Inhomogeneous harmonic plane waves in viscoelastic media. Analytical solutions

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Summary

The computation of the slowness vector of an inhomogeneous plane wave propagating in a homogeneous viscoelastic anisotropic medium is based on the solutions of an eigenvalue problem for 3×3 or 6×6 complex-valued matrices. Individual approaches leading to these eigenvalue problems differ in the specification of the slowness vector (directional specification, componental specification, mixed specification). In two important cases, complete analytical solutions for the slowness vector are obtained. This applies to P and S inhomogeneous plane waves propagating in an isotropic viscoelastic medium, and to qSH inhomogeneous plane waves propagating in the plane of symmetry of a monoclinic (orthorhombic, hexagonal) anisotropic viscoelastic medium. The analytical solutions for these two cases are derived and discussed in this paper. These solutions offer a simple physical insight into the problem of inhomogeneous plane waves, and a valuable comparison of advantages and drawbacks of the individual approaches. For isotropic viscoelastic media, all the three specifications of the slowness vector yield analogous results, and can be used fully alternatively. For qSH inhomogeneous plane waves propagating in the plane of symmetry of a monoclinic anisotropic viscoelastic medium, however, the situation is different. The approaches based on the componental and mixed specifications of the slowness vector offer simpler and more powerful analytical solutions than the approach based on the directional specification. They yield explicit equations for the slowness vector, the phase velocity, the attenuation amplitude factor and the attenuation angle, and avoid fully the problems of forbidden directions.

Keywords
Viscoelastic media, inhomogeneous plane waves, phase velocity, attenuation angle, slowness vector.

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1 Introduction

The slowness vector of an inhomogeneous plane wave propagating in viscoelastic, isotropic or anisotropic media may be determined in several ways. Individual approaches to the solution of this problem differ in the specification of the slowness vector of the inhomogeneous plane wave under consideration. For a detailed discussion of the problem, derivation of all relevant equations, notations used and appropriate references see Červený (2003).

For general viscoelastic anisotropic media, numerical methods must be used to solve the proposed equations and to compute the slowness vector. Only in exceptional cases, these equations can be completely solved analytically. Two such analytical solutions are discussed in this paper. The first analytical solution corresponds to P and S waves propagating in a viscoelastic isotropic medium, and the second to qSH waves propagating in the plane of symmetry of a monoclinic anisotropic viscoelastic medium. Of course, the latter solution may also be used in the plane of symmetry of an orthorhombic (or hexagonal) anisotropy system.

We shall consider a time-harmonic inhomogeneous plane wave

$$u_j(x_k, t) = U_j \exp[-\mathrm{i}\omega(t - p_n x_n)] \ , \tag{1}$$

where u_j, p_j and U_j are Cartesian components of the complex-valued displacement vector $\mathbf{u}$, slowness vector $\mathbf{p}$ and polarisation vector $\mathbf{U}$, respectively. Moreover, t is time and ω is a fixed, positive circular frequency. Equation (1) represents a plane wave if, and only if, U_j and p_j are chosen in such a way that (1) satisfies the elastodynamic equation. This requirement yields the constraint relations

$$(c_{ijkl}/\rho) p_j p_l U_k = U_i \ , \qquad i = 1, 2, 3 \ . \tag{2}$$

Here c_{ijkl} are complex-valued viscoelastic moduli, and ρ the real-valued density. The condition of solvability of the system of linear equations (2) for U_1, U_2, U_3 reads

$$\det[(c_{ijkl}/\rho) p_j p_l - \delta_{ik}] = 0 \ . \tag{3}$$

We shall now introduce certain notations for slowness vector $\mathbf{p}$. It is complex-valued,

$$\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A} \ , \tag{4}$$

where $\mathbf{P}$ is the propagation vector (perpendicular to the plane of constant phase) and $\mathbf{A}$ the attenuation vector (perpendicular to the plane of constant amplitude). We also introduce the real-valued unit vectors $\mathbf{N}$ and $\mathbf{M}$ in the directions of $\mathbf{P}$ and $\mathbf{A}$, phase velocity $\mathcal{C}$, attenuation amplitude factor δ and attenuation angle γ as

$$\mathbf{N} = \mathbf{P}/|\mathbf{P}| \ , \qquad \mathbf{M} = \mathbf{A}/|\mathbf{A}| \ , \tag{5}$$

$$\mathcal{C} = 1/|\mathbf{P}| \ , \qquad \delta = |\mathbf{A}|/|\mathbf{P}| \ , \qquad \cos\gamma = \mathbf{N}\cdot\mathbf{M} \ . \tag{6}$$

The plane waves are called homogeneous for $\gamma = 0$, and inhomogeneous for $\gamma \neq 0$. We further introduce the propagation-attenuation plane $\Sigma^{||}$ as a plane specified by unit vectors $\mathbf{N}$ and $\mathbf{M}$.

In the directional specification of slowness vector $\mathbf{p}$, we express the slowness vector in terms of à priori known real-valued unit vectors $\mathbf{N}$ and $\mathbf{M}$,

$$\mathbf{p} = \frac{1}{\mathcal{C}}(\mathbf{N} + \mathrm{i}\delta\mathbf{M}) \ . \tag{7}$$

The unknown quantities $\mathcal{C}$ and δ must be determined by inserting (7) into (3). However, attenuation angle γ is also à priori known, as $\cos\gamma = \mathbf{N}\cdot\mathbf{M}$, see (6). If, for given $\mathbf{N}$ and $\mathbf{M}$, Equation (3) does not have any real-valued solution $\mathcal{C} \geq 0$ and $\delta \geq 0$, we speak of forbidden directions $\mathbf{N}$, $\mathbf{M}$.

In the componental specification, slowness vector $\mathbf{p}$ is expressed in terms of an à priori known real-valued unit vector $\mathbf{n}$ and a complex-valued vector $\mathbf{p}^{\Sigma}$ as follows:

$$\mathbf{p} = \sigma\mathbf{n} + \mathbf{p}^{\Sigma} \ , \qquad \text{with} \quad \mathbf{p}^{\Sigma}\cdot\mathbf{n} = 0 \ . \tag{8}$$

In fact, $\mathbf{p}^{\Sigma}$ represents a known vectorial component of slowness vector $\mathbf{p}$ in plane Σ perpendicular to $\mathbf{n}$. The unknown complex-valued quantity σ must be determined by inserting (8) into (3).

In the mixed specification of the slowness vector, the slowness vector is expressed in terms of à priori known real-valued vectors $\mathbf{n}$ and $\mathbf{d}$ as follows

$$\mathbf{p} = \sigma\mathbf{n} + \mathrm{i}\mathbf{d} \ , \qquad \text{with} \quad \mathbf{d}\cdot\mathbf{n} = 0 \ . \tag{9}$$

Here $\mathbf{n}$ has the same meaning as in the componental specification, and $\mathbf{d}$ is an arbitrary real-valued vector perpendicular to $\mathbf{n}$. The unknown complex-valued quantity σ must be determined by inserting (9) into (3). As we can see from (8) and (9), the mixed specification is a special case of the componental specification, with $\mathbf{p}^{\Sigma}$ purely imaginary,

$$\mathbf{p}^{\Sigma} = \mathrm{i}\mathbf{d} \ . \tag{10}$$

Plane Σ represents the plane of constant phase (wavefront), as $\mathrm{Re}(\mathbf{p}^{\Sigma}) = \mathbf{0}$ along it. Consequently, $\mathbf{n} = \pm\mathbf{N}$. It is easy to see that $\mathbf{d}$ represents the vectorial component of attenuation vector $\mathbf{A}$ in the wavefront, $\mathbf{d} = \mathbf{n}\times(\mathbf{A}\times\mathbf{n}) = \mathbf{N}\times(\mathbf{A}\times\mathbf{N})$. For this reason, we call $\mathbf{d}$ the wavefront attenuation vector. We also introduce the strength of inhomogeneity d, and unit vectors $\mathbf{e}^{||}$ and $\mathbf{e}^{\perp}$, as

$$\mathbf{d} = d\,\mathbf{e}^{||} \ , \qquad \mathbf{e}^{\perp} = \mathbf{n}\times\mathbf{e}^{||} \ . \tag{11}$$

The real-valued unit vectors $\mathbf{e}^{||}$, $\mathbf{e}^{\perp}$, $\mathbf{n}$ form a triplet of three mutually orthogonal, right-handed unit vectors, with $\mathbf{n}$ and $\mathbf{e}^{||}$ situated in the propagation-attenuation plane $\Sigma^{||}$.

In this paper, we shall derive and discuss the analytical solutions for slowness vector $\mathbf{p}$, using all the above three specifications, (7), (8) and (9), of the slowness vector. It would be possible to derive these analytical solutions considering special types of viscoelastic media from the beginning. Here, however, we shall obtain them from equations derived in Červený (2003), valid for general anisotropic viscoelastic media. The reason is that we wish to demonstrate clearly the advantages and drawbacks of the individual specifications. Further, we also wish to compare the results obtained in this way with those known from the seismological literature.

In Section 2, we shall consider inhomogeneous plane waves propagating in a viscoelastic isotropic medium, and in Section 3 the inhomogeneous plane waves propagating in the symmetry plane of a monoclinic anisotropic viscoelastic medium. In both sections, the results obtained under different specifications of the slowness vector are mutually compared and discussed.

2 Viscoelastic isotropic media

In this section, we shall consider inhomogeneous plane waves propagating in a viscoelastic isotropic medium. The complex-valued viscoelastic moduli c_{ijkl} can then be expressed in terms of complex-valued Lamé's moduli λ and μ using the relation

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \ . \tag{12}$$

The inhomogeneous plane waves propagating in isotropic viscoelastic media have also been broadly discussed in the seismological literature. See Buchen (1971), Borcherdt (1973), Aki and Richards (1980), Caviglia and Morro (1992), Brokešová and Červený (1998), where many other references can also be found. Mostly a conventional approach based on the elastic Lamé's potentials has been used.

2.1 Directional specification of the slowness vector

The application of the directional specification of the slowness vector in the determination of the slowness vector of an inhomogeneous plane wave propagating in a general viscoelastic anisotropic medium is described in detail in Červený (2003, Section 3). We shall apply the approach called semi-analytic in that reference. We first introduce the 3×3 complex-valued matrix $\Gamma_{ik}(p_n) = (c_{ijkl}/\rho)p_jp_l$, and determine its eigenvalues $G^{(m)}(p_n)$, $m = 1, 2, 3$, analytically. It is not difficult to see that the constraint relation (3) is satisfied for a given m if

$$G^{(m)}(p_n) = 1 \ . \tag{13}$$

Equation (13) represents a new form of the constraint relation for a selected inhomogeneous plane wave, specified by a fixed $m\,(m = 1,2,\text{ or }3)$. As $G^{(m)}(p_n)$ is a homogeneous function of the second degree in p_n, (13) yields two equations:

$$\mathrm{Re}[G^{(m)}(N_n + \mathrm{i}\delta M_n)] = \mathcal{C}^2 \ , \quad \mathrm{Im}[G^{(m)}(N_n + \mathrm{i}\delta M_n)] = 0 \ . \tag{14}$$

For given unit vectors $\mathbf{N}$ and $\mathbf{M}$, the second equation of (14) can be used to determine δ. Once δ is determined, we can insert it into the first equation of (14), and obtain $\mathcal{C}^2$. If $\mathcal{C}^2 < 0$ and/or $\delta < 0$, the solution is not physically acceptable. We then speak of forbidden directions $\mathbf{N}, \mathbf{M}$.

In isotropic viscoelastic media, the 3×3 matrix $\mathbf{\Gamma}$ reads

$$\Gamma_{ik}(p_n) = \frac{\lambda+\mu}{\rho}\,p_ip_k + \frac{\mu}{\rho}\,p_lp_l \ , \tag{15}$$

and its eigenvalues $G^{(m)}(p_n)$, $m = 1, 2, 3$, are given as

$$G^{(1)}(p_n) = G^{(2)}(p_n) = \beta^2 p_l p_l \ , \qquad G^{(3)}(p_n) = \alpha^2 p_l p_l \ , \tag{16}$$

see Červený (2001a, Section 2.2.6). Eigenvalues $G^{(1)}(p_n)$ and $G^{(2)}(p_n)$ correspond to S plane waves, $G^{(3)}(p_n)$ to P plane waves. The complex-valued velocities α and β are given by relations

$$\alpha^2 = V_P^2(1 - \mathrm{i}/Q_P) \ , \qquad \beta^2 = V_S^2(1 - \mathrm{i}/Q_S) \ , \tag{17}$$

where V_P, V_S, Q_P and Q_S are real-valued positive quantities:

$$\begin{aligned} V_P &= \sqrt{(\lambda^R + 2\mu^R)/\rho} \ , & V_S &= \sqrt{\mu^R/\rho} \ , \\ Q_P &= -(\lambda^R + 2\mu^R)/(\lambda^I + 2\mu^I) \ , & Q_S &= -\mu^R/\mu^I \ . \end{aligned} \tag{18}$$

Here λ^R and μ^R denote the real parts of Lamé's elastic moduli, λ^I and μ^I their imaginary parts, V_P and V_S the real-valued velocities of P and S waves, and Q_P and Q_S the relevant quality factors. The eigenvalues $G^{(1)}(p_n)$ and $G^{(2)}(p_n)$ of S waves coincide, so that the case of S waves is degenerate.

For simplicity, we shall consider only P waves here. The case of S waves will be summarized at the end of this section.

For P waves, Equations (13) and (16) yield the constraint relation

$$G^{(3)}(p_n) = \alpha^2 p_l p_l = 1 \ . \tag{19}$$

Using (17) and (7), we obtain the analytic expression for $G^{(3)}(N_n + \mathrm{i}\delta M_n)$:

$$G^{(3)}(N_n + \mathrm{i}\delta M_n) = V_P^2(1 - \mathrm{i}/Q_P)(N_l + \mathrm{i}\delta M_l)(N_l + \mathrm{i}\delta M_l) \ . \tag{20}$$

This yields two equations, see (14),

$$1 - \delta^2 + 2\delta\cos\gamma/Q_P = \mathcal{C}^2/V_P^2 \ , \tag{21}$$
$$\delta^2 + 2\delta Q_P\cos\gamma - 1 = 0 \ . \tag{22}$$

Using (22), Equation (21) for $\mathcal{C}^2$ can also be simplified to yield

$$2\delta\cos\gamma(Q_P + 1/Q_P) = \mathcal{C}^2/V_P^2 \ . \tag{23}$$

Equation (22) can be solved for the attenuation amplitude factor δ in terms of à priori known Q_P and $\cos\gamma = \mathbf{N}\cdot\mathbf{M}$. It is convenient to consider independently two cases: Q_P finite (viscoelastic medium), and Q_P infinite (perfectly elastic medium).

1. **The case of infinite $\mathbf{Q}_P$** (perfectly elastic medium). Equation (22) then yields

$$\delta\cos\gamma = 0 \ . \tag{24}$$

We shall first consider the specific case of $\gamma = \frac{1}{2}\pi$. Then $\cos\gamma = 0$, and (24) is satisfied for arbitrary δ. Equation (21) then yields the solution for phase velocity $\mathcal{C}$:

$$\mathcal{C} = V_P\sqrt{1 - \delta^2} \ . \tag{25}$$

As the phase velocity must be real-valued and nonnegative, we have taken the sign "+" in (25), and δ within the limits $0 \leq \delta \leq 1$. Thus, δ is a free parameter of the inhomogeneous plane wave, with $0 \leq \delta \leq 1$. Phase velocity $\mathcal{C}$ vanishes for $\delta = 1$, increases with decreasing δ, and equals V_P for $\delta = 0$. For $\delta = 0$, however, the imaginary part of the slowness vector vanishes, and the plane wave is not inhomogeneous. Consequently, δ must be finally chosen within the limits $0 < \delta \leq 1$ for inhomogeneous plane waves.

Now we shall consider all other attenuation angles γ, different from $\frac{1}{2}\pi$. Equation (24) is then satisfied only if $\delta = 0$. This proves that inhomogeneous plane waves with attenuation angles $\gamma \neq \frac{1}{2}\pi$ do not propagate in perfectly elastic media.

