Strength of anisotropy, which generates triplications in transversely isotropic media

Václav Vavryčuk

Geophysical Institute, Academy of Sciences of the Czech Republic, Boční II/1401, 141 31 Praha 4, Czech Republic, E-mail: vv@ig.cas.cz

Abstract

Triplications and cusp edges can occur in homogeneous transverse isotropy (TI) provided the strength of anisotropy exceeds a critical value. The critical strength of anisotropy is 9.50% for axial triplication, 9.71% for basal triplication, 8.86% for oