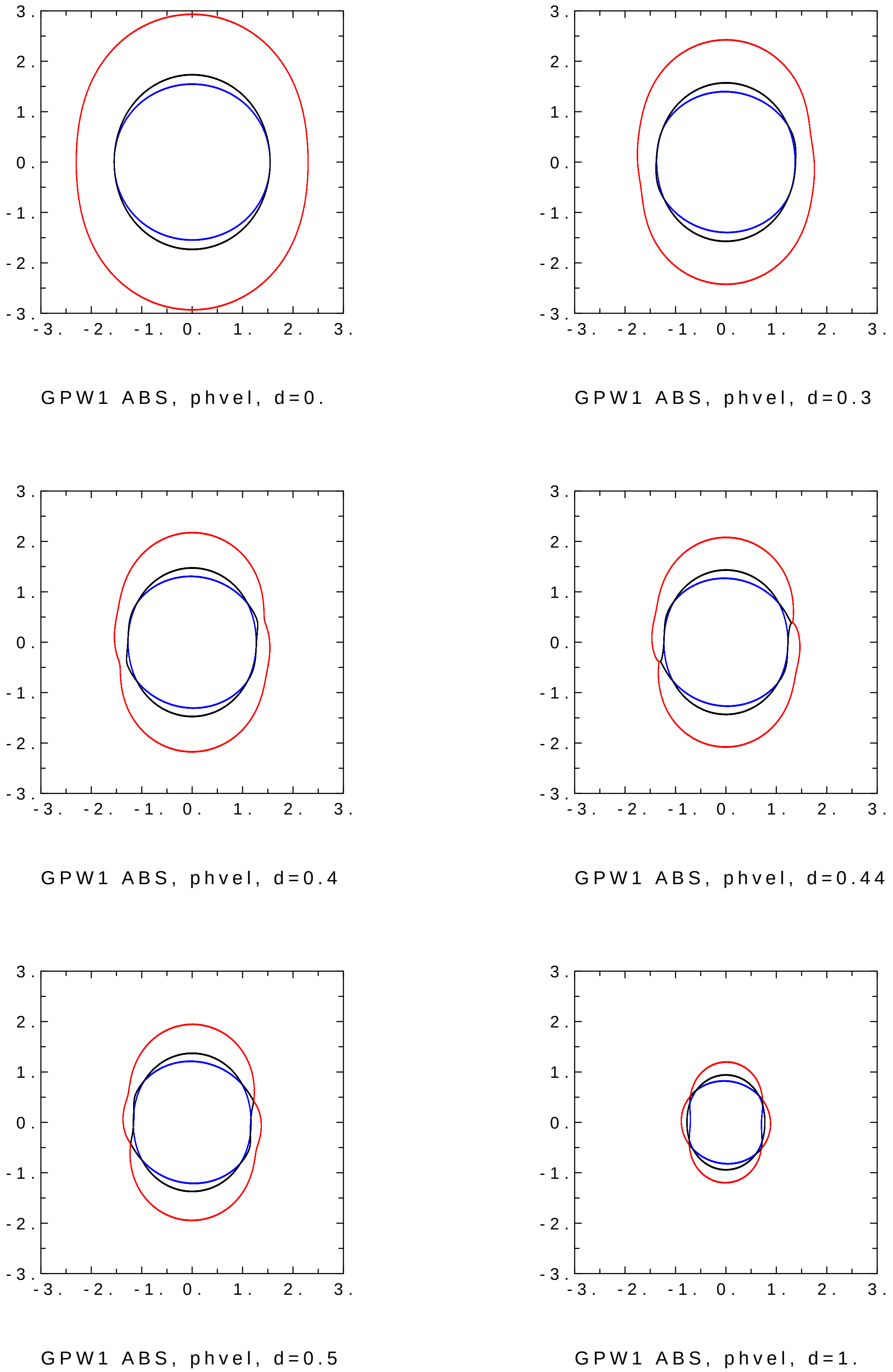

Fig.1: Variation of the phase velocities with inhomogeneity strength d.

