Homogeneous harmonic plane waves in viscoelastic anisotropic media

Vlastislav Červený

Department of Geophysics, Faculty of Mathematics and Physics, Charles University, Ke Karlovu 3, 121 16 Praha 2, Czech Republic. E-mail vcerveny@seis.karlov.mff.cuni.cz

Summary

The determination of the slowness vector of homogeneous plane waves propagating in an arbitrary direction in a homogeneous viscoelastic anisotropic medium is discussed. Whereas the determination of the slowness vector of an inhomogeneous plane wave requires the solution of an eigenvalue problem for a 6×6 complex-valued matrix, it is sufficient to solve an eigenvalue problem for a 3×3 complex-valued problem for homogeneous plane waves. Expressions for phase velocities and for the ratios of the lengths of attenuation and propagation vectors are derived.

Keywords
Viscoelastic anisotropic media, homogeneous plane waves, phase velocity, slowness vector, polarisation vector.

1 Introduction

We shall consider a homogeneous viscoelastic anisotropic medium, specified by complex-valued viscoelastic moduli c_{ijkl} and by density ρ. The time-harmonic plane waves, propagating in this medium, are described by the expression

$$u_j(x_k, t) = U_j \exp[-\mathrm{i}\omega(t - p_n x_n)] \ , \tag{1}$$

where p_n and U_n satisfy the constraint relations

$$(c_{ijkl}/\rho) p_j p_l U_k = U_i \ , \quad i = 1, 2, 3 \ . \tag{2}$$

t is the time, ω a fixed, positive circular frequency, u_j, p_j and U_j are Cartesian components of displacement vector $\mathbf{u}$, of slowness vector $\mathbf{p}$, and of polarisation vector $\mathbf{U}$,

Seismic Waves in Complex 3-D Structures, Report 13, Department of Geophysics, Faculty of Mathematics and Physics, Charles University, Praha 2003

respectively. The constraint relations (2) follow from the insertion of (1) into the elastodynamic equation. Equations (2) represent a system of three linear algebraic equations for U_i, $i = 1, 2, 3$. The condition of the solvability of system (2) reads

$$\det[(c_{ijkl}/\rho)p_j p_l - \delta_{ik}] = 0 \ . \tag{3}$$

In viscoelastic media, slowness vector $\mathbf{p}$ is complex-valued, $\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A}$. Here $\mathbf{P}$ is the real-valued propagation vector, perpendicular to the plane of constant phases, and $\mathbf{A}$ is the real-valued attenuation vector, perpendicular to the plane of constant amplitudes. For $\mathbf{P}$ parallel to $\mathbf{A}$, the plane wave is called homogeneous, and for $\mathbf{P}$ and $\mathbf{A}$ nonparallel, it is called inhomogeneous. The procedures of determining the slowness vector $\mathbf{p}$ of a homogeneous plane wave propagating in a viscoelastic anisotropic medium, satisfying (2) and (3), are discussed in this paper.

An analogous problem for general inhomogeneous plane waves is discussed in Červený (2003), included in this volume. The procedures proposed therein remain valid also for homogeneous plane waves. As the homogeneous plane waves have been commonly considered in various seismological applications, we shall simplify the general procedures to simpler procedures, valid specifically for homogeneous plane waves. For a more detailed discussion of the problem, for relevant notations and references, see Červený (2003).

To determine the slowness vector $\mathbf{p}$ of an inhomogeneous plane wave propagating in a viscoelastic anisotropic medium, three different specifications of the slowness vector have been used: the directional specification, the componental specification and the mixed specification. The procedures based on the directional specification employ 3×3 complex-valued matrices, but the two other specifications use 6×6 complex-valued matrices. It is shown in Červený (2003) that the most general and straightforward procedure for inhomogeneous plane waves is based on the solution of an eigenvalue problem for a 6×6 complex-valued matrix. The directional specification, based on 3×3 complex-valued matrices, yields very cumbersome procedures in this case. For homogeneous plane waves, however, the situation is considerably simpler. In this case the procedures based on the solution of the eigenvalue problem for a 3×3 complex-valued matrix are quite sufficient. It will be shown in this paper that in this case also the 6×6 matrices of the projection and mixed specifications reduce to 3×3 matrices.

In Section 2, we apply the directional specification of the slowness vector and show that it yields the eigenvalue problem for the 3×3 complex-valued Christoffel matrix. In Section 3, we show that the componental and mixed specifications yield the same 3×3 Christoffel matrix. We shall not repeat the derivations of the equations valid for general inhomogeneous plane waves here, but merely adopt these equations from Červený (2003).