Thus, we can conclude that inhomogeneous plane waves can propagate even in perfectly elastic media, but attenuation angle γ must in this case equal $\frac{1}{2}\pi$.

2. **The case of finite Q_P** (viscoelastic medium): For given finite Q_P and given $\cos\gamma$, Equation (22) can be solved for δ,

$$\delta_{1,2} = -Q_P \cos\gamma \pm \sqrt{Q_P^2 \cos^2\gamma + 1} \ . \tag{26}$$

As δ should always be real-valued and nonnegative, we have to retain only the "+" sign in (26),

$$\delta = -Q_P \cos\gamma + \sqrt{Q_P^2 \cos^2\gamma + 1} \ . \tag{27}$$

The basic role in the discussion of the slowness vector of inhomogeneous plane waves propagating in viscoelastic isotropic media is played by the sign of $\cos\gamma$. First consider $\cos\gamma < 0$. Then δ would increase with increasing Q_P, and Equation (23) would yield negative $\mathcal{C}^2$. This is not physically acceptable. Consequently, the case of $\cos\gamma = \mathbf{N}\cdot\mathbf{M} < 0$ represents a forbidden direction. Now consider $\cos\gamma > 0$. Then δ decreases with increasing Q_P, see (27), and (23) yields positive $\mathcal{C}^2$. Finally, for $\cos\gamma = 0$, (27) yields $\delta = 1$, and (23) yields zero phase velocity, $\mathcal{C} = 0$.

Thus, we only have to consider such directions $\mathbf{N}$, $\mathbf{M}$, for which

$$\cos\gamma = \mathbf{N}\cdot\mathbf{M} > 0 \ . \tag{28}$$

The final result for P waves is then as follows:

$$\delta^{(3)} = -Q_P \cos\gamma + \sqrt{Q_P^2 \cos^2\gamma + 1} = \sqrt{\frac{\sqrt{1 + 1/Q_P^2 \cos^2\gamma} - 1}{\sqrt{1 + 1/Q_P^2 \cos^2\gamma} + 1}} \ , \tag{29}$$

$$\mathcal{C}^{(3)} = \frac{\sqrt{2}\sqrt{1 + 1/Q_P^2}\, V_P}{\sqrt{1 + \sqrt{1 + 1/Q_P^2 \cos^2\gamma}}} \ . \tag{30}$$

The relevant expression for the slowness vector reads

$$\mathbf{p}^{(3)} = \frac{1}{\sqrt{2}V_P\sqrt{1 + 1/Q_P^2}}\left[\mathbf{N}\sqrt{\sqrt{1 + 1/Q_P^2 \cos^2\gamma} + 1} + \mathrm{i}\mathbf{M}\sqrt{\sqrt{1 + 1/Q_P^2 \cos^2\gamma} - 1}\right] . \tag{31}$$

The solutions (29)–(31) exist for any direction $\mathbf{N}$ and $\mathbf{M}$, satisfying (28). They remain valid also for homogeneous plane waves, for which attenuation angle γ vanishes. For the limiting case of $\gamma = \frac{1}{2}\pi$, zero phase velocity is obtained, $\mathcal{C} = 0$, and the modulus of the slowness vector is infinite, $|\mathbf{p}^{(3)}| = \infty$.

Equations (29)–(31) remain valid even for S waves; only V_P and Q_P should be replaced by V_S and Q_S. Note that

$$\mathcal{C}^{(1)} = \mathcal{C}^{(2)} , \quad \delta^{(1)} = \delta^{(2)} , \quad \mathbf{p}^{(1)} = \mathbf{p}^{(2)} \tag{32}$$

for S waves. The differences between the expressions for S and P waves consist only in different polarisation vectors.

2.2 Componental specification of the slowness vector

The application of the componental specification (8) of the slowness vector in determining the slowness vector of an inhomogeneous plane wave propagating in a viscoelastic anisotropic medium is described in detail in Červený (2003). Here we shall not derive all these equations again; we shall only refer to the main result. The complex-valued quantity σ in (8) is an eigenvalue of the 6×6 complex-valued matrix $\mathbf{A}$,

$$\det(\mathbf{A} - \sigma\mathbf{I}) = 0 . \tag{33}$$

$\mathbf{A}$ is given by the relation

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} , \tag{34}$$

where the 3×3 complex-valued partition matrices $\mathbf{A}_{IJ}$ read

$$\begin{array}{ll} \mathbf{A}_{11} = -\mathbf{C}^{(1)-1}\mathbf{C}^{(2)} = \mathbf{A}_{22}^T , & \mathbf{A}_{12} = -\mathbf{C}^{(1)-1} , \\ \mathbf{A}_{21} = -\rho\mathbf{I} + \mathbf{C}^{(4)} - \mathbf{C}^{(3)}\mathbf{C}^{(1)-1}\mathbf{C}^{(2)} , & \mathbf{A}_{22} = -\mathbf{C}^{(3)}\mathbf{C}^{(1)-1} , \end{array} \tag{35}$$

with

$$\begin{array}{ll} C_{ik}^{(1)} = c_{ijkl}n_j n_l , & C_{ik}^{(2)} = c_{ijkl}n_j p_l^{\Sigma} , \\ C_{ik}^{(3)} = c_{ijkl}p_j^{\Sigma} n_l , & C_{ik}^{(4)} = c_{ijkl}p_j^{\Sigma} p_l^{\Sigma} . \end{array} \tag{36}$$

As usual, $\mathbf{I}$ in (33) is a 6×6 identity matrix and in (35) a 3×3 identity matrix. The three components p_i^{Σ}, $i = 1, 2, 3$, are not arbitrary; they must satisfy the orthogonality relation $\mathbf{p}^{\Sigma} \cdot \mathbf{n} = 0$, see (8).

Equations (33) – (36) are valid generally, even for anisotropic viscoelastic media. We shall now specify them for isotropic viscoelastic media. Without any loss of generality, we shall choose the Cartesian coordinate system x_i in such a way that the x_3-axis coincides with the normal $\mathbf{n}$ to plane Σ. Then $\mathbf{n} \equiv (0, 0, 1)$, and plane Σ is represented by coordinate plane x_1, x_2. We also obtain

$$p_3^{\Sigma} = 0 . \tag{37}$$

The orthogonality relation $\mathbf{p}^{\Sigma} \cdot \mathbf{n} = 0$ is in this case satisfied automatically.

The 6×6 complex-valued matrix $\mathbf{A}$, given by (34)–(36), specified for the isotropic viscoelastic medium (12) and for the Cartesian coordinate system described above, reads

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & -p_1^\Sigma & -1/\beta^2 & 0 & 0 \\ 0 & 0 & -p_2^\Sigma & 0 & -1/\beta^2 & 0 \\ -\Delta_2 p_1^\Sigma & -\Delta_2 p_2^\Sigma & 0 & 0 & 0 & -1/\alpha^2 \\ \Delta_3 + \Delta_1 \beta^2 p_1^{\Sigma 2} & \Delta_1 \beta^2 p_1^\Sigma p_2^\Sigma & 0 & 0 & 0 & -\Delta_2 p_1^\Sigma \\ \Delta_1 \beta^2 p_1^\Sigma p_2^\Sigma & \Delta_3 + \Delta_1 \beta^2 p_2^{\Sigma 2} & 0 & 0 & 0 & -\Delta_2 p_2^\Sigma \\ 0 & 0 & -1 & -p_1^\Sigma & -p_2^\Sigma & 0 \end{pmatrix} , \tag{38}$$

where

$$\Delta_1 = 3 - 4\beta^2/\alpha^2 , \quad \Delta_2 = 1 - 2\beta^2/\alpha^2 , \quad \Delta_3 = -1 + \beta^2 (p_1^{\Sigma 2} + p_2^{\Sigma 2}) . \tag{39}$$

Here α^2 and β^2 are given by (17). For a more detailed derivation of (38) refer to Červený (2001b). After some cumbersome algebra, the characteristic equation (33) can be expressed in the following simple form:

$$\det(\mathbf{A} - \sigma \mathbf{I}) = [\sigma^2 - (1/\alpha^2 - p_1^{\Sigma 2} - p_2^{\Sigma 2})][\sigma^2 - (1/\beta^2 - p_1^{\Sigma 2} - p_2^\Sigma)]^2 = 0 . \tag{40}$$

As we can see from (40), the characteristic equation (40) has two single roots for P waves,

$$\sigma = \pm (1/\alpha^2 - p_1^{\Sigma 2} - p_2^{\Sigma 2})^{1/2} , \tag{41}$$

and two double roots for S waves,

$$\sigma = \pm (1/\beta^2 - p_1^{\Sigma 2} - p_2^{\Sigma 2})^{1/2} . \tag{42}$$

Thus, we obtain six eigenvalues σ. They correspond to inhomogeneous P and S plane waves, propagating to both sides of plane Σ.

Let us briefly discuss the physical meaning of eigenvalues σ in our case. Equations (8) and (37) yield $p_3 = \sigma n_3 + p_3^\Sigma = \sigma$. Thus, eigenvalue σ represents the component p_3 of slowness vector $\mathbf{p}$, $\sigma = p_3$. We can then see that (41) and (42) satisfy the standard constraint relations (13) with (16).

We again emphasize that p_1^Σ and p_2^Σ are complex-valued, $p_I^\Sigma = P_I^\Sigma + \mathrm{i} A_I^\Sigma$ $(I = 1, 2)$. The real-valued components of the propagation vector (P_1^Σ and P_2^Σ), and of the attenuation vector (A_1^Σ and A_2^Σ) may be taken arbitrarily, $-\infty < P_I < \infty$, $-\infty < A_I < \infty$, $I = 1, 2$. For an arbitrarily chosen combination of P_1^Σ, P_2^Σ, A_1^Σ and A_2^Σ, the eigenvalues $\sigma = p_3 = P_3 + \mathrm{i} A_3$ are strictly determined by (41) and (42).

We shall now discuss the slowness vector $\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A}$ for inhomogeneous plane P waves. The results for inhomogeneous plane S waves are the same as for P waves; only α should be replaced by β.

For inhomogeneous plane P waves, Equations (8), (37) and (41) yield

$$\begin{array}{ll} P_1 = \mathrm{Re} p_1^\Sigma , & A_1 = \mathrm{Im} p_1^\Sigma , \\ P_2 = \mathrm{Re} p_2^\Sigma , & A_2 = \mathrm{Im} p_2^\Sigma , \\ P_3 = \pm \frac{1}{\sqrt{2}} \sqrt{\sqrt{a^2 + b^2} + a} , & A_3 = \pm \frac{1}{\sqrt{2}} \mathrm{sgn}(b) \sqrt{\sqrt{a^2 + b^2} - a} , \end{array} \tag{43}$$

where

$$a = \mathrm{Re}(1/\alpha^2 - p_1^{\Sigma 2} - p_2^{\Sigma 2}) \ , \quad b = \mathrm{Im}(1/\alpha^2 - p_1^{\Sigma 2} - p_2^{\Sigma 2}) \ . \tag{44}$$

For $|\mathbf{P}|$ and $|\mathbf{A}|$, the above equations yield

$$\begin{aligned}
|\mathbf{P}| &= [(\mathrm{Re}p_1^{\Sigma})^2 + (\mathrm{Re}p_2^{\Sigma})^2 + \tfrac{1}{2}(\sqrt{a^2+b^2} + a)]^{1/2} \\
&= [(\mathrm{Im}p_1^{\Sigma})^2 + (\mathrm{Im}p_2^{\Sigma})^2 + \tfrac{1}{2}\sqrt{a^2+b^2} + \mathrm{Re}(1/\alpha^2)]^{1/2} \ , \\
|\mathbf{A}| &= [(\mathrm{Im}p_1^{\Sigma})^2 + (\mathrm{Im}p_2^{\Sigma})^2 + \tfrac{1}{2}(\sqrt{a^2+b^2} - a)]^{1/2} \ .
\end{aligned} \tag{45}$$

Unit vectors $\mathbf{N}$ and $\mathbf{M}$ are then obtained simply from (43)–(45):

$$N_i = P_i/|\mathbf{P}| \ , \qquad M_i = A_i/|\mathbf{A}| \ . \tag{46}$$

Let us emphasize that unit vectors $\mathbf{N}$ and $\mathbf{M}$ are not known in advance (as in the directional specification), but they are obtained as a result of the solution of the eigenvalue problem for the 6×6 complex-valued matrix $\mathbf{A}$, given by (38).

Using (43)–(45), we can simply determine phase velocity $\mathcal{C}$, the attenuation amplitude factor δ and attenuation angle γ for the inhomogeneous plane P wave,

$$\mathcal{C} = 1/|\mathbf{P}| \ , \quad \delta = |\mathbf{A}|/|\mathbf{P}| = \mathcal{C}|\mathbf{A}| \ , \ \cos\gamma = P_i A_i/|\mathbf{P}||\mathbf{A}| = \tfrac{1}{2}\mathrm{Im}(1/\alpha^2)/|\mathbf{P}|\mathbf{A}| \ . \tag{47}$$

Equations (43)–(47) represent the final solution of the problem. They are valid universally, even for inhomogeneous plane waves propagating in perfectly elastic isotropic media. We only insert $Q_P = \infty$, i. e. $\mathrm{Im}(1/\alpha^2) = 0$, into (43)–(47). Equation (47) for $\cos\gamma$ then immediately yields an important result: $\cos\gamma = 0$. Thus, the attenuation angle of an inhomogeneous plane wave propagating in a perfectly elastic isotropic medium is always $\frac{1}{2}\pi$. Inhomogeneous plane waves with $\gamma \neq \frac{1}{2}\pi$ cannot propagate in perfectly elastic isotropic media. This fully corresponds to the results of Section 2.1, obtained under the directional specification of the slowness vector. The result is valid for arbitrarily chosen $\mathrm{Re}(p_1^{\Sigma})$, $\mathrm{Re}(p_2^{\Sigma})$, $\mathrm{Im}(p_1^{\Sigma})$ and $\mathrm{Im}(p_2^{\Sigma})$.

We shall now briefly discuss Equation (47) for phase velocity $\mathcal{C}$, $\mathcal{C} = 1/|\mathbf{P}|$. As we can deduce from (45) and (44), $|\mathbf{P}|$ is always positive and finite, for any finite $\mathrm{Re}p_1^{\Sigma}$, $\mathrm{Re}p_2^{\Sigma}$, $\mathrm{Im}p_1^{\Sigma}$ and $\mathrm{Im}p_2^{\Sigma}$. Consequently, phase velocity $\mathcal{C}$ is positive for any finite $\mathrm{Re}p_I^{\Sigma}$ and $\mathrm{Im}p_I^{\Sigma}$, $I = 1, 2$. If any of these four quantities grows above all limits, the phase velocity approaches zero.

Now we shall discuss the attenuation angle γ for inhomogeneous plane waves propagating in viscoelastic isotropic media given by (47). For $Q_P \neq \infty$, (17) yields

$$\mathrm{Im}(1/\alpha^2) = V_P^{-2}(1 + Q_P^{-2})^{-1}Q_P^{-1} \ . \tag{48}$$

For Q_P finite, $\cos\gamma$ may vanish only if any of the four quantities $|\mathrm{Re}p_I^{\Sigma}|$ and $|\mathrm{Im}p_I^{\Sigma}|$, $I = 1, 2$, grows above all limits. Consequently, $\cos\gamma$ is always positive, for any finite $\mathrm{Re}p_I^{\Sigma}$ and $\mathrm{Im}p_I^{\Sigma}$.