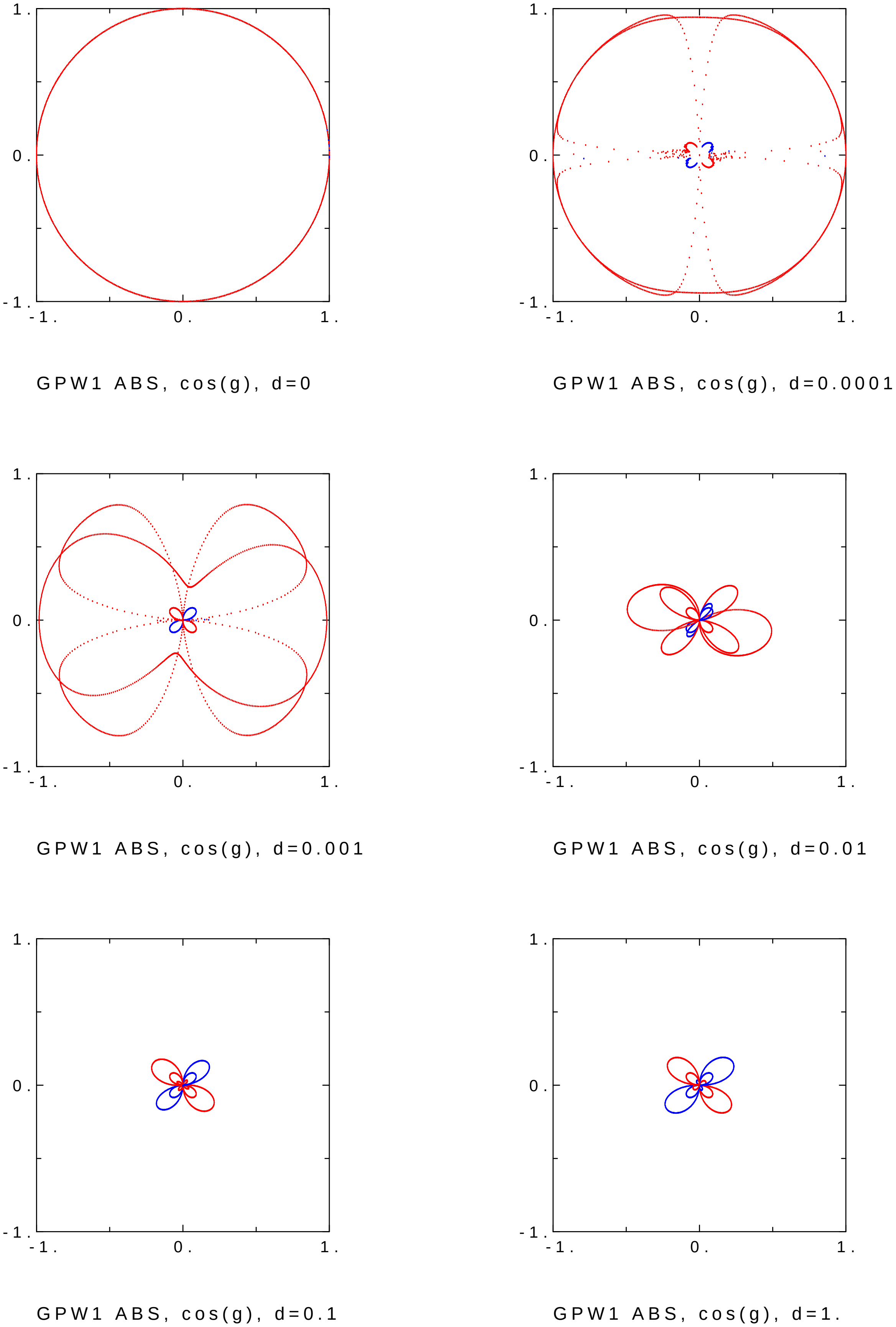

Fig.2: Variation of $\cos(\gamma)$ with inhomogeneity strength d.