2 Directional specification of the slowness vector

We shall use the directional specification of the slowness vector,

$$\mathbf{p} = \frac{1}{\mathcal{C}}(\mathbf{N} + \mathrm{i}\delta\mathbf{M}) \ . \tag{4}$$

Here $\mathbf{N}$ and $\mathbf{M}$ are real-valued unit vectors in the direction of $\mathbf{P}$ and $\mathbf{A}$, $\mathcal{C}$ is the real-valued phase velocity, and δ the so-called attenuation amplitude ratio (also real-valued). For homogeneous plane waves, $\mathbf{N} \equiv \mathbf{M}$, and (4) yields

$$\mathbf{p} = \frac{1 + \mathrm{i}\delta}{\mathcal{C}} \mathbf{N} \; . \tag{5}$$

Equations (2) and (3) then yield

$$[\Gamma_{ik}(N_n) - \mathcal{C}^2(1+\mathrm{i}\delta)^{-2}\delta_{ik}]U_k \;=\; 0 \; , \qquad i = 1,2,3 \; , \tag{6}$$

$$\det[\Gamma_{ik}(N_n) - \mathcal{C}^2(1+\mathrm{i}\delta)^{-2}\delta_{ik}] \;=\; 0 \; . \tag{7}$$

Here

$$\Gamma_{ik}(N_n) = (c_{ijkl}/\rho)N_j N_l \tag{8}$$

is the complex-valued Christoffel matrix. Let us emphasize that c_{ijkl} are complex-valued, but N_i real-valued. For a known model (c_{ijkl}) and known direction of propagation $\mathbf{N}$, it is simple to calculate $\Gamma_{ik}(N_n)$.

Equations (6) and (7) represent a conventional 3×3 complex-valued eigenvalue problem. We denote the complex-valued eigenvalues of the Christoffel matrix $\Gamma_{ik}(N_n)$ by $G^{(m)}(N_n)$, $m = 1,2,3$. These eigenvalues can be determined from the characteristic equation

$$\det[\Gamma_{ik}(N_n) - G^{(m)}(N_n)\delta_{ik}] = 0 \; . \tag{9}$$

Using conventional approaches, we can determine eigenvalues $G^{(1)}(N_n)$, $G^{(2)}(N_n)$ and $G^{(3)}(N_n)$ from (9). These eigenvalues correspond to the three homogeneous plane waves which can propagate in the viscoelastic anisotropic medium (qS1, qS2, qP) in the direction of $\mathbf{N}$. It is simple to see from (7) and (9) that $G^{(m)}(N_n)$ are related to $\mathcal{C}^{(m)}$ and $\delta^{(m)}$ as follows:

$$G^{(m)}(N_n) = \mathcal{C}^{(m)2}(N_n)(1 + \mathrm{i}\delta^{(m)}(N_n))^{-2} \; . \tag{10}$$

Consequently, once $G^{(m)}(N_n)$ have been determined, we can simply determine the three relevant phase velocities $\mathcal{C}^{(m)}(N_n)$, attenuation amplitude ratios $\delta^{(m)}(N_n)$, and slowness vectors $\mathbf{p}^{(m)}(N_n)$. $\mathcal{C}^{(m)}(N_n)$ and $\delta^{(m)}(N_n)$ can be expressed in terms of $G^{(m)}(N_n)$ as follows:

$$\frac{1}{\mathcal{C}^{(m)}(N_n)} = \frac{\mathrm{Re}\sqrt{G^{(m)}(N_n)}}{|G^{(m)}(N_n)|} \; , \quad \delta^{(m)}(N_n) = -\frac{\mathrm{Im}\sqrt{G^{(m)}(N_n)}}{\mathrm{Re}\sqrt{G^{(m)}(N_n)}} \; . \tag{11}$$

Here the square root $\sqrt{G^{(m)}(N_n)}$ is determined in such a way that its real part is positive:

$$\sqrt{a + \mathrm{i}b} = \frac{1}{\sqrt{2}}\left(\sqrt{\sqrt{a^2+b^2}+a} + \mathrm{i}\,\mathrm{sgn}(b)\sqrt{\sqrt{a^2+b^2}-a}\right) \; . \tag{12}$$

The slowness vector $\mathbf{p}^{(m)}(N_n)$ of a homogeneous plane wave is then given by the relation,

$$p_i^{(m)}(N_n) = N_i\left[\mathrm{Re}\sqrt{G^{(m)}(N_n)} - \mathrm{i}\,\mathrm{Im}\sqrt{G^{(m)}(N_n)}\right]/|G^{(m)}(N_n)| \; . \tag{13}$$

Equations (11)–(13) represent the final results for homogeneous plane waves, propagating in an arbitrary direction in an arbitrary viscoelastic anisotropic medium.