Thus, if we use the componental specification of the slowness vector with arbitrary finite $\mathrm{Re}p_I^{\Sigma}$ and $\mathrm{Im}p_I^{\Sigma}$, $I = 1, 2$, **no forbidden directions exist**. We automatically obtain

$\mathcal{C} > 0$ and such unit vectors $\mathbf{N}$ and $\mathbf{M}$, for which $\cos\gamma = \mathbf{N}\cdot\mathbf{M} > 0$. In the directional specification of the slowness vector, however, $\mathbf{N}$ and $\mathbf{M}$ are specified in advance. The condition $\cos\gamma = \mathbf{N}\cdot\mathbf{M} > 0$ is not valid automatically. If $\mathbf{N}$ and $\mathbf{M}$ are specified in such a way that $\cos\gamma = \mathbf{N}\cdot\mathbf{M} < 0$, the obtained phase velocity $\mathcal{C}$ is not real-valued positive. Consequently, $\mathbf{N}$ and $\mathbf{M}$ represent forbidden directions.

For $\cos\gamma = \mathbf{N}\cdot\mathbf{M} > 0$, the results of this section may easily be transformed to those obtained in Section 2.1. We can do this in the following way: First, we obtain $|\mathbf{P}|^2 - |\mathbf{A}|^2 = \mathrm{Re}(1/\alpha^2)$ from (45). We then use Equation (47) for $\cos\gamma$ and obtain $|\mathbf{P}||\mathbf{A}| = P_iA_i/\cos\gamma$. Thus, we arrive at two equations for $|\mathbf{P}|$ and $|\mathbf{A}|$:

$$|\mathbf{P}|^2 - |\mathbf{A}|^2 = \mathrm{Re}(1/\alpha^2)\ , \qquad |\mathbf{P}||\mathbf{A}| = \mathrm{Im}(1/\alpha^2)/2\cos\gamma\ . \tag{49}$$

Solving these equations for $|\mathbf{P}|$ and $|\mathbf{A}|$, we obtain

$$\begin{aligned} |\mathbf{P}| &= \sqrt{\sqrt{1+1/Q_P^2\cos^2\gamma}+1}\Big/\sqrt{2}V_P\sqrt{1+Q_P^{-2}}\ , \\ |\mathbf{A}| &= \sqrt{\sqrt{1+1/Q_P^2\cos^2\gamma}-1}\Big/\sqrt{2}V_P\sqrt{1+Q_P^{-2}}\ . \end{aligned} \tag{50}$$

Taking into account that $\mathcal{C} = 1/|\mathbf{P}|$ and $\delta = |\mathbf{A}|/|\mathbf{P}|$, we can see that these equations fully correspond to those obtained under the directional specification of the slowness vector, see (29) and (30).

2.3 Mixed specification of the slowness vector

We shall again consider only P waves; the equations for S waves would be obtained from those for P waves by replacing V_P and Q_P by V_S and Q_S. We shall use the same notations as in Section 2.2, where the projection specification of the slowness vector is discussed. We shall again consider Cartesian coordinate system x_i, in such a way that the x_3-axis coincides with the unit vector $\mathbf{n}$ to plane Σ. All the results of Section 2.2 remain valid even in the mixed specification (9) of the slowness vector, if we use (10). More explicitly, we can express (10) in the following form:

$$\mathrm{Re}p_1^\Sigma = \mathrm{Re}p_2^\Sigma = 0\ , \quad \mathrm{Im}p_1^\Sigma = d_1\ , \quad \mathrm{Im}p_2^\Sigma = d_2\ . \tag{51}$$

The eigenvalues σ of the 6×6 complex-valued matrix $\mathbf{A}$, corresponding to P waves, are then given by a simple relation, see (41),

$$\sigma = \pm\sqrt{1/\alpha^2 + d^2} = \pm\sqrt{(1+\mathrm{i}/Q_P)/V_P^2(1+Q_P^{-2}) + d^2}\ , \tag{52}$$

where d is the inhomogeneity strength, $d = \sqrt{d_1^2 + d_2^2}$. The specification of the inhomogeneity strength, $d = \sqrt{d_1^2 + d_2^2}$, corresponds to the assumed direction of unit vector $\mathbf{n}$ along the x_3-axis. It is, however, simple to see that (52) is valid generally, for any direction of $\mathbf{n}$; we must only define inhomogeneity strength d properly:

$$d^2 = d_1^2 + d_2^2 + d_3^2\ , \qquad \text{with}\ \ d_in_i = 0\ . \tag{53}$$

As expected, the eigenvalues σ of inhomogeneous plane waves propagating in viscoelastic isotropic media do not depend on the direction of propagation $\mathbf{n}$, and on the direction of the wavefront attenuation vector $\mathbf{d}$ (only relation $\mathbf{d}\cdot\mathbf{n}=0$ must be satisfied). They depend only on inhomogeneity strength d, see (53), and on the parameters of the medium, V_P and Q_P.

The Cartesian components of slowness vector $\mathbf{p}=\mathbf{P}+\mathrm{i}\mathbf{A}$ are given by relations, see (8) and (51),

$$\begin{aligned} P_i &= \pm\mathrm{Re}\sqrt{1/\alpha^2+d^2}\;n_i=\pm\tfrac{1}{\sqrt{2}}\sqrt{\sqrt{a^2+b^2}+a}\;n_i\;, \\ A_i &= \pm\mathrm{Im}\sqrt{1/\alpha^2+d^2}\;n_i+d_i=\pm\tfrac{1}{\sqrt{2}}\sqrt{\sqrt{a^2+b^2}-a}\;n_i+d_i\;, \end{aligned} \tag{54}$$

where a and b read

$$a=1/V_P^2(1+Q_P^{-2})+d^2\;,\qquad b=1/V_P^2Q_P(1+Q_P^{-2})\geq 0\;. \tag{55}$$

For $|\mathbf{P}|$ and $|\mathbf{A}|$, we obtain,

$$|\mathbf{P}|=\sqrt{\sqrt{a^2+b^2}+a}/\sqrt{2}\;,\qquad |\mathbf{A}|=\sqrt{d^2+\tfrac{1}{2}(\sqrt{a^2+b^2}-a)}\;. \tag{56}$$

Equation $N_i=P_i/|\mathbf{P}|$ then yields an important result: $\mathbf{N}=\pm\mathbf{n}$. Thus, eigenvalue σ_1 corresponds to the inhomogeneous plane P wave propagating along $\mathbf{n}$ (with $\mathbf{N}=\mathbf{n}$), and eigenvalue σ_2 corresponds to the inhomogeneous plane P wave propagating against $\mathbf{n}$ (with $\mathbf{N}=-\mathbf{n}$).

The phase velocity $\mathcal{C}$, the attenuation amplitude ratio δ and the attenuation angle γ of these two inhomogeneous plane P waves are given by (47). They depend only on inhomogeneity strength d and on the parameters of the medium, V_P and Q_P. They are the same for both directions of propagation, $\mathbf{N}=\mathbf{n}$ and $\mathbf{N}=-\mathbf{n}$. Equations (47), (48) and (56) yield

$$\begin{aligned} \mathcal{C} &= \sqrt{2}/\sqrt{\sqrt{a^2+b^2}+a}\;, \\ \delta &= 2\sqrt{[d^2+\tfrac{1}{2}(\sqrt{a^2+b^2}-a)]/(\sqrt{a^2+b^2}+a)}\;, \\ \cos\gamma &= 1/\sqrt{1+2d^2(\sqrt{a^2+b^2}+a)/b^2}\;. \end{aligned} \tag{57}$$

Here a and b are given by (55). As we can see from (57), $\mathcal{C}$, δ and $\cos\gamma$ are nonnegative for any inhomogeneity strength d. The last equation of (57) can also be expressed in an alternative form,

$$\tan\gamma=\sqrt{2}d\sqrt{\sqrt{a^2+b^2}+a}\Big/b\geq 0\;. \tag{58}$$

Consequently, $0\leq\gamma<\frac{1}{2}\pi$.

If we insert a and b from (55) into (57) and (58), we express $\mathcal{C}$, δ, $\cos\gamma$ and $\tan\gamma$ in terms of V_P Q_P and d. These expressions can be simplified, if we introduce two new quantities $\mathcal{C}_0$ and d_0:

$$\mathcal{C}_0=V_P\sqrt{1+Q_P^{-2}}\;,\qquad d_0=dV_P\sqrt{1+Q_P^{-2}}\;. \tag{59}$$

Here d_0 is a modified inhomogeneity strength, which is dimensionless, and $0 \leq d_0 < \infty$. Expressions (57) then yield

$$\begin{aligned} \mathcal{C} &= \sqrt{2}\mathcal{C}_0/\sqrt{1+d_0^2}\sqrt{1+\sqrt{1+1/Q_P^2(1+d_0^2)^2}} \ , \\ \delta &= (\mathcal{C}/\mathcal{C}_0)\sqrt{d_0^2+\tfrac{1}{4}Q_P^{-2}(\mathcal{C}/\mathcal{C}_0)^2} \ , \\ \cos\gamma &= \tfrac{1}{2}(\mathcal{C}/\mathcal{C}_0)Q_P^{-1}/\sqrt{d_0^2+\tfrac{1}{4}Q_P^{-2}(\mathcal{C}/\mathcal{C}_0)^2} \ . \end{aligned} \tag{60}$$

Analogously, (58) yields a simple relation for $\tan\gamma$,

$$\tan\gamma = 2d_0 Q_P^{-1}(\mathcal{C}/\mathcal{C}_0)^{-1} \ . \tag{61}$$

Equations (60) and (61) are quite general and universal. They can be used for inhomogeneous (or homogeneous) plane P waves, propagating in isotropic viscoelastic (or isotropic perfectly elastic) media.

We shall now briefly discuss several special cases of (60).

a) **For perfectly elastic isotropic media**, $Q_P^{-1} = 0$. Equations (60) then yield, for any d_0,

$$\mathcal{C} = V_P/\sqrt{1+d_0^2} \ , \quad \delta = d_0/\sqrt{1+d_0^2} \ , \quad \cos\gamma = 0 \ . \tag{62}$$

Thus, attenuation angle γ is always $\gamma = \frac{1}{2}\pi$ in perfectly elastic isotropic media. Inhomogeneous plane waves with $\gamma \neq \frac{1}{2}\pi$ **cannot** propagate there. Phase velocity $\mathcal{C}$ for $\gamma = \frac{1}{2}\pi$ depends on the inhomogeneity strength: It is maximum for $d_0 = 0$, where it equals V_P. With increasing d_0 it decreases, and completely vanishes for $d_0 \to \infty$. The relevant attenuation amplitude ratio δ equals zero for $d_0 = 0$, increases with increasing d_0, and equals unity for $d_0 \to \infty$. Homogeneous plane waves (with nonvanishing attenuation vector $\mathbf{A}$ parallel to $\mathbf{P}$) do not exist in perfectly elastic isotropic media, as $\delta = 0$ for $d_0 = 0$.

These conclusions fully correspond to the conclusions of Section 2.1. The only difference is that phase velocity $\mathcal{C}$ is expressed there in terms of δ, not in terms of d_0 as it is here. It is, however, simple to prove that both expressions are fully equivalent.

b) **For homogeneous plane P waves** propagating in viscoelastic isotropic media ($d_0 = 0$, $Q_P^{-1} \neq 0$), Equations (60) and (61) yield

$$\mathcal{C}_{\text{hom}} = \sqrt{2}\mathcal{C}_0/\sqrt{1+\sqrt{1+Q_P^{-2}}} \ , \quad \delta_{\text{hom}} = (\mathcal{C}_{\text{hom}}/\mathcal{C}_0)^2/2Q_P \ , \quad \gamma_{\text{hom}} = 0 \ . \tag{63}$$

These expressions again correspond to (29) with (30), taken for $\gamma = 0$.

c) **For the limiting case of** $d_0 \to \infty$ in viscoelastic isotropic media ($d_0 \to \infty$, $Q_P^{-1} \neq 0$), Equations (60) and (61) yield:

$$\mathcal{C} \doteq \mathcal{C}_0/d_0 \to 0 \ , \quad \delta \doteq (\mathcal{C}/\mathcal{C}_0)d_0 \to 1 \ , \quad \gamma \to \tfrac{1}{2}\pi \ .$$

Consequently, the attenuation angle $\gamma = \frac{1}{2}\pi$ corresponds to the zero phase velocity ($\mathcal{C} = 0$) and represents the boundary attenuation angle γ^*.

d) Now consider $0 \leq d_0 < \infty$ in viscoelastic isotropic media. With increasing d_0, attenuation angle γ increases from 0 to $\frac{1}{2}\pi$, phase velocity $\mathcal{C}$ decreases from $\mathcal{C}_{\text{hom}}$ to zero, and the attenuation amplitude ratio δ increases from δ_{hom} to unity. Here $\mathcal{C}_{\text{hom}}$ and δ_{hom} correspond to homogeneous plane waves and are given by (63).

2.4 Discussion of results

The properties of inhomogeneous plane waves propagating in viscoelastic isotropic media are simple. They can be alternatively derived under any of the three specifications of the slowness vector, see (7), (8) and (9).

The simplicity of the results is mainly due to the simplicity of the boundary attenuation angle γ^*, which corresponds to the zero phase velocity ($\mathcal{C} = 0$). In viscoelastic isotropic media, the boundary attenuation angle γ^* always equals $\frac{1}{2}\pi$, independently of complex-valued Lamé's moduli λ and μ, of the direction of propagation vector $\mathbf{P}$, and of the orientation of the propagation-attenuation plane $\Sigma^{\|}$. For any inhomogeneous plane wave with a nonvanishing phase velocity $\mathcal{C}$, attenuation angle γ is less than the boundary attenuation angle $\gamma^* = \frac{1}{2}\pi$. Inhomogeneous plane waves with $\gamma > \gamma^* = \frac{1}{2}\pi$ cannot propagate in viscoelastic isotropic media.

In viscoelastic isotropic media, any of the three specifications of the slowness vector yields physically reliable results for phase velocity $\mathcal{C}$, attenuation amplitude ratio δ, attenuation angle γ and slowness vector $\mathbf{p}$. If we use the directional specification (7), we have to choose unit vectors $\mathbf{N}$ and $\mathbf{M}$ in such a way that $\cos\gamma = \mathbf{N}\cdot\mathbf{M} > 0$. If we use the componental specification (8), we can choose the real-valued unit vector $\mathbf{n}$ and the complex-valued vector $\mathbf{p}^{\Sigma}$ quite arbitrarily; they must only satisfy the condition of orthogonality $\mathbf{p}^{\Sigma}\cdot\mathbf{n} = 0$. Finally, if we use the mixed specification (9), we can also choose the real-valued unit vector $\mathbf{n}$ and real-valued vector $\mathbf{d}$ quite arbitrarily; they must only satisfy orthogonality condition $\mathbf{d}\cdot\mathbf{n} = 0$.

Let us now consider an inhomogeneous plane wave propagating in a viscoelastic isotropic medium, with propagation vector $\mathbf{P}$ oriented along an à priori known real-valued unit vector $\mathbf{N}$. We can then use either the directional specification (and also specify unit vector $\mathbf{M}$, satisfying condition $\mathbf{N}\cdot\mathbf{M} > 0$), or the mixed specification (and also specify the real-valued wavefront attenuation vector $\mathbf{d}$, perpendicular to $\mathbf{N}$). The results obtained under the mixed specification can always be easily transformed to the results obtained under the directional specification, and vice versa. We only express attenuation angle γ in terms of inhomogeneity strength d (or d_0), or vice versa. The mutual relations between γ and d_0 can be simply obtained from (60) and (61):

$$\tan\gamma = 2d_0 Q_P^{-1}(\mathcal{C}/\mathcal{C}_0)^{-1} , \qquad d_0 = \sin\gamma\sqrt{\sqrt{1 + 1/Q_P^2\cos^2\gamma} - 1} \ .$$

Here $\mathcal{C} = \mathcal{C}(d_0)$ is given by the first equation of (60).