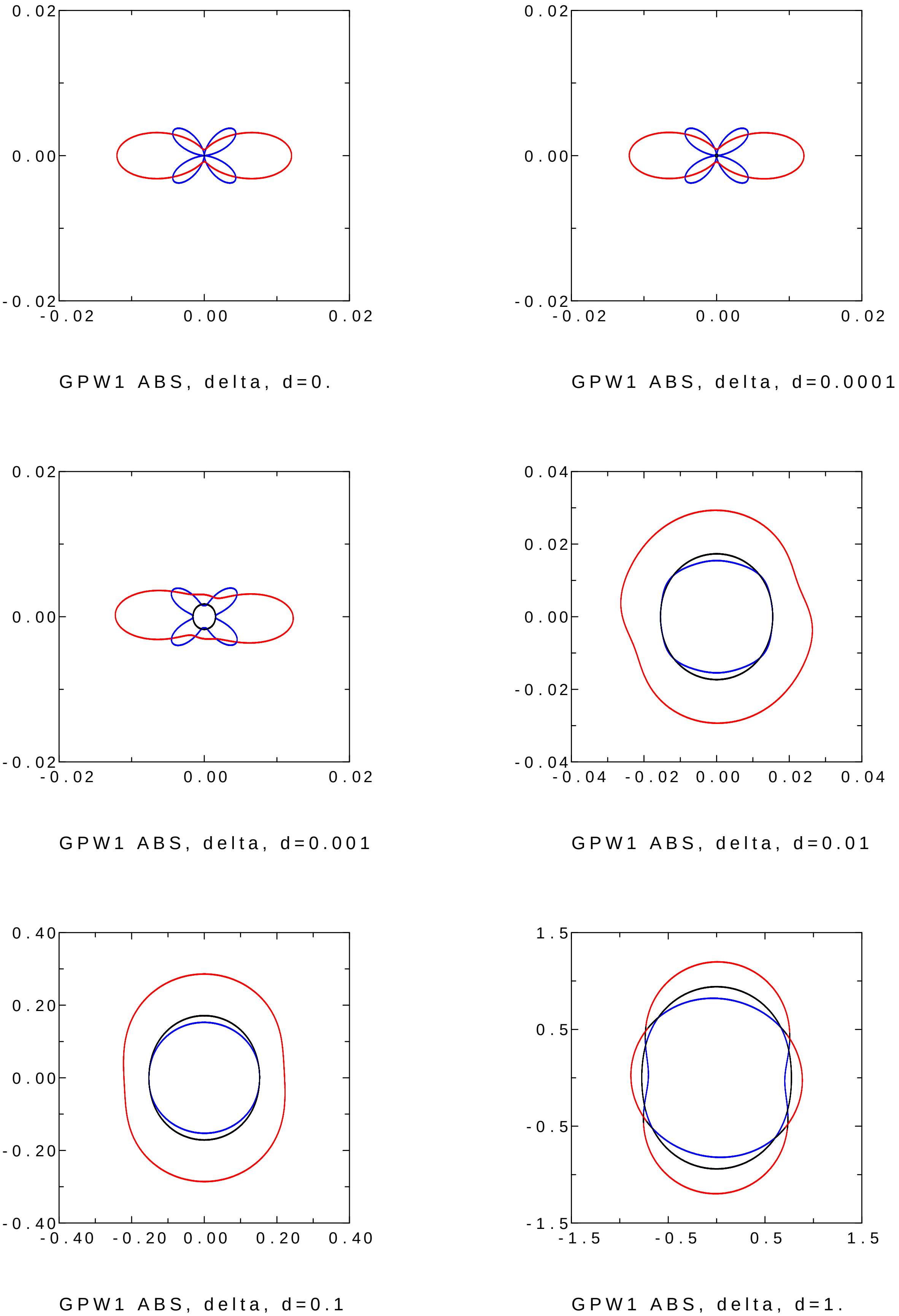

Fig.3: Variation of the attenuation amplitude ratio δ with inhomogeneity strength d.