3 Componental and mixed specifications of the slowness vector

In this section, we show that the componental and mixed specifications again yield the eigenvalue problem for the complex-valued Christoffel matrix (8) for homogeneous plane waves.

The componental specification of the slowness vector is given by the relation

$$\mathbf{p} = \sigma\mathbf{n} + \mathbf{p}^{\Sigma} \ , \tag{14}$$

where $\mathbf{n}$ and $\mathbf{p}^{\Sigma}$ are assumed to be known. Here $\mathbf{n}$ is an arbitrarily chosen real-valued unit vector, perpendicular to plane Σ, and $\mathbf{p}^{\Sigma}$ is an arbitrary complex-valued vector, situated in plane Σ. Actually, it represents a normal projection of slowness vector $\mathbf{p}$ into Σ. Complex-valued vector $\mathbf{p}^{\Sigma}$ must satisfy the orthogonality relation

$$\mathbf{p}^{\Sigma} \cdot \mathbf{n} = 0 \ . \tag{15}$$

For known $\mathbf{n}$ and $\mathbf{p}^{\Sigma}$, quantity σ is an eigenvalue of a 6×6 complex-valued matrix $\mathbf{A}$,

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} \ , \tag{16}$$

where the 3×3 partition matrices $\mathbf{A}_{11}$, $\mathbf{A}_{12}$, $\mathbf{A}_{21}$ and $\mathbf{A}_{22}$ are given by relations

$$\begin{array}{ll} \mathbf{A}_{11} = -\mathbf{C}^{(1)-1}\mathbf{C}^{(2)} \ , & \mathbf{A}_{12} = -\mathbf{C}^{(1)-1} \ , \\ \mathbf{A}_{21} = -\rho\mathbf{I} + \mathbf{C}^{(4)} - \mathbf{C}^{(3)}\mathbf{C}^{(1)-1}\mathbf{C}^{(2)} \ , & \mathbf{A}_{22} = -\mathbf{C}^{(3)}\mathbf{C}^{(1)-1} \ , \end{array} \tag{17}$$

and

$$\begin{array}{ll} C^{(1)}_{ik} = c_{ijkl}n_j n_l \ , & C^{(2)}_{ik} = c_{ijkl}n_j p^{\Sigma}_l \ , \\ C^{(3)}_{ik} = c_{ijkl}p^{\Sigma}_j n_l \ , & C^{(4)}_{ik} = c_{ijkl}p^{\Sigma}_j p^{\Sigma}_l \ . \end{array} \tag{18}$$

In (17), $\mathbf{I}$ denotes the 3×3 identity matrix. For more details see Červený (2003, Section 5.4.7).

The mixed specification of the slowness vector is a special case of (14), in which $\mathbf{p}^{\Sigma}$ is purely imaginary,

$$\mathbf{p}^{\Sigma} = \mathrm{i}\mathbf{d} \ , \quad \text{with} \quad \mathbf{d} \cdot \mathbf{n} = 0 \ . \tag{19}$$

The componental specification (14) corresponds to general inhomogeneous plane waves. It reduces to the case of a homogeneous plane wave with the propagation vector along $\mathbf{n}$ if we put

$$\mathbf{p}^{\Sigma} = \mathbf{0} \ , \tag{20}$$

where $\mathbf{0}$ denotes the null vector. In this case, Σ represents both the plane of a constant phase and the plane of a constant amplitude. Consequently,

$$\mathbf{n} = \pm\mathbf{N} \ ,$$

where $\mathbf{N}$ is the unit vector in the direction of propagation. Waves propagating to both sides of Σ are considered, so that $\mathbf{N}$ coincides with $\mathbf{n}$, or is opposite to it. The componental specification (14) of slowness vector $\mathbf{p}$ then reads

$$\mathbf{p} = \sigma \mathbf{n} \; . \tag{21}$$

The same equation is obtained for the mixed specification (19), if we put $\mathbf{d} = \mathbf{0}$. For this reason, we shall only consider the componental specification (14), with (20).