Although Equation (60) (obtained under the directional specification of the slowness vector) and Equations (29)–(31) (obtained under the mixed specification of the slowness vector) are alternative, the form of Equations (60) is preferable in numerical discussions. The reason is that Equations (29)–(31) contain the unpleasant product $Q_P\cos\gamma$, which becomes indefinite for $Q_P \to \infty$ (as $\cos\gamma \to 0$ in this case). There is no such indefinite product in Equations (60)–(61). Consequently, Equations (60)–(61) can be used even for inhomogeneous plane waves propagating in perfectly elastic isotropic media ($Q_P^{-1} = 0$), whereas Equations (29)–(31) fail in this case. The case of inhomogeneous plane waves propagating in perfectly elastic isotropic media must be treated independently in the directional specification, see (25). In the mixed specification, product $Q_P \cos\gamma$ can be simply expressed in terms of d_0 using an explicit relation:

$$Q_P \cos\gamma = \tfrac{1}{2}(\mathcal{C}/\mathcal{C}_0)\sqrt{d_0^2 + \tfrac{1}{4}Q_P^{-2}(\mathcal{C}/\mathcal{C}_0)^2}\ ,$$

see (60). We also obtain

$$\delta\cos\gamma = \tfrac{1}{2}(\mathcal{C}/\mathcal{C}_0)Q_P^{-1}\ ,$$

which yields (24) for $Q_P^{-1} = 0$.

Equations (60)–(61) are also preferable to (29)–(31) if we study inhomogeneous plane waves propagating in weakly dissipative isotropic media (Q_P high). In this case, product $Q_P \cos\gamma$ does not vanish, but it is very sensitive to small changes of Q_P^{-1} and $\cos\gamma$. Equations (60) and (61), however, can be safely used in this case, for arbitrarily high Q_P. Note that the weakly dissipative media play a very important role in practical applications.

Finally, a note on the componental specification. In the componental specification, neither $\mathbf{N}$, nor $\mathbf{M}$ are à priori known. The componental specification may be very useful in such applications, in which plane Σ does not represent a wavefront ($\mathrm{Re}\mathbf{p}^{\Sigma} \neq 0$). Let us present two examples. In the first example, the problem of inhomogeneous plane waves generated at a plane initial surface Σ (with known $\mathbf{p}^{\Sigma}$ along Σ) is to be solved. In the second example, the reflection/transmission problem for inhomogeneous plane waves at a plane structural interface is to be treated.

3 Plane of symmetry of a monoclinic anisotropic viscoelastic medium

In this section, we shall consider inhomogeneous plane waves propagating in the plane of symmetry of a monoclinic anisotropic viscoelastic medium. We choose the Cartesian coordinate system x_i in such a way that the plane of symmetry corresponds to coordinate plane x_1x_3. In the Voigt notation, the elastic moduli $C_{\mu\nu}$ ($\mu, \nu = 1, 2, ..., 6$) for the monoclinic anisotropic medium are then given by the table

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & C_{15} & 0 \\ C_{12} & C_{22} & C_{23} & 0 & C_{25} & 0 \\ C_{13} & C_{23} & C_{33} & 0 & C_{35} & 0 \\ 0 & 0 & 0 & C_{44} & 0 & C_{46} \\ C_{15} & C_{25} & C_{35} & 0 & C_{55} & 0 \\ 0 & 0 & 0 & C_{46} & 0 & C_{66} \end{pmatrix} . \tag{64}$$

See Carcione (2001, Equation (1.37)). For a viscoelastic monoclinic medium, the table remains the same, only $\mathcal{C}_{\mu\nu}$ are complex-valued. As special cases of (64), we can consider an orthorhombic viscoelastic medium (with $C_{15} = C_{25} = C_{35} = C_{46} = 0$), a hexagonal medium, etc.

3.1 Directional specification of the slowness vector

We shall use the directional specification (7) of the slowness vector, and the approach described in Section 2.1, based on Equations (13) and (14). The 3×3 complex-valued matrix $\Gamma_{ik}(p_n) = (c_{ijkl}/\rho)p_j p_l$ in a monoclinic anisotropic viscoelastic medium is given by relations,

$$
\begin{aligned}
\Gamma_{11}(p_n) &= [C_{11}p_1^2 + C_{66}p_2^2 + C_{55}p_3^2 + 2C_{15}p_1p_3]/\rho \ , \\
\Gamma_{22}(p_n) &= [C_{66}p_1^2 + C_{22}p_2^2 + C_{44}p_3^2 + 2C_{46}p_1p_3]/\rho \ , \\
\Gamma_{33}(p_n) &= [C_{55}p_1^2 + C_{44}p_2^2 + C_{33}p_3^2 + 2C_{35}p_1p_3]/\rho \ , \\
\Gamma_{12}(p_n) &= \Gamma_{21}(p_n) = [(C_{11} + C_{66})p_1p_2 + (C_{46} + C_{25})p_2p_3]/\rho \ , \\
\Gamma_{13}(p_n) &= \Gamma_{31}(p_n) = [C_{15}p_1^2 + C_{46}p_2^2 + C_{35}p_3^2 + (C_{13} + C_{55})p_1p_3]/\rho \ , \\
\Gamma_{23}(p_n) &= \Gamma_{32}(p_n) = [(C_{46} + C_{25})p_1p_2 + (C_{23} + C_{44})p_1p_3]/\rho \ ,
\end{aligned}
\tag{65}
$$

We now need to find analytical expressions for the eigenvalues $G^{(m)}(p_n)$, $m = 1, 2, 3$, of the 3×3 matrix $\mathbf{\Gamma}(p_n)$. In the general case of (65), eigenvalues $G^{(m)}(p_n)$ of $\mathbf{\Gamma}(p_n)$ must be found by solving a cubic algebraic equation. The three eigenvalues $G^{(m)}(p_n)$, $m = 1, 2, 3$, correspond to qS1, qS2 and qP inhomogeneous plane waves. The analytic solution of the cubic equation is always possible, but it may be rather cumbersome.

The treatment simplifies, if we assume

$$p_2 = 0 \ . \tag{66}$$

Equations (65) then read:

$$
\begin{aligned}
\Gamma_{11}(p_n) &= [C_{11}p_1^2 + C_{55}p_3^2 + 2C_{15}p_1p_3]/\rho \ , \\
\Gamma_{22}(p_n) &= [C_{66}p_1^2 + C_{44}p_3^2 + 2C_{46}p_1p_3]/\rho \ , \\
\Gamma_{33}(p_n) &= [C_{55}p_1^2 + C_{33}p_3^2 + 2C_{35}p_1p_3]/\rho \ , \\
\Gamma_{13}(p_n) &= \Gamma_{31}(p_n) = [C_{15}p_1^2 + C_{35}p_3^2 + (C_{13} + C_{55})p_1p_3]/\rho \ , \\
\Gamma_{12}(p_n) &= \Gamma_{21}(p_n) = \Gamma_{23}(p_n) = \Gamma_{32}(p_n) = 0 \ .
\end{aligned}
\tag{67}
$$

The characteristic equation for the 3×3 matrix $\mathbf{\Gamma}(p_n)$ is then factorized:

$$[G(p_n) - \Gamma_{22}][G^2(p_n) + (\Gamma_{11} + \Gamma_{33})G(p_n) + \Gamma_{11}\Gamma_{33} - \Gamma_{13}^2] = 0 \ . \tag{68}$$

Explicit analytical solutions for eigenvalues $G^{(1)}(p_n)$, $G^{(2)}(p_n)$ and $G^{(3)}(p_n)$ are as follows:

$$G^{(1,3)}(p_n) = \tfrac{1}{2}\left[\Gamma_{11} + \Gamma_{33} \mp \sqrt{(\Gamma_{11} + \Gamma_{33})^2 - 4(\Gamma_{11}\Gamma_{33} - \Gamma_{13}^2)}\right]^{1/2} \ , \tag{69}$$

$$G^{(2)}(p_n) = \Gamma_{22} \ . \tag{70}$$

In (68)–(70), $\Gamma_{ij} = \Gamma_{ij}(p_n)$. Eigenvalue $G^{(1)}(p_n)$ corresponds to the qSV waves, $G^{(2)}(p_n)$ to qSH waves, and $G^{(3)}(p_n)$ to qP waves. For any selected inhomogeneous plane wave, the constraint relation $G^{(m)}(p_n) = 1$ can be used to determine phase velocity $\mathcal{C}$, the attenuation amplitude ratio δ and slowness vector $\mathbf{p}$, see (13).

3.1.1 The case of qSH waves

Equations (14) for $\mathcal{C}$ and δ for the qSH inhomogeneous plane waves read

$$\mathrm{Re}[\Gamma_{22}(N_n + \mathrm{i}\delta M_n)] = \mathcal{C}^2 \ , \qquad \mathrm{Im}[\Gamma_{22}(N_n + \mathrm{i}\delta M_n)] = 0 \ . \tag{71}$$

Here $\Gamma_{22}(N_n+\mathrm{i}\delta M_n)$ are given by the second equation of (67), where $N_n+\mathrm{i}\delta M_n$ is inserted for p_n. These two equations can be expressed in the following form:

$$\begin{aligned} a_1 + \delta a_2 - \delta^2 a_3 &= \mathcal{C}^2 \ , &(72)\\ -a_4 + \delta a_5 + \delta^2 a_6 &= 0 \ , &(73) \end{aligned}$$

where

$$\begin{aligned}
a_1 &= [C^R_{66}N_1^2 + C^R_{44}N_3^2 + 2C^R_{46}N_1N_3]/\rho \ ,\\
a_2 &= -2[C^I_{66}N_1M_1 + C^I_{44}N_3M_3 + C^I_{46}(M_1N_3 + N_1M_3)]/\rho \ ,\\
a_3 &= [C^R_{66}M_1^2 + C^R_{44}M_3^2 + 2C^R_{46}M_1M_3]/\rho \ ,\\
a_4 &= -[C^I_{66}N_1^2 + C^I_{44}N_3^2 + 2C^I_{46}N_1N_3]/\rho \ ,\\
a_5 &= 2[C^R_{66}N_1M_1 + C^R_{44}N_3M_3 + C^R_{46}(N_1M_3 + M_1N_3)]/\rho \ ,\\
a_6 &= -[C^I_{66}M_1^2 + C^I_{44}M_3^2 + 2C^I_{46}M_1M_3]/\rho \ .
\end{aligned} \tag{74}$$

Here we have used the notations $C^R_{ij} = \mathrm{Re}(C_{ij})$, $C^I_{ij} = \mathrm{Im}(C_{ij})$.

For a given model of a viscoelastic anisotropic monoclinic medium (64), and for given unit vectors $\mathbf{N} \equiv (N_1, 0, N_3)$ and $\mathbf{M} \equiv (M_1, 0, M_3)$, the quantities $a_1 - a_6$ in (74) are assumed to be known. Thus, Equations (72) and (73) can be used to determine δ and $\mathcal{C}^2$.

An analogous system of equations for a qSH inhomogeneous plane wave propagating in a viscoelastic transversely isotropic medium was first derived and numerically discussed by Krebes and Le (1994). For qSH inhomogeneous plane waves propagating in the symmetry plane of a monoclinic anisotropic viscoelastic medium, see Carcione and Cavallini (1995), Carcione (2001).

A brief note on the coefficients $a_1 - a_6$ in (74). It is reasonable to assume that the two real-valued symmetric matrices,

$$\begin{pmatrix} C^R_{66} & C^R_{46} \\ C^R_{46} & C^R_{44} \end{pmatrix} , - \begin{pmatrix} C^I_{66} & C^I_{46} \\ C^I_{46} & C^I_{44} \end{pmatrix} , \tag{75}$$

are positive definite. Then a_1, a_3, a_4 and a_6 are positive, but a_2 and a_5 may be positive, zero or negative.

Equation (73) for δ is quadratic, and yields the solutions,

$$\delta_{1,2} = -a_5/2a_6 \pm \sqrt{(a_5/2a_6)^2 + a_4/a_6} \ . \tag{76}$$

The discriminant in (76) is always positive, so that δ_1 and δ_2 are real-valued. As δ must be nonnegative, we have to use the "+" sign in (76),

$$\delta = -a_5/2a_6 + \sqrt{(a_4/2a_6)^2 + a_4/a_6} \ . \tag{77}$$

Inserting this δ into (72) yields the square of the phase velocity, $\mathcal{C}^2$.

Thus, for given directions $\mathbf{N}$ and $\mathbf{M}$, (77) and (72) represent the final equations for the determination of phase velocity $\mathcal{C}$ and for the attenuation amplitude ratio δ. Slowness vector $\mathbf{p}$ is then given by (7), and attenuation angle γ by the relation $\cos\gamma = \mathbf{N}\cdot\mathbf{M}$.

There is, however, one problem in the determination of $\mathcal{C}^2$ and δ using (72) and (77). For given $\mathbf{N}$ and $\mathbf{M}$, Equations (77) with (72) can yield $\mathcal{C}^2 < 0$. Such $\mathbf{N}$ and $\mathbf{M}$, which yield $\mathcal{C}^2 < 0$, are called forbidden directions, see Krebes and Le (1994), Carcione and Cavallini (1995), Carcione (2001). Carcione and Cavallini (1995) speak of "stop bands", where there is no wave propagation. The determination of forbidden directions in anisotropic viscoelastic media is not as simple as in viscoelastic isotropic media. Let us denote γ^* the attenuation angle γ for which phase velocity $\mathcal{C}$ vanishes ($\mathcal{C} = 0$), and call it the *boundary attenuation angle*. Directions $\mathbf{N}$ and $\mathbf{M}$ represent forbidden directions if the relevant attenuation angle γ satisfies the relation

$$\cos\gamma = \mathbf{N}\cdot\mathbf{M} < \cos\gamma^* \ .$$

In isotropic viscoelastic media, the situation is simple, as the boundary attenuation angle γ^* always equals $\frac{1}{2}\pi$. In anisotropic viscoelastic media, however, the boundary attenuation angle γ^* is, in general, different from $\frac{1}{2}\pi$, and also depends on other parameters. For a more detailed discussion, see Section 3.3.1.

Equations (77) and (72) do not allow the boundary attenuation angle γ^* to be computed in advance. If we wish to determine it, we must seek numerically such $\mathbf{N}$ and $\mathbf{M}$ from (72) and (73), for which $\mathcal{C} = 0$ is obtained. This is the disadvantage of the directional specification of the slowness vector. If we use the mixed specification of the slowness vector, see Section 3.3, the boundary attenuation angle γ^* can be determined explicitly.

If directions $\mathbf{N}$ and $\mathbf{M}$ are chosen in such a way that

$$\cos\gamma = \mathbf{N}\cdot\mathbf{M} > \cos\gamma^* \ ,$$

Equations (77), (72) and (7) represent the final solution of the problem.

3.1.2 The case of qSV and qP waves

For qSV and qP inhomogeneous plane waves, propagating in the plane of symmetry of a monoclinic anisotropic viscoelastic medium, Equations (14) for $\mathcal{C}^2$ and δ read

$$\begin{aligned} \mathrm{Re}[G^{(1,3)}(N_n + \mathrm{i}\delta M_n)] &= \mathcal{C}^2 \ , && (78)\\ \mathrm{Im}[G^{(1,3)}(N_n + \mathrm{i}\delta M_n)] &= 0 \ . && (79) \end{aligned}$$

Here $G^{(1)}(N_n + \mathrm{i}\delta M_n)$ and $G^{(3)}(N_n + \mathrm{i}\delta M_n)$ are given by (69) with (67), where $N_n + \mathrm{i}\delta M_n$ are inserted for p_n. Eigenvalue $G^{(1)}$ corresponds to qSV waves, $G^{(3)}$ to qP waves. For given $\mathbf{N}$ and $\mathbf{M}$, it is first necessary to solve (79) for δ. Inserting the found δ into (78), we can calculate $\mathcal{C}^2$. We can consider only such δ and $\mathcal{C}$, which satisfy the conditions $\delta \geq 0$ and $\mathcal{C}^2 \geq 0$. The unit vectors $\mathbf{N}$ and $\mathbf{M}$, for which $\mathcal{C}^2 < 0$ is obtained, represent the forbidden directions. The boundary attenuation angles γ^* correspond to such $\mathbf{N}$ and $\mathbf{M}$, for which the phase velocity vanishes.

However, Equation (79) for qSV and qP inhomogeneous plane waves is rather complicated. Analytical methods can be hardly used to solve it; the equations should be solved by numerical methods. In addition, the boundary attenuation angle γ^* is not known in advance, and the chosen $\mathbf{N}$, $\mathbf{M}$ may represent forbidden directions.

3.2 Componental specification of the slowness vector

In the componental specification, the slowness vector $\mathbf{p}$ is expressed in the form $\mathbf{p} = \sigma\mathbf{n}+\mathbf{p}^\Sigma$, see (8). Here $\mathbf{n}$ is a real-valued unit vector, perpendicular to an arbitrarily chosen plane Σ, and $\mathbf{p}^\Sigma$ is an arbitrary complex-valued vector situated in Σ. Consequently, $\mathbf{p}^\Sigma$ satisfies orthogonality condition $\mathbf{p}^\Sigma \cdot \mathbf{n} = 0$. Finally, σ is an eigenvalue of the 6×6 complex-valued matrix $\mathbf{A}$, given by (34)–(36). It represents a component of the slowness vector, perpendicular to plane Σ,

$$\sigma = \mathbf{p} \cdot \mathbf{n} \; . \tag{80}$$

Once an eigenvalue σ of the 6×6 matrix $\mathbf{A}$ is found, we can determine the relevant real-valued propagation vector $\mathbf{P}$, attenuation vector $\mathbf{A}$, unit vectors $\mathbf{N}$ and $\mathbf{M}$, phase velocity $\mathcal{C}$, attenuation amplitude ratio δ and attenuation angle γ. They are given by relations, see Červený (2003),

$$\begin{array}{ll}
\mathbf{P} = \mathbf{n}(\mathrm{Re}\sigma) + \mathrm{Re}\mathbf{p}^\Sigma \; , & \mathbf{A} = \mathbf{n}(\mathrm{Im}\sigma) + \mathrm{Im}\mathbf{p}^\Sigma \; , \\
|\mathbf{P}| = [(\mathrm{Re}\sigma)^2 + (\mathrm{Re}\mathbf{p}^\Sigma)(\mathrm{Re}\mathbf{p}^\Sigma)]^{1/2} \; , & |\mathbf{A}| = [(\mathrm{Im}\sigma)^2 + (\mathrm{Im}\mathbf{p}^\Sigma)(\mathrm{Im}\mathbf{p}^\Sigma)]^{1/2} \; , \\
\mathbf{N} = \mathbf{P}/|\mathbf{P}| \; , & \mathbf{M} = \mathbf{A}/|\mathbf{A}| \; , \\
\mathcal{C} = 1/|\mathbf{P}| \; , & \delta = |\mathbf{A}|/|\mathbf{P}| \; , \\
\cos\gamma = [(\mathrm{Re}\mathbf{p}^\Sigma)(\mathrm{Im}\mathbf{p}^\Sigma) + (\mathrm{Re}\sigma)(\mathrm{Im}\sigma)]/|\mathbf{P}||\mathbf{A}| \; . &
\end{array} \tag{81}$$

We shall now apply the componental specification to inhomogeneous plane waves propagating in the plane of symmetry x_1x_3 of the monoclinic anisotropic viscoelastic media, see (64). We assume that the unit normal $\mathbf{n}$ to plane Σ is situated in the plane of symmetry x_1x_3, so that $n_2 = 0$. Finally, we also assume $p_2^\Sigma = 0$. Plane $\Sigma^{||}$ then corresponds to plane x_1x_3. From (36), we obtain

$$C_{12}^{(1)} = C_{21}^{(1)} = C_{23}^{(1)} = C_{32}^{(1)} = 0 \; ,$$

and similarly for $\mathbf{C}^{(2)}$, $\mathbf{C}^{(3)}$ and $\mathbf{C}^{(4)}$. Analogously, we obtain

$$(\mathbf{A}_{ij})_{12} = (\mathbf{A}_{ij})_{21} = (\mathbf{A}_{ij})_{23} = (\mathbf{A}_{ij})_{32} = 0 \; , \tag{82}$$

where $i, j = 1, 2, 3$, see (35). The eigenvalue problem (33) for the 6×6 complex-valued matrix $\mathbf{A}$ then factorizes to two simpler eigenvalue problems; one for a 2×2 "transverse" matrix $\mathbf{A}^\perp$, and one for the 4×4 "in-plane" matrix $\mathbf{A}^{||}$. We shall consider both cases independently. In both cases, we can use (81) to determine $\mathbf{P}, \mathbf{A}, \mathbf{N}, \mathbf{M}, \mathcal{C}, \delta$ and $\cos\gamma$, as soon as we determine eigenvalue σ. We only insert $p_2^\Sigma = 0$ and use orthogonality relation $p_1^\Sigma n_1 + p_3^\Sigma n_3 = 0$. For example, for $n_3 \neq 0$, we can exclude p_3^Σ using relation $p_3^\Sigma = -p_1^\Sigma n_1/n_3$. For the readers' convenience, we shall present the final form of (81) for such case:

$$
\begin{array}{ll}
P_1 = n_1(\mathrm{Re}\sigma) + \mathrm{Re}p_1^\Sigma \ , & A_1 = n_1(\mathrm{Im}\sigma) + \mathrm{Im}p_1^\Sigma \ , \\
P_3 = n_3(\mathrm{Re}\sigma) - (\mathrm{Re}p_1^\Sigma)n_1/n_3 \ , & A_3 = n_3(\mathrm{Im}\sigma) - (\mathrm{Im}p_1^\Sigma)n_1/n_3 \ , \\
|\mathbf{P}| = [(\mathrm{Re}\sigma)^2 + (\mathrm{Re}p_1^\Sigma)^2/n_3^2]^{1/2} \ , & |\mathbf{A}| = [(\mathrm{Im}\sigma)^2 + (\mathrm{Im}p_1^\Sigma)^2/n_3^2]^{1/2} \ , \\
N_i = P_i/|\mathbf{P}| \ , & M_i = A_i/|\mathbf{A}| \ , \\
\mathcal{C} = 1/[(\mathrm{Re}\sigma)^2 + (\mathrm{Re}p_1^\Sigma)^2/n_3^2]^{1/2} \ , & \\
\delta = \sqrt{[(\mathrm{Im}\sigma)^2 + (\mathrm{Im}p_1^\Sigma)^2/n_3^2]/[(\mathrm{Re}\sigma)^2 + (\mathrm{Re}p_1^\Sigma)^2/n_3^2]} \ , & \\
\cos\gamma = [(\mathrm{Re}\sigma)(\mathrm{Im}\sigma) + (\mathrm{Re}p_1^\Sigma)(\mathrm{Im}p_1^\Sigma)/n_3^2]/|\mathbf{P}||\mathbf{A}| \ . &
\end{array}
\tag{83}
$$

We further obtain $P_2 = 0,\ A_2 = 0,\ N_2 = 0,\ M_2 = 0$.

3.2.1 qSH waves

The 2×2 transverse matrix $\mathbf{A}^\perp$ corresponds to the qSH waves. The characteristic equation for $\mathbf{A}^\perp$ reads,

$$
\det(\mathbf{A}^\perp - \sigma\mathbf{I}) = 0 \ . \tag{84}
$$

Here $\mathbf{I}$ is a 2×2 identity matrix, and transverse matrix $\mathbf{A}^\perp$ is given by the relations, see (35),

$$
\begin{array}{ll}
A_{11}^\perp = -C_{22}^{(1)-1}C_{22}^{(2)} \ , & A_{12}^\perp = -C_{22}^{(1)-1} \ , \\
A_{21}^\perp = -\rho + C_{22}^{(4)} - C_{22}^{(3)}C_{22}^{(1)-1}C_{22}^{(2)} \ , & A_{22}^\perp = -C_{22}^{(3)}C_{22}^{(1)-1} \ .
\end{array}
\tag{85}
$$

Here $C_{22}^{(n)}$, $n = 1,2,3,4$, can be expressed in the following form, see (36),

$$
C_{22}^{(1)} = \bar{C}^{(1)} \ , \quad C_{22}^{(2)} = C_{22}^{(3)} = (p_1^\Sigma/n_3)\bar{C}^{(2)} \ , \quad C_{22}^{(4)} = (p_1^\Sigma/n_3)^2\bar{C}^{(4)} \ , \tag{86}
$$

with

$$
\begin{aligned}
\bar{C}^{(1)} &= C_{66}n_1^2 + C_{44}n_3^2 + 2C_{46}n_1n_3 \ , \\
\bar{C}^{(2)} &= (C_{66} - C_{44})n_1n_3 + C_{46}(n_3^2 - n_1^2) \ , \\
\bar{C}^{(4)} &= C_{66}n_3^2 + C_{44}n_1^2 - 2C_{46}n_1n_3 \ .
\end{aligned}
\tag{87}
$$

Here we have used orthogonality relation $n_1p_1^\Sigma + n_3p_3^\Sigma = 0$. Note that $\bar{C}^{(1)}$, $\bar{C}^{(2)}$ and $\bar{C}^{(4)}$ depend on viscoelastic moduli C_{44}, C_{66} and C_{46}, and on n_1 and n_3 only; they do not depend on p_1^Σ, p_3^Σ and ρ.

The characteristic equation (84) for qSH inhomogeneous plane waves then reduces to the simple equation

$$
[A_{11}^\perp - \sigma]^2 - A_{12}^\perp A_{21}^\perp = 0 \ . \tag{88}
$$

The solution for σ reads

$$
\begin{aligned}
\sigma_{1,2} &= A_{11}^\perp \pm \sqrt{A_{12}^\perp A_{21}^\perp} \\
&= -\frac{p_1^\Sigma}{n_3}\frac{\bar{C}^{(2)}}{\bar{C}^{(1)}} \pm \left[\frac{\rho}{\bar{C}^{(1)}} - \left(\frac{p_1^\Sigma}{n_3}\right)^2 \frac{\Delta}{\bar{C}^{(1)2}}\right]^{1/2} \ .
\end{aligned}
\tag{89}
$$

Quantity Δ is given by the relation

$$\Delta = \bar{C}^{(4)}\bar{C}^{(1)} - \bar{C}^{(2)2} \; . \tag{90}$$

Quantity Δ does not depend on p_1^Σ, p_3^Σ and ρ. Using (87), it is also possible to prove that Δ is invariant with respect to rotation, so that it does not depend on n_1 and n_3:

$$\Delta = C_{44}C_{66} - C_{46}^2 \; . \tag{91}$$

For $p_1^\Sigma = 0$, Equation (89) correspond to a homogeneous plane wave. This immediately implies that $\rho/\bar{C}^{(1)}$ represents $1/\beta_{hom}^2$, where β_{hom} is the complex-valued phase velocity of the homogeneous plane wave propagating in the direction of $\mathbf{n} \equiv (n_1, 0, n_3)$.

Inserting (89) into (83), we obtain the relevant equations for propagation vector $\mathbf{P}$, attenuation vector $\mathbf{A}$, unit vectors $\mathbf{N}$ and $\mathbf{M}$, phase velocity $\mathcal{C}$, attenuation amplitude ratio δ and attenuation angle γ.

3.2.2 qSV and qP waves

The 4×4 in-plane matrix $\mathbf{A}^{\|}$ corresponds to qSV and qP waves. The characteristic equation for $\mathbf{A}^{\|}$ is quartic, and reads

$$\det(\mathbf{A}^{\|} - \sigma\mathbf{I}) = 0 \; , \tag{92}$$

where $\mathbf{I}$ is a 4×4 identity matrix. The 4×4 complex-valued matrix $\mathbf{A}^{\|}$ is constructed from the 6×6 matrix $\mathbf{A}$ given by (34)– (36), in such a way that the second and fifth rows and columns are removed. We can again take into account orthogonality condition $p_1^\Sigma n_1 + p_3^\Sigma n_3 = 0$ and exclude p_3^Σ, so that $\mathbf{A}^{\|}$ depends on p_1^Σ only. (Of course, it would also be possible to exclude p_1^Σ and express $\mathbf{A}^{\|}$ in terms of p_3^Σ.) The quartic characteristic equation (92) can be solved by conventional numerical methods. It is not necessary to seek roots of any additional equation as in the directional specification of the slowness vector, see (79).

Once eigenvalue σ of the 4×4 matrix $\mathbf{A}^{\|}$ is determined from (92), we can again determine all quantities related to the inhomogeneous plane wave under consideration from (83).

3.3 Mixed specification of the slowness vector

In the mixed specification, slowness vector $\mathbf{p}$ is expressed in the form $\mathbf{p} = \sigma\mathbf{n} + \mathrm{i}\mathbf{d}$, see Equation (9). Here $\mathbf{n}$ is the real-valued unit vector perpendicular to an arbitrarily chosen plane Σ, and $\mathbf{d}$ is an arbitrary real-valued vector, satisfying orthogonality relation $\mathbf{d} \cdot \mathbf{n} = 0$. The mixed specification may be treated as a special case of the componental specification discussed in Section 3.2, if we put $\mathbf{p}^\Sigma = \mathrm{i}\mathbf{d}$. Finally, σ is an eigenvalue of the 6×6 complex-valued matrix $\mathbf{A}$, given by (34)– (36), where we have inserted $\mathbf{p}^\Sigma = \mathrm{i}\mathbf{d}$. All the quantities related to the inhomogeneous plane wave under consideration can then be determined from Equations (81), where we have put $\mathbf{p}^\Sigma = \mathrm{i}\mathbf{d}$, with $d_i n_i = 0$.

We shall now consider inhomogeneous qS1, qS2 and qP plane waves propagating in the plane of symmetry x_1x_3 of a monoclinic anisotropic viscoelastic medium, see (64). Similarly as in Section 3.2, we assume that the unit normal $\mathbf{n}$ to plane Σ is situated in the plane of symmetry x_1, x_3, so that $\mathbf{n} \equiv (n_1, 0, n_3)$. We also assume that $d_2 = 0$, so that the orthogonality relation reads $d_1n_1 + d_3n_3 = 0$. The eigenvalue problem for the 6×6 complex-valued matrix $\mathbf{A}$ can then be factorized into two simpler eigenvalue problems; one for the 2×2 "transverse" matrix $\mathbf{A}^{\perp}$, and the other for the 4×4 "in-plane" matrix $\mathbf{A}^{\|}$. Once we determine eigenvalue σ, we can determine all quantities related to the inhomogeneous qSV, qSH or qP plane wave under consideration using the following equations:

$$
\begin{array}{ll}
P_1 = n_1(\mathrm{Re}\sigma) \ , & A_1 = n_1(\mathrm{Im}\sigma) + n_3 D \ , \\
P_3 = n_3(\mathrm{Re}\sigma) \ , & A_3 = n_3(\mathrm{Im}\sigma) - n_1 D \ , \\
|\mathbf{P}| = |\mathrm{Re}\sigma| \ , & |\mathbf{A}| = [(\mathrm{Im}\sigma)^2 + D^2]^{1/2} \ , \\
N_1 = \epsilon n_1 \ , & M_1 = [n_1(\mathrm{Im}\sigma) + n_3 D]/[(\mathrm{Im}\sigma)^2 + D^2]^{1/2} \ , \\
N_3 = \epsilon n_3 \ , & M_3 = [n_3(\mathrm{Im}\sigma) - n_1 D]/[(\mathrm{Im}\sigma)^2 + D^2]^{1/2} \ , \\
\mathcal{C} = 1/|\mathrm{Re}\sigma| \ , & \delta = [(\mathrm{Im}\sigma)^2 + D^2]^{1/2}/|\mathrm{Re}\sigma| \ , \\
\cos\gamma = \epsilon\, \mathrm{Im}\sigma/[(\mathrm{Im}\sigma)^2 + D^2]^{1/2} \ . &
\end{array}
\qquad (93)
$$

These equations immediately follow from (83), if we use $p_1^{\Sigma} = \mathrm{i}d_1$. We have also used the notation:

$$\epsilon = (\mathrm{Re}\sigma)/|\mathrm{Re}\sigma| \ , \qquad D = d_1/n_3 \ . \qquad (94)$$

We further obtain $P_2 = 0$, $A_2 = 0$, $N_2 = 0$, $M_2 = 0$.