Using (20), (18) yields,

$$C^{(1)}_{ik} = c_{ijkl} n_j n_l = \rho \Gamma_{ik} \; , \quad C^{(2)}_{ik} = C^{(3)}_{ik} = C^{(4)}_{ik} = 0 \; . \tag{22}$$

Here Γ_{ik} are the elements of the 3×3 complex-valued Christoffel matrix (8). Equations (17) then yield

$$\mathbf{A}_{11} = \mathbf{0} \; , \quad \mathbf{A}_{12} = -\rho^{-1} \mathbf{\Gamma}^{-1} \; , \quad \mathbf{A}_{21} = -\rho \mathbf{I} \; , \quad \mathbf{A}_{22} = \mathbf{0} \; . \tag{23}$$

Consequently, the 6×6 matrix $\mathbf{A}$ is given by a simple expression

$$\mathbf{A} = \begin{pmatrix} \mathbf{0} & -\rho^{-1}\mathbf{\Gamma}^{-1} \\ -\rho \mathbf{I} & \mathbf{0} \end{pmatrix} \tag{24}$$

The eigenvalues σ of $\mathbf{A}$ are solutions of the characteristic equation

$$\det \begin{pmatrix} -\sigma \mathbf{I} & -\rho^{-1}\mathbf{\Gamma}^{-1} \\ -\rho \mathbf{I} & -\sigma \mathbf{I} \end{pmatrix} = 0 \; . \tag{25}$$

Equation (25) represents an algebraic equation of the sixth order for eigenvalues σ. There are six such eigenvalues; three with positive real parts, and three with negative real parts. The eigenequation corresponding to $\mathbf{A}$ can then be expressed in the following form

$$\mathbf{A}\mathbf{W} = \mathbf{W}\boldsymbol{\sigma} \; . \tag{26}$$

Here $\boldsymbol{\sigma}$ is the diagonal 6×6 matrix with the individual eigenvalues σ on diagonal, and $\mathbf{W}$ is the 6×6 matrix composed of 6×1 eigenvectors. The eigenequations (26) can be partitioned in terms of 3×3 matrices

$$\begin{pmatrix} \mathbf{0} & -\rho^{-1}\mathbf{\Gamma}^{-1} \\ -\rho \mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} \\ \mathbf{W}_{21} & \mathbf{W}_{22} \end{pmatrix} = \begin{pmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} \\ \mathbf{W}_{21} & \mathbf{W}_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma}^{+} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}^{-} \end{pmatrix} \; , \tag{27}$$

where $\boldsymbol{\sigma}^{+}$ and $\boldsymbol{\sigma}^{-}$ are 3×3 diagonal matrices, with positive and negative real parts of σ in diagonal terms, respectively. Explicitly, (27) yields four equations

$$\begin{aligned} -\rho^{-1}\mathbf{\Gamma}^{-1}\mathbf{W}_{21} &= \mathbf{W}_{11}\boldsymbol{\sigma}^{+} \; , \quad -\rho^{-1}\mathbf{\Gamma}^{-1}\mathbf{W}_{22} = \mathbf{W}_{12}\boldsymbol{\sigma}^{-} \; , \\ -\rho \mathbf{W}_{11} &= \mathbf{W}_{21}\boldsymbol{\sigma}^{+} \; , \quad -\rho \mathbf{W}_{12} = \mathbf{W}_{22}\boldsymbol{\sigma}^{-} \; . \end{aligned} \tag{28}$$

Eliminating 3×3 matrices $\mathbf{W}_{21}$ and $\mathbf{W}_{22}$ from (28), we obtain two 3×3 matrix equations

$$\mathbf{\Gamma}^{-1}\mathbf{W}_{11} = \mathbf{W}_{11}(\boldsymbol{\sigma}^{+})^2 \; , \quad \mathbf{\Gamma}^{-1}\mathbf{W}_{12} = \mathbf{W}_{12}(\boldsymbol{\sigma}^{-})^2 \; . \tag{29}$$

The characteristic equation for any σ then reads

$$\det(\mathbf{\Gamma}^{-1} - \sigma^2 \mathbf{I}) = 0 \ . \tag{30}$$

This is the final 3×3 version of the 6×6 characteristic equation (25), corresponding to homogeneous plane waves propagating in a viscoelastic anisotropic medium.

Note that (30) can also be obtained directly from (25), if we expand the 6×6 determinant in terms of 3×3 determinants using the Laplace method. The relevant 3×3 determinants in the expansion are constructed from the first three columns and from the last three columns.