Before we discuss eigenvalues σ, we shall make several comments on (93), valid for any qSH, qSV and qP inhomogeneous plane wave propagating in the plane of symmetry x_1, x_3 of a monoclinic anisotropic viscoelastic medium.

a) As we can see from (93), $\mathbf{N} = \epsilon\, \mathbf{n}$, where ϵ equals either +1, or -1. Thus, the unit vector $\mathbf{N}$ specifying the direction of propagation vector $\mathbf{P}$ (perpendicular to the wavefront) equals $\mathbf{n}$, or is opposite to it. In the directional specification, unit vector $\mathbf{N}$ is specified in advance, and we seek plane waves whose wavefront propagates in the direction of $\mathbf{N}$. In the mixed specification, however, unit vector $\mathbf{n}$ is specified in advance, and we seek *all plane waves* whose wavefronts propagate in the directions of $\mathbf{N} = +\mathbf{n}$ and $\mathbf{N} = -\mathbf{n}$. The actual orientation of $\mathbf{N}$ depends on the sign of $\mathrm{Re}\sigma$. For $\mathrm{Re}\sigma > 0$, we obtain $\mathbf{N} = \mathbf{n}$, and for $\mathrm{Re}\sigma < 0$ $\mathbf{N} = -\mathbf{n}$.

b) Unit vectors $\mathbf{N}$ and $\mathbf{d}$ are situated in plane x_1, x_3. Consequently, symmetry plane x_1, x_3 represents the propagation-attenuation plane $\Sigma^{\|}$.

c) The wavefront attenuation vector $\mathbf{d}$ is real-valued, with $d_2 = 0$. Both its Cartesian coordinates d_1 and d_3 are quite arbitrary, $d_1 \epsilon (-\infty, \infty)$ and $d_3 \epsilon (-\infty, \infty)$. Assuming that unit vector $\mathbf{n}$ is given, the wavefront attenuation vector $\mathbf{d}$ can be specified uniquely by one of its Cartesian components, e.g., by d_1. The second Cartesian component of $\mathbf{d}$ can then be determined using orthogonality relation $d_1n_1 + d_3n_3 = 0$. Let us assume that d_1 is given. Then $d_3 = -d_1n_1/n_3$, and inhomogeneity strength $d = |\mathbf{d}|$ is given by relation $d = |d_1/n_3|$. Instead of d_1, we can also use quantity $D = d_1/n_3$, see (94). Similarly as d_1, D may be an arbitrary real-valued number, $D \epsilon (-\infty, \infty)$. Quantity $|D| = \sqrt{d_1^2 + d_3^2}$ then represents directly the inhomogeneity strength,

$$d = |D| = |d_1/n_3| \ . \tag{95}$$

The signs of D and d_1 are the same for $n_3 > 0$, and opposite for $n_3 < 0$. The sign of D can be strictly defined in the following way:

$$\begin{array}{ll} D > 0 \quad \text{if} & \mathbf{d} \cdot (\mathbf{e}_2 \times \mathbf{n}) > 0 \ , \\ D < 0 \quad \text{if} & \mathbf{d} \cdot (\mathbf{e}_2 \times \mathbf{n}) < 0 \ . \end{array} \tag{96}$$

Here $\mathbf{e}_2$ is the fixed basis vector of the used Cartesian coordinate system x_i, oriented along the x_2-axis (perpendicular to the plane of symmetry $\Sigma^{||}$). Thus, if we look at the plane of symmetry $\Sigma^{||}$ in the direction of $\mathbf{e}_2$, quantity D is positive for $\mathbf{d}$ pointing to the right of $\mathbf{n}$. Any inhomogeneous plane wave propagating in the plane of symmetry $\Sigma^{||}$ may then be fully specified by $\mathbf{n}$ and D.

d) Attenuation angle γ was introduced here as a positive angle between unit vectors $\mathbf{N}$ and $\mathbf{M}$, $0 \leq \gamma \leq \pi$, in such a way that $\cos\gamma = \mathbf{N} \cdot \mathbf{M}$. In the planar case that we are treating, it has been usual to use an "oriented attenuation angle γ" in such a way that also $\sin\gamma$ is used to define γ (although the word "oriented" has been considered only tacitly). The oriented attenuation angle γ would then be uniquely defined in the range $-\pi < \gamma < \pi$. Such a definition of γ has been particularly useful in the directional specification of the slowness vector, where γ is used as an input parameter of the inhomogeneous plane wave under consideration. Here, however, we use D as the input parameter, and the correct orientation is fully specified by the sign of D. Thus, we do not need to consider an oriented attenuation angle at all. We shall always define γ by (93), in the range $0 \leq \gamma \leq \pi$. For $0 \leq \cos\gamma \leq 1$, we obtain $0^0 \leq \gamma \leq 90^0$, and for $-1 \leq \cos\gamma \leq 0$, we get $90^0 \leq \gamma \leq 180^0$.

3.3.1 qSH waves

The eigenvalues $\sigma_{1,2}$ corresponding to qSH inhomogeneous plane waves propagating in the symmetry plane x_1, x_3 of a monoclinic anisotropic viscoelastic medium are given by simple analytical expressions. We insert $p_1^{\Sigma} = \mathrm{i}d_1$ into (89), where d_1 is the Cartesian x_1-component of the wavefront attenuation vector $\mathbf{d}$, and obtain

$$\sigma_{1,2} = -\mathrm{i}\frac{d_1}{n_3}\,\frac{\bar{C}^{(2)}}{\bar{C}^{(1)}} \pm \left[\frac{\rho}{\bar{C}^{(1)}} + \left(\frac{d_1}{n_3}\right)^2 \frac{\Delta}{\bar{C}^{(1)2}}\right]^{1/2} \ . \tag{97}$$

Quantities $\bar{C}^{(1)}$ and $\bar{C}^{(2)}$ depend on à priori known real-valued unit vector $\mathbf{n} \equiv (n_1, 0, n_3)$, perpendicular to the wavefront, and are given by (87). Quantity Δ is independent of $\mathbf{n}$, and is given by (91). Unit vector $\mathbf{n}$ may be chosen arbitrarily, with $n_2 = 0$. Instead of d_1, it is more useful to use quantity $D = d_1/n_3$, see (94). Equation (97) then reads

$$\sigma_{1,2} = -\mathrm{i}D\bar{C}^{(2)}/\bar{C}^{(1)} \pm [\rho/\bar{C}^{(1)} + D^2\Delta/\bar{C}^{(1)2}]^{1/2} \ . \tag{98}$$

Let us again emphasize that D may be an arbitrary real-valued quantity, $-\infty < D < \infty$, that $|D|$ represents the inhomogeneity strength (the length of the wavefront attenuation vector $\mathbf{d}$), and that the sign of D specifies the orientation of the wavefront attenuation vector $\mathbf{d}$ (see (96)).

Equation (98) implies

$$\begin{aligned} \mathrm{Re}\sigma_{1,2} &= D\mathrm{Im}(\bar{C}^{(2)}/\bar{C}^{(1)}) \pm \mathrm{Re}[\rho/\bar{C}^{(1)} + D^2\Delta/\bar{C}^{(1)2}]^{1/2} , \\ \mathrm{Im}\sigma_{1,2} &= -D\mathrm{Re}(\bar{C}^{(2)}/\bar{C}^{(1)}) \pm \mathrm{Im}[\rho/\bar{C}^{(1)} + D^2\Delta/\bar{C}^{(1)2}]^{1/2} . \end{aligned} \tag{99}$$

It is reasonable to assume that the following relations are valid for any D:

$$\mathrm{Re}\sigma_1 > 0 , \quad \mathrm{Re}\sigma_2 < 0 . \tag{100}$$

For homogeneous plane waves, the validity of (100) is obvious from (99). The validity of (100) is also obvious for inhomogeneous plane waves propagating in weakly dissipative media, in which $\mathrm{Im}(\bar{C}^{(2)}/\bar{C}^{(1)})$ is small with respect to $\mathrm{Re}[\rho/\bar{C}^{(1)}D^2 + \Delta/\bar{C}^{(1)2}]^{1/2}$. In general, if relations (100) were not satisfied, phase velocity $\mathcal{C}$ would be infinite for some D. This would not be physically acceptable. Consequently, eigenvalue $\sigma = \sigma_1$ corresponds to $\mathbf{N} = \mathbf{n}$ (the wavefront propagating along $\mathbf{n}$), and eigenvalue $\sigma = \sigma_2$ corresponds to $\mathbf{N} = -\mathbf{n}$ (the wavefront propagating against $\mathbf{n}$).

We shall now denote by $\mathcal{C}_I(n, D)$, $\delta_I(n, D)$ and $\gamma_I(n, D)$ respectively phase velocity $\mathcal{C}$, attenuation amplitude factor δ and the attenuation angle γ, corresponding to eigenvalue $\sigma_I(\mathbf{n}, D)$, $I = 1, 2$. Equations (98) and (99) can then be used to determine certain general properties and symmetries of inhomogeneous qSH plane waves.

One can immediately see from (98) that the following relations are valid for any $\bar{C}^{(2)} \neq 0$ and $D \neq 0$:

$$\sigma_1(\mathbf{n}, D) \neq \sigma_1(\mathbf{n}, -D) , \quad \sigma_2(\mathbf{n}, D) \neq \sigma_2(\mathbf{n}, D) . \tag{101}$$

This implies

$$\begin{array}{ll} \mathcal{C}_1(\mathbf{n}, D) \neq \mathcal{C}_1(\mathbf{n}, -D) , & \mathcal{C}_2(\mathbf{n}, D) \neq \mathcal{C}_2(\mathbf{n}, -D) , \\ \delta_1(\mathbf{n}, D) \neq \delta_1(\mathbf{n}, -D) , & \delta_2(\mathbf{n}, D) \neq \delta_2(\mathbf{n}, -D) , \\ \cos\gamma_1(\mathbf{n}, D) \neq \cos\gamma_1(\mathbf{n}, -D) , & \cos\gamma_2(\mathbf{n}, D) \neq \cos\gamma_2(\mathbf{n}, -D) . \end{array} \tag{102}$$

This is a great difference with respect to inhomogeneous plane waves propagating in viscoelastic isotropic media, where these quantities do not depend on the sign of D.

We shall now rotate unit vector $\mathbf{n}$ to the opposite direction $-\mathbf{n}$. Quantities $\bar{C}^{(1)}$, $\bar{C}^{(2)}$ and Δ in (98) do not change. Keeping D constant, we obtain

$$\sigma_1(\mathbf{n}, D) = \sigma_1(-\mathbf{n}, D) , \quad \sigma_2(\mathbf{n}, D) = \sigma_2(-\mathbf{n}, D) . \tag{103}$$

Analogous equations are obtained also for $\mathcal{C}_I(\mathbf{n}, D)$, $\delta_I(\mathbf{n}, D)$ and $\cos\gamma_I(\mathbf{n}, D)$. Thus, σ_I, $\mathcal{C}_I$, δ_I and $\cos\gamma_I$ do not change if $\mathbf{n}$ changes its sign. Let us, however, emphasize that the same D for $\mathbf{n}$ and $-\mathbf{n}$ implies opposite directions of the wavefront attenuation vector $\mathbf{d}$ for $\mathbf{n}$ and $-\mathbf{n}$. Consequently, σ_I, $\mathcal{C}_I$, δ_I and $\cos\gamma_I$ ($I = 1, 2$) do not change if both $\mathbf{n}$ and $\mathbf{d}$ change their signs.

Other important relations apply to mutual symmetries between σ_1 and σ_2 for fixed $\mathbf{n}$:

$$\sigma_2(\mathbf{n}, -D) = -\sigma_1(\mathbf{n}, D) . \tag{104}$$

This implies

$$\mathcal{C}_2(\mathbf{n}, -D) = \mathcal{C}_1(\mathbf{n}, D) \ , \qquad \delta_2(\mathbf{n}, -D) = \delta_1(\mathbf{n}, D) \ , \qquad \cos\gamma_2(\mathbf{n}, -D) = \cos\gamma_1(\mathbf{n}, D) \ . \tag{105}$$

Let us emphasize again that solution σ_1 corresponds to plane waves propagating in the direction of $\mathbf{n}$, and solution σ_2 to plane waves propagating against $\mathbf{n}$.

Equations (93) with (98) or (99) are very convenient for discussing the properties of inhomogeneous qSH plane waves. We shall now treat several cases, in which these general equations simplify.

1. In **isotropic viscoelastic media**, $\bar{C}^{(2)} = 0$ and $\Delta = \bar{C}^{(1)2}$. Equation (98) then yields:

$$\sigma_{1,2} = \pm[1/\beta^2 + D^2]^{1/2} = \pm[(1 + \mathrm{i}Q_S)/V_S^2(1 + Q_S^{-2}) + D^2]^{1/2} \ . \tag{106}$$

Here β is the complex-valued velocity of S waves.

Equation (106) is fully analogous to (52), derived directly for viscoelastic isotropic media in Section 2.3. As shown in Section 2.3, this equation can be used both for homogeneous ($D = 0$) and inhomogeneous ($D \neq 0$) plane waves, propagating in perfectly elastic ($Q_S^{-1} = 0$) and viscoelastic ($Q_S^{-1} > 0$) isotropic media. Inserting (106) into (93), we obtain the same expressions for phase velocity $\mathcal{C}$, attenuation amplitude ratio δ and attenuation angle γ as in (60) with (64). These quantities do not depend on the signs of $\mathbf{n}$ and D. The attenuation angle γ of inhomogeneous plane waves propagating in viscoelastic isotropic medium may be an arbitrary angle less than $\frac{1}{2}\pi$, $0 \leq \gamma < \frac{1}{2}\pi$. Thus, the boundary attenuation angle γ^* equals $\frac{1}{2}\pi$ in this case, and is independent of $\mathbf{n}$ and $\mathbf{d}$. The inhomogeneous plane waves can also propagate in perfectly elastic isotropic media, but the attenuation angle must equal $\frac{1}{2}\pi$.This discrete attenuation angle does not depend on $\mathbf{n}$ and $\mathbf{d}$, but phase velocity $\mathcal{C}$ depends on inhomogeneity strength $d = |D|$. The solutions σ_1, corresponding to inhomogeneous plane waves propagating along $\mathbf{n}$, are suitably supplemented by the solutions σ_2, corresponding to inhomogeneous plane waves propagating against $\mathbf{n}$. Quantities $\mathcal{C}$, δ and γ remain the same for both directions of propagation. See also (105).

2. In **perfectly elastic anisotropic media**, elastic moduli C_{44}, C_{66} and C_{46} are real-valued. Consequently, also $\bar{C}^{(1)}$, $\bar{C}^{(2)}$ and Δ are real-valued, and (98) yields

$$\mathrm{Re}\sigma_{1,2} = \pm[1 + D^2\Delta/\rho^2 V_S^2]^{1/2}/V_S \ , \qquad \mathrm{Im}\sigma_{1,2} = -D\bar{C}^{(2)}/\rho V_S^2 \ . \tag{107}$$

Quantities $\bar{C}^{(2)}$ and V_S depend on $\mathbf{n}$, but Δ is independent of it. V_S is the real-valued velocity of plane qSH waves, $V_S^2 = \bar{C}^{(1)}/\rho = [C_{66}n_1^2 + C_{44}n_3^2 + 2C_{46}n_1n_3]/\rho$. As matrices (75) are assumed to be positive definite, V_S is positive for any $\mathbf{n}$. Quantity $\bar{C}^{(2)}$, however, may be any real-valued number. As we can see from (107), the relations (104) between $\sigma_1(\mathbf{n}, D)$ and $\sigma_2(\mathbf{n}, -D)$ are fully satisfied.

The first equation of (107) gives a very simple relation for phase velocities $\mathcal{C}_1$ and $\mathcal{C}_2$:

$$\mathcal{C}_{1,2} = V_S/[1 + D^2\Delta/\rho^2 V_S^2]^{1/2} \ . \tag{108}$$

Phase velocities $\mathcal{C}_1$ and $\mathcal{C}_2$ correspond to propagation along/against $\mathbf{n}$ and are the same. They do not depend on the sign of D. For given $\mathbf{n}$, the phase velocity is maximum for

$D = 0$, when it equals V_S. With increasing $|D|$ it decreases, and for the limiting case of infinite inhomogeneity strength $d = |D|$ it equals zero.