For $\sigma \neq 0$ and $\det \mathbf{\Gamma} \neq 0$, characteristic equation (30) is equivalent to the equation

$$\det(\mathbf{\Gamma} - \sigma^{-2} \mathbf{I}) = 0 \ . \tag{31}$$

Thus, the final result is, see (9),

$$\sigma^{-2} = G^{(m)} \ , \tag{32}$$

where $G^{(m)}$ are the eigenvalues of the 3×3 complex-valued Christoffel matrix (8). Consequently,

$$\sigma = \pm \frac{1}{\sqrt{G^{(m)}}} = \pm \frac{\sqrt{G^{(m)*}}}{|G^{(m)}|} = \pm \left[\frac{\mathrm{Re}\sqrt{G^{(m)}}}{|G^{(m)}|} - \mathrm{i} \ \frac{\mathrm{Im}\sqrt{G^{(m)}}}{|G^{(m)}|} \right] \ . \tag{33}$$

Using (21), we finally obtain the slowness vector of the homogeneous plane wave in the following form:

$$p_i^{(m)} = \pm n_i \left[\frac{\mathrm{Re}\sqrt{G^{(m)}}}{|G^{(m)}|} - \mathrm{i} \ \frac{\mathrm{Im}\sqrt{G^{(m)}}}{|G^{(m)}|} \right] \ . \tag{34}$$

Equation (34) fully agrees with (13), obtained by the directional specification of the slowness vector. Consequently, phase velocity $\mathcal{C}^{(m)}$, and the attenuation amplitude factor $\delta^{(m)}$ can be determined from (11).

Note. As we have shown, the eigenvalues σ of the 6×6 matrix $\mathbf{A}$, given by (16)–(18), and the eigenvalues G of the 3×3 matrix $\mathbf{\Gamma}$, given by (8), satisfy the mutual relation (32) for homogeneous plane waves propagating in viscoelastic anisotropic media. We have proved the relation (32) by a direct computation of eigenvalues of matrix $\mathbf{A}$. Indeed, Equation (32) can also be proved in a considerably simpler way, if we take into account some properties of eigenvalues σ. It was shown in Červený (2001, Section 5.4.7) that the eigenvalues σ of the 6×6 matrix $\mathbf{A}$ satisfy the equation:

$$\det[a_{ijkl}(p_j^{\Sigma} + \sigma n_j)(p_l^{\Sigma} + \sigma n_l) - \delta_{ik}] = 0 \ . \tag{35}$$

For homogeneous plane waves, $p_i^{\Sigma} = 0$, see (20), and Equation (35) reduces, if $\sigma \neq 0$, to

$$\det[a_{ijkl} n_j n_l - \sigma^{-2} \delta_{ik}] = 0 \ . \tag{36}$$

This directly implies the basic relation (32).

4 Concluding remarks

The computation of the slowness vectors of plane waves propagating in a general viscoelastic anisotropic medium is considerably simpler in the case of homogeneous plane waves than in the case of inhomogeneous plane waves. The conventional eigenvalue equation for the 3×3 complex-valued Christoffel matrix can be used in this case. Of course, this does not mean that the eigenvalue equations for the 6×6 complex-valued matrix $\mathbf{A}$ cannot be used. These equations, however, are numerically less efficient and do not offer any advantages.

In a perfectly elastic, isotropic and anisotropic medium, homogeneous plane waves (with a nonvanishing attenuation vector parallel to the propagation vector) cannot propagate. The explanation is simple. The Christoffel matrix $\Gamma_{ik}(N_n)$, given by (8), is real valued for real-valued c_{ijkl}. Further, it is symmetric and positive definite. Consequently, its eigenvalues $G^{(m)}(N_n)$, $m = 1, 2, 3$, are real-valued and positive. The second equation of (11) then implies that the attenuation amplitude factor $\delta^{(m)}$ is zero. Consequently, also the attenuation vector vanishes. This conclusion, of course, applies only to homogeneous plane waves. Inhomogeneous plane waves can propagate even in perfectly elastic media.

The reflection/transmission problem of plane waves at a plane interface between two viscoelastic anisotropic media cannot be solved in terms of homogeneous plane waves. The reason is that the incident homogeneous plane wave generates inhomogeneous reflected/transmitted waves at the interface. This is well-known for isotropic viscoelastic media, and also remains valid for anisotropic viscoelastic media. In fact, condition (20) cannot be used at the interface for an oblique angle of incidence. The exception is only the normal incidence.

Reference

Červený, V. (2001). *Seismic ray theory.* Cambridge: Cambridge University Press.

Červený, V. (2003). Inhomogeneous plane waves in viscoelastic anisotropic media. This volume.