Let us now discuss the attenuation angles γ_1 and γ_2 of plane waves, propagating along/against $\mathbf{n}$ in a perfectly elastic anisotropic medium. Using (107) and (93), we obtain:

$$\cos\gamma_{1,2} = \mp\frac{D}{|D|}\,\frac{\bar{C}^{(2)}/\rho V_S^2}{[(\bar{C}^{(2)}/\rho V_S^2)^2+1]^{1/2}}\ . \tag{109}$$

This can be alternatively expressed in a simpler form

$$\mathrm{cotan}\gamma_{1,2} = \mp(D/|D|)\bar{C}^{(2)}/\rho V_S^2\ . \tag{110}$$

Thus, we have obtained an interesting result. The attenuation angles γ_1 and γ_2 of an inhomogeneous qSH plane wave propagating in the symmetry plane x_1, x_3 of a perfectly elastic monoclinic anisotropic medium do not depend on inhomogeneity strength d, but only on the sign of D. This is a result similar to that obtained for inhomogeneous plane waves propagating in a perfectly elastic isotropic medium, where the attenuation angles also do not depend on inhomogeneity strength d. There is, however, a basic difference between both cases. In the perfectly elastic isotropic medium, attenuation angles γ_1 and γ_2 always equal $\frac{1}{2}\pi$, and are independent of $\mathbf{n}$ and $\mathbf{d}$. In the perfectly elastic anisotropic medium treated here, the attenuation angles equal $\frac{1}{2}\pi$ only for such $\mathbf{n}$, for which $\bar{C}^{(2)} = 0$, i. e. for $\mathbf{n}$ satisfying the relation $(C_{66} - C_{44})n_1 n_3 + C_{46}(n_3^2 - n_1^2) = 0$. For all other directions $\mathbf{n}$, attenuation angles γ are different from $\frac{1}{2}\pi$, although they may be close to it (for small $\bar{C}^{(2)}$). They depend on $\mathbf{n}$ and on the sign of D. If $\mathbf{n}$ is fixed, we obtain $\mathrm{cotan}\gamma_1 > 0$ (i. e. $\gamma_1 < \frac{1}{2}\pi$) for one sign of D, and $\mathrm{cotan}\gamma_1 < 0$ (i. e. $\gamma_1 > \frac{1}{2}\pi$) for the other sign of D. Both values of $\mathrm{cotan}\gamma_1$ (for $D > 0$ and for $D < 0$) are, however, the same in magnitude. A quite analogous result is obtained also for γ_2.

Let us now discuss the mutual relations between γ_1 and γ_2 (corresponding to the wavefront propagating along/against $\mathbf{n}$). For fixed $\mathbf{n}$ and for fixed wavefront attenuation vector $\mathbf{d}$, we obtain $\mathrm{cotan}\gamma_1(\mathbf{n}, D) = -\mathrm{cotan}\gamma_2(\mathbf{n}, D)$. Thus, we can express attenuation angles γ_1 and γ_2 using the relation $\gamma_{1,2} = \frac{1}{2}\pi \pm \bar{\gamma}$, where $\bar{\gamma}$ is fixed. This indeed shows that the direction of attenuation vector $\mathbf{A}$ is the same both for $\mathbf{N} = \mathbf{n}$ and $\mathbf{N} = -\mathbf{n}$, but that this vector is not perpendicular to the propagation vector (as in isotropic perfectly elastic media). Moreover, for the opposite direction of the wavefront attenuation vector $\mathbf{d}$, an opposite direction of $\mathbf{A}$ is obtained.

3. **Weakly inhomogeneous plane waves**. We shall now discuss the properties of the inhomogeneous waves corresponding to small D, and call these waves weakly inhomogeneous plane waves. For small D, the square root in (98) can be expanded in terms of D^2. Retaining only the first two terms, we obtain

$$\sigma_{1,2} \doteq -\mathrm{i}D\bar{C}^{(2)}/\bar{C}^{(1)} \pm [\rho/\bar{C}^{(1)}]^{1/2} \pm \tfrac{1}{2}D^2\Delta/\rho^{1/2}\bar{C}^{(1)3/2}\ . \tag{111}$$

For $\mathrm{Re}\sigma_{1,2}$, we obtain from (111)

$$\mathrm{Re}\sigma_{1,2} \doteq \pm a + bD \pm cD^2\ , \tag{112}$$

where

$$\begin{aligned} a &= \mathrm{Re}(\rho/\bar{C}^{(1)})^{1/2} , \qquad b = \mathrm{Im}(\bar{C}^{(2)}/\bar{C}^{(1)}) , \\ c &= \mathrm{Re}(\Delta/\bar{C}^{(1)3/2})/2\sqrt{\rho} . \end{aligned} \tag{113}$$

Thus, the real parts of σ_1 and σ_2 can be approximated by quadratic parabolas for small D. The minimum (maximum) of these parabolas correspond to $D = D^M_{1,2}$, where $D^M_{1,2}$ are given by the relation

$$D^M_{1,2} = \mp b/2c . \tag{114}$$

The phase velocities $\mathcal{C}^M_{1,2}$ corresponding to $D^M_{1,2}$ are

$$\mathcal{C}^M_{1,2} = 1/(a - b^2/4c) . \tag{115}$$

Equations (114) and (115) have certain important consequences. For simplicity, we shall only consider eigenvalues σ_1 (corresponding to $\mathbf{N} = \mathbf{n}$); the results for σ_2 could be obtained from those for σ_1 in a simple way. We remind the reader that $\mathrm{Re}\sigma_1 > 0$ and $\mathrm{Re}\sigma_2 < 0$.

The basic role in the discussion of phase velocities $\mathcal{C}_1$ is played by quantity $b = \mathrm{Im}(\bar{C}^{(2)}/\bar{C}^{(1)})$, see (113) and (87). For $b = 0$, the minimum of the parabola (112) corresponds to $D = D^M_1 = 0$, see (114). Thus, the maximum phase velocity is obtained for homogeneous plane waves ($D = 0$), and is given by relation $\mathcal{C}^M_1 = \mathcal{C}_{\mathrm{hom}}$,

$$\mathcal{C}_{\mathrm{hom}} = 1/a = 1/\mathrm{Re}[\rho/(C_{66}n_1^2 + C_{44}n_3^2 + 2C_{46}n_1n_3)]^{1/2} . \tag{116}$$

With $|D|$ increasing, the phase velocity decreases. The situation, however, is quite different for $b \neq 0$. The maximum phase velocity $\mathcal{C}^M_1$ does not then correspond to the homogeneous plane wave, but to an inhomogeneous plane wave with D given by relation $D^M_1 = b/2c$, and is given by (115). It is always larger than the phase velocity $\mathcal{C}_{\mathrm{hom}}$, corresponding to the homogeneous plane wave:

$$\mathcal{C}^M_1 \geq \mathcal{C}_{\mathrm{hom}} . \tag{117}$$

The general conclusion for the phase velocity of plane waves propagating in an anisotropic viscoelastic medium is then as follows: The phase velocity of an inhomogeneous plane wave may be larger than the phase velocity of the homogeneous plane wave propagating in the same direction.

Quantity b also plays an important role in discussing the attenuation angles γ_1 and γ_2 of inhomogeneous plane waves. It immediately follows from (112) and (93) that

$$\gamma_1(\mathbf{n}, D) \neq \gamma_1(\mathbf{n}, -D) , \tag{118}$$

if $b \neq 0$. Both attenuation angles $\gamma_1(\mathbf{n}, D)$ and $\gamma_2(\mathbf{n}, -D)$ are equal only if $b = 0$. See also (102).

4. **Infinite inhomogeneity strength** d. We shall now briefly discuss the limiting case of inhomogeneous qSH plane waves with infinite inhomogeneity strength ($d \to \infty$), propagating in a viscoelastic anisotropic medium. The limiting case of $d \to \infty$ has a

close relation to the zero phase velocities ($\mathcal{C} = 0$), and to the boundary attenuation angle $\gamma = \gamma^*$. For $d \to \infty$, the relations (99) for eigenvalues $\sigma_{1,2}$ simplify:

$$\begin{aligned} \mathrm{Re}\sigma_{1,2} &= D\,\mathrm{Im}(\bar{C}^{(2)}/\bar{C}^{(1)}) \pm |D|\,\mathrm{Re}(\Delta/\bar{C}^{(1)2})^{1/2} , \\ \mathrm{Im}\sigma_{1,2} &= -D\,\mathrm{Re}(\bar{C}^{(2)}/\bar{C}^{(1)}) \pm |D|\,\mathrm{Im}(\Delta/\bar{C}^{(1)2})^{1/2} . \end{aligned} \tag{119}$$

The first equation of (119) implies a simple relation for the phase velocities $\mathcal{C}_{1,2}$, corresponding to $d \to \infty$:

$$\mathcal{C}_{1,2} = \frac{1}{|D|}\,\frac{1}{|\mathrm{Re}(\Delta/\bar{C}^{((1)2})^{1/2} \pm (D/|D|)\mathrm{Im}(\bar{C}^{(2)}/\bar{C}^{(1)})|} \to 0 . \tag{120}$$

Thus, for $d \to \infty$, we obtain zero phase velocities ($\mathcal{C}_{1,2} = 0$). Of course, zero phase velocity $\mathcal{C} = 0$ is not realistic; it corresponds to the limiting case of $d \to \infty$, which cannot in fact accur. This limiting case, however, plays an important role in the discussion of inhomogeneous plane waves, as it corresponds to the boundary attenuation angles $\gamma^*_{1,2}$.

Using (93) and (119), we obtain

$$\mathrm{cotan}\gamma^*_{1,2} = \mp(D/|D|)\mathrm{Re}(\bar{C}^{(2)}/\bar{C}^{(1)}) + \mathrm{Im}(\Delta/\bar{C}^{(1)2})^{1/2} . \tag{121}$$

It is interesting to compare (121) with (110). As we can see, (121) yields (110) for perfectly elastic anisotropic media. The boundary attenuation angles $\gamma^*_{1,2}$ of inhomogeneous plane waves propagating in viscoelastic anisotropic media reduce to the discrete attenuation angles $\gamma_{1,2}$ of inhomogeneous plane waves propagating in perfectly elastic anisotropic media, if the imaginary parts of the viscoelastic moduli vanish. Thus, Equation (110) can be used as the first estimate of the boundary attenuation angles $\gamma^*_{1,2}$ also in weakly dissipative media.

The boundary attenuation angles γ^*_1 and γ^*_2 of inhomogeneous plane waves propagating in viscoelastic anisotropic media are, in general, different from $\frac{1}{2}\pi$. They depend on $\mathbf{n}$ and on the sign of D. The boundary attenuation angle γ^*_1 corresponds to the inhomogeneous plane waves propagating in the direction of $\mathbf{N} = \mathbf{n}$, and the boundary attenuation angle γ^*_2 to the plane waves propagating in the opposite direction, $\mathbf{N} = -\mathbf{n}$. There is no symmetricity between $\gamma^*_1(n, D)$ and $\gamma^*_1(n, -D)$. The same applies to $\gamma^*_2(n, D)$ and $\gamma^*_2(n, -D)$. It is, however, simple to see from (119) that the boundary attenuation angles γ^*_1 and γ^*_2 satisfy the mutual symmetry relations $\mathrm{cotan}\gamma^*_2(\mathbf{n}, -D) = \mathrm{cotan}\gamma^*_1(\mathbf{n}, D)$; see also (105). The simple relation $\mathrm{cotan}\gamma_1(\mathbf{n}, D) = -\mathrm{cotan}\gamma_2(\mathbf{n}, D)$, valid in anisotropic perfectly elastic media, however, is no longer valid in viscoelastic anisotropic media.

Now consider an arbitrary fixed $\mathbf{n}$ and nonnegative $D's$, $0 \le D < \infty$. The attenuation angle $\gamma_1(\mathbf{n}, D)$, corresponding to an inhomogeneous plane wave propagating along $\mathbf{n}$ (i. e. $\mathbf{N} = \mathbf{n}$), is well defined for any D, and is continuous with respect to D. For $D = 0$, attenuation angle $\gamma_1(\mathbf{n}, D)$ vanishes, and for the limiting case of $D = \infty$, it equals $\gamma^*_1(\mathbf{n}, D)$. There are no "forbidden regions" with respect to D. The situation is quite analogous for $-\infty < D \le 0$; only the boundary attenuation angle $\gamma^*_1(\mathbf{n}, D)$ should be replaced by $\gamma^*_1(\mathbf{n}, -D)$. As $\mathrm{cotan}\,\gamma_2(\mathbf{n}, D) = \mathrm{cotan}\,\gamma_1(\mathbf{n}, -D)$, the same conclusions are also valid for $\gamma_2(\mathbf{n}, D)$ and $\gamma_2(\mathbf{n}, -D)$.

One important comment. The sign of D specifies the orientation of the wavefront attenuation vector with respect to $\mathbf{n}$, (not $\mathbf{N}$), see (96). Thus, for $D > 0$, the wavefront attenuation vector $\mathbf{d}$ is always oriented "to the right" of $\mathbf{n}$, both for $\mathbf{N} = \mathbf{n}$ and $\mathbf{N} = -\mathbf{n}$. This is a different situation from that considered in Equations (103), where unit vector $\mathbf{n}$ was rotated to the opposite direction.

5. **Solutions in a conveniently chosen Cartesian coordinate system**. The computation of eigenvalues σ simplifies if we choose convenient Cartesian coordinate system. Of course, we have to apply proper transformation equations for viscoelastic moduli c_{ijkl} when passing from one Cartesian coordinate system to another. The general equations for computing eigenvalues σ in various Cartesian coordinate systems are given in Červený (2003). The method is particularly useful in more complex anisotropy systems. For inhomogeneous qSH plane waves propagating in the plane of symmetry of a monoclinic anisotropic viscoelastic medium, the relevant simplifications are not great. Nevertheless, we shall briefly outline the procedure here, as the comparison of the two approaches is very transparent and simple in this case.

We choose the Cartesian coordinate system x_i in such a way that the x_3-axis coincides with unit vector $\mathbf{n}$,

$$\mathbf{n} \equiv (0, 0, 1) \ . \tag{122}$$

The x_2-axis is again taken perpendicular to the plane of symmetry, and axis x_1 in the plane of symmetry. The wavefront then coincides with the x_1, x_2 plane, and the Cartesian component d_1 of the wavefront attenuation vector $\mathbf{d}$ equals D. Without going into details, we shall only present the expressions for $\sigma_{1,2}$ for the case of (122). The derivation of eigenvalues $\sigma_{1,2}$ would be the same as the general derivation for $\mathbf{n} \equiv (n_1, 0, n_3)$ used above, only the individual expressions involved would be simplified. For more details refer to Section 3.3.2, and Červený (2003). The expressions for $\sigma_{1,2}$ for the case of (122) read

$$\sigma_{1,2} = -\mathrm{i}DC_{46}/C_{44} \pm (\rho/C_{44} + D^2\Delta/C_{44}^2)^{1/2} \ . \tag{123}$$

Here $\Delta = C_{44}C_{66} - C_{46}^2$, and C_{44}, C_{66} and C_{46} are complex-valued viscoelastic moduli. Of course, Equation (123) could be simply obtained from (98) using (122), but we wish to demonstrate the opposite case: to derive the general equation (98) from the special equation (123).

If we wish to compute $\sigma_{1,2}$ for an arbitrarily chosen unit normal $\mathbf{n}$, $\mathbf{n} \equiv (n_1, 0, n_3)$, we must first rotate $(n_1, 0, n_3)$ into $(0, 0, 1)$. This is done by applying the transformation matrix

$$\begin{pmatrix} n_3 & 0 & -n_1 \\ 0 & 1 & 0 \\ n_1 & 0 & n_3 \end{pmatrix} \ . \tag{124}$$

This transformation matrix transforms the unrotated viscoelastic moduli C_{44}, C_{66} and C_{46} into new viscoelastic moduli C'_{44}, C'_{66}, C'_{46}:

$$\begin{aligned} C'_{44} &= n_3^2 C_{44} + n_1^2 C_{66} + 2n_1 n_3 C_{46} \ , \\ C'_{66} &= n_3^2 C_{66} + n_1^2 C_{44} - 2n_1 n_3 C_{46} \ , \\ C'_{46} &= (C_{66} - C_{44}) n_1 n_3 + C_{46}(n_3^2 - n_1^2) \ . \end{aligned} \tag{125}$$

Equation (123) then reads

$$\sigma_{1,2} = -\mathrm{i}DC'_{46}/C'_{44} \pm (\rho/C'_{44} + D^2\Delta'/C'^2_{44})^{1/2} \ . \tag{126}$$

Here $\Delta' = C'_{44}C'_{66} - C'^2_{46}$. It is easy to realize that the quantity Δ' is invariant with respect to rotation in plane x_1, x_3. This yields $\Delta' = \Delta = C_{44}C_{66} - C^2_{46}$, and (126) fully coincides with (98), as expected. The difference is only in the derivation; the numerical algorithms to compute $\sigma_{1,2}$ will be the same.

3.3.2 qSV and qP waves

Similarly as in the projection specification, eigenvalues σ can be determined by solving the characteristic equation (92), where the 4×4 complex-valued in-plane matrix $\mathbf{A}^{\|}$ is constructed from the 6×6 matrix $\mathbf{A}$, given by (34)–(36), by removing the second and fifth rows and columns. We shall again use the Cartesian coordinate system with $\mathbf{n} \equiv (0,0,1)$ and with $p_1^\Sigma = \mathrm{i}D$, $p_2^\Sigma = p_3^\Sigma = 0$. We further introduce the 2×2 complex-valued matrices $\mathbf{D}^{\|}_{IJ}$ as

$$\begin{array}{ll} \mathbf{A}^{\|}_{11} = -\mathrm{i}D\mathbf{D}^{\|}_{11} \ , & \mathbf{A}^{\|}_{12} = -\mathbf{D}^{\|}_{12} \ , \\ \mathbf{A}^{\|}_{21} = -\rho\mathbf{I} + D^2\mathbf{D}^{\|}_{21} \ , & \mathbf{A}^{\|}_{22} = -\mathrm{i}D\mathbf{D}^{\|}_{22} \ , \end{array} \tag{127}$$

where $\mathbf{A}^{\|}_{IJ}$ are 2×2 partition matrices of the 4×4 in-plane matrix $\mathbf{A}^{\|}$. The characteristic equation (92) for qP and qSV plane waves then takes the form

$$\det\begin{pmatrix} \mathrm{i}D\mathbf{D}^{\|}_{11} + \sigma\mathbf{I} & \mathbf{D}^{\|}_{12} \\ \rho\mathbf{I} + D^2\mathbf{D}^{\|}_{21} & \mathrm{i}D\mathbf{D}^{\|}_{22} + \sigma\mathbf{I} \end{pmatrix} = 0 \ . \tag{128}$$

$\mathbf{I}$ is the 2×2 identity matrix, and $\mathbf{D}^{\|}_{IJ}$ are given as

$$\begin{array}{llll} \mathbf{D}^{\|}_{11} = \dfrac{1}{a}\begin{pmatrix} b & a \\ c & 0 \end{pmatrix} \ , & \mathbf{D}^{\|}_{12} = \dfrac{1}{a}\begin{pmatrix} C_{33} & -C_{35} \\ -C_{35} & C_{55} \end{pmatrix} \ , \\ \mathbf{D}^{\|}_{21} = \dfrac{1}{a}\begin{pmatrix} C_{11}a - d & 0 \\ 0 & 0 \end{pmatrix} \ , & \mathbf{D}^{\|}_{22} = \dfrac{1}{a}\begin{pmatrix} b & c \\ a & 0 \end{pmatrix} \ . \end{array} \tag{129}$$

Quantities a, b, c, d depend on the viscoelastic moduli only, and

$$\begin{array}{ll} a = C_{33}C_{55} - C^2_{35} \ , & b = C_{33}C_{15} - C_{55}C_{13} \ , \\ c = C_{55}C_{13} - C_{15}C_{35} \ , & d = C_{33}C^2_{15} + C_{55}C^2_{13} \ . \end{array} \tag{130}$$

Explicitly, the characteristic equation (128) can be expressed as follows:

$$\det\begin{pmatrix} \mathrm{i}Db + \sigma a & \mathrm{i}Da & C_{33} & -C_{35} \\ \mathrm{i}Dc & \sigma a & -C_{35} & C_{55} \\ \rho a + D^2(C_{11}a - d) & 0 & \mathrm{i}Db + \sigma a & \mathrm{i}Dc \\ 0 & \rho a & \mathrm{i}Da & \sigma a \end{pmatrix} = 0 \ . \tag{131}$$

The characteristic equation (131) represents a quartic algebraic equation in σ. In principle, it can be solved analytically, but the analytic solutions are rather cumbersome. It can, however, be solved simply by conventional numerical methods.

For the special case of $C_{15} = 0$ and $C_{35} = 0$, quantities a, b, c and d are given by simple relations:

$$a = C_{33}C_{55} \ , \quad b = -C_{13}C_{55} \ , \quad c = C_{13}C_{55} \ , \quad d = C_{13}^2 C_{55} \ . \tag{132}$$

The quartic characteristic equation (131) for σ then reduces to a quadratic equation for σ^2, and can be simply solved analytically. The case of $C_{15} = 0$ and $C_{35} = 0$ corresponds to the plane of symmetry of an orthorhombic medium.

Similarly as (123), Equation (131) can be used to determine the eigenvalues of inhomogeneous qSV and qP plane waves propagating along the x_3-axis, as $\mathbf{n}$ pressumably equals $(0, 0, 1)$. To obtain the eigenvalues for a general orientation of $\mathbf{n}$ in symmetry plane x_1, x_3, we must rotate the Cartesian coordinate system in such a way as to transform $\mathbf{n} \equiv (n_1, 0, n_3)$ into $\mathbf{n} \equiv (0, 0, 1)$. The relevant transformation matrix is given by (124). Applying this transformation matrix to viscoelastic moduli C_{11}, C_{33}, C_{13}, C_{15}, C_{35} and C_{55}, we obtain new viscoelastic moduli C'_{11}, C'_{33}, C'_{13}, C'_{15}, C'_{35} and C'_{55}. The viscoelastic moduli $C_{\alpha,\beta}$ in (131) must then be replaced by $C'_{\alpha,\beta}$, and eigenvalues σ must be determined for the primed viscoelastic moduli. It is important to realize that the case of $C_{15} = 0$ and $C_{35} = 0$ does not imply $C'_{15} = 0$ and $C'_{35} = 0$. Thus, if we wish to consider an arbitrary orientation of $\mathbf{n}$ in symmetry plane x_1, x_3, the computation of eigenvalues σ for qSV and qP waves requires that $C_{15} \neq 0$ and $C_{35} \neq 0$ in (131) also be considered for the orthorhombic medium. Analogously, the quartic characteristic equation (131) should also be used for inhomogeneous qSV and qP plane waves propagating in a transversely isotropic medium.

The characteristic equation (131) is very general. It can be used both for homogeneous and inhomogeneous plane waves, for viscoelastic and perfectly elastic media, for isotropic and anisotropic media. Quantity D may be chosen arbitrarily, $-\infty < D < \infty$. The characteristic equation (131) depends on D in a very simple way, as a, b, c, d do not depend on D at all. Once eigenvalues σ have been determined from (131), Equations (93) can be used to determine all other quantities related to the inhomogeneous plane waves under consideration ($\mathbf{P}, \mathbf{A}, \mathbf{N}, \mathbf{M}, \mathcal{C}, \delta$ and $\cos\gamma$).

For $D \neq 0$, the characteristic equation (131) can be modified by introducing new eigenvalues w:

$$\sigma = Dw \ . \tag{133}$$

Equation (131) yields the following quartic characteristic equation for w,

$$\det \begin{pmatrix} ib + wa & ia & C_{33} & -C_{35} \\ ic & wa & -C_{35} & C_{55} \\ \rho a/D^2 + (C_{11}a - d) & 0 & ib + wa & ic \\ 0 & \rho a/D^2 & ia & wa \end{pmatrix} = 0 \ . \tag{134}$$

Both characteristic equations, (131) and (134), are fully equivalent, they can be used alternatively. The characteristic equation (134) depends on D in an even simpler form than (131). Only two terms of sixteen depend on D in such a way that they contain factor $1/D^2$. Equations (93) remain valid also in this case; we only insert $\mathrm{Re}\sigma = D\,\mathrm{Re}w$ and $\mathrm{Im}\sigma = D\,\mathrm{Im}w$. Particularly simple expressions are obtained for phase velocity $\mathcal{C}$, amplitude attenuation δ and attenuation angle γ:

$$\mathcal{C} = 1/|D||\mathrm{Re}w| \ , \qquad \delta = [(\mathrm{Im}w)^2 + 1]^{1/2}/|\mathrm{Re}w| \ , \quad \mathrm{cotan}\gamma = \epsilon_w \mathrm{Im}w \ . \tag{135}$$

Here $\epsilon_w = \mathrm{Re}w/|\mathrm{Re}w| = \pm 1$.

Let us now consider the limiting case of $|D| \to \infty$. The relevant characteristic equation for w is then simply obtained from (134),

$$\det \begin{pmatrix} ib + wa & ia & C_{33} & -C_{35} \\ ic & wa & -C_{35} & C_{55} \\ C_{11}a - d & 0 & ib + wa & ic \\ 0 & 0 & ia & wa \end{pmatrix} = 0 \ . \tag{136}$$

It immediately follows from (135) and (136) that, in this case, phase velocity $\mathcal{C}$ vanishes, $\mathcal{C} = 0$. The corresponding attenuation angle γ specifies the boundary attenuation angle γ^*, and is given by the relation $\mathrm{cotan}\gamma^* = \epsilon_w \mathrm{Im}w$, where w is a solution of (136).

3.4 Discussion of results

The properties of inhomogeneous qSH plane waves propagating in the plane of symmetry of a monoclinic anisotropic viscoelastic medium are not as simple as the properties of inhomogeneous plane waves propagating in a viscoelastic isotropic medium. The theoretical treatment based on the mixed specification of the slowness vector has several important advantages in comparison with the theoretical treatment based on the directional specification. It offers explicit expressions for slowness vector $\mathbf{p}$, phase velocity $\mathcal{C}$, attenuation amplitude factor δ, and attenuation angle γ, as functions of the viscoelastic moduli, direction of propagation $\mathbf{n}$, and wavefront attenuation vector $\mathbf{d}$. The wavefront attenuation vector can be specified by inhomogeneity strength d, $0 \leq d < \infty$, or by the related parameter D, $-\infty < D < \infty$, see (94)– (96). The directional specification of the slowness vector does not yield such explicit equations. It only yields a system of equations which must be solved numerically to obtain the relevant solutions. For certain attenuation angles γ, the system of equations does not give any physically meaningful solution (forbidden directions).

We shall now briefly discuss the basic difference between the directional and mixed specifications of the slowness vector. The difference is only formal in viscoelastic isotropic media, but plays a very important role in viscoelastic anisotropic media. In the directional specification, the unit vectors $\mathbf{N}$ and $\mathbf{M}$ are à priori known. Consequently, attenuation angle γ (defined as the angle made by $\mathbf{N}$ and $\mathbf{M}$) is also à priori known, and may be treated as an input parameter in the problem. Attenuation angle γ, however, cannot be chosen arbitrarily, it must be less than the boundary attenuation angle γ^*. The boundary attenuation angle γ^*, however, is not known à priori in anisotropic viscoelastic media (although it is known in isotropic viscoelastic media, where it always equals $\frac{1}{2}\pi$). If we choose $\gamma > \gamma^*$, we obtain a non-physical solution, corresponding to a forbidden direction, and we must ignore the solution. The situation is considerably simpler in the mixed specification; input parameter D (related to inhomogeneity strength d) may be chosen quite arbitrarily, $-\infty < D < \infty$. Proper solutions exist for any choice of D; no choice of D generates forbidden directions. Attenuation angle γ is not pressumably known, but it is obtained as a result of computation. It is always less than the boundary attenuation angle γ^*.

Another advantage of the mixed specification is that the whole treatment is based on the solution of a conventional eigenvalue problem. For general anisotropic viscoelastic media, we must determine the eigenvalues of a 6×6 complex-valued matrix $\mathbf{A}$, see (34)–(36), for qP and qSV plane waves propagating in the symmetry plane of a monoclinic anisotropic viscoelastic media, the 6×6 matrix $\mathbf{A}$ reduces to a 4×4 matrix $\mathbf{A}^{\parallel}$, see (127)–(130), and for qSH plane waves in the same symmetry plane, $\mathbf{A}$ reduces to a 2×2 matrix $\mathbf{A}^{\perp}$, see (84)– (87). Contrary to it, the directional specification requires a complicated equation, containing *analytical* expressions for complex-valued eigenvalues of a 3×3 complex-valued matrix to be solved numerically. See (14) for the general case. For qP and qSV plane waves propagating in the symmetry plane of a monoclinic anisotropic viscoelastic medium, it is sufficient to know the analytical expressions for the eigenvalues of a 2×2 complex-valued matrix, see (79) with (69). These equations are very complicated, and their numerical solutions would be rather time consuming. It is not guaranteed that the equation has a physically meaningful solution (forbidden directions), and that it has only one solution. Only for inhomogeneous qSH plane waves, does the equation simplify, see (73) with (74). Even in this case, however, the problem of forbidden directions is not removed.

The final equations obtained under the mixed specification are valid quite generally and universally. They are valid for inhomogeneous plane waves propagating in viscoelastic anisotropic and isotropic media, only the medium parameters should be specified properly. The same equations are valid for homogeneous ($D = 0$) and inhomogeneous ($D \neq 0$) plane waves, even for the limiting case of infinite inhomogeneity strength ($|D| \to \infty$). Infinite inhomogeneity strength yields zero phase velocities ($\mathcal{C} = 0$) and the boundary attenuation angle γ^*.

Now we shall compare certain properties of inhomogeneous plane waves propagating in anisotropic and isotropic viscoelastic media. Although we have derived these properties only for qSH inhomogeneous plane waves propagating in the plane of symmetry of a monoclinic anisotropic viscoelastic media, the properties also apply to general anisotropic viscoelastic media.

a) In *anisotropic viscoelastic media*, the phase velocities $\mathcal{C}$ of certain inhomogeneous plane waves are higher than the phase velocities of the relevant homogeneous plane waves, propagating in the same direction. The reader is reminded that the phase velocities of inhomogeneous plane waves are always smaller than the phase velocities of the relevant homogeneous plane waves in viscoelastic isotropic media.

b) In *anisotropic viscoelastic media*, attenuation angle γ is always less than the boundary attenuation angle γ^*, $0 \leq \gamma < \gamma^*$, which is, in general, different from $\frac{1}{2}\pi$, and depends on the direction of propagation $\mathbf{N}$ and on the direction of attenuation vector $\mathbf{d}$. Contrary to this, the boundary attenuation angle γ^* always equals $\frac{1}{2}\pi$ and is independent of $\mathbf{N}$ and $\mathbf{d}$ in viscoelastic isotropic media.

c) The attenuation angle γ of inhomogeneous plane waves propagating in *anisotropic perfectly elastic media* does not depend on inhomogeneity strength d. However, it depends on the directions of $\mathbf{N}$ and $\mathbf{d}$, and on the elastic moduli. Indeed, this attenuation angle γ corresponds to the limiting value of the boundary attenuation angle γ^* in viscoelastic media, if we decrease the imaginary parts of the viscoelastic moduli to zero. In isotropic perfectly elastic media, this angle always equals $\frac{1}{2}\pi$.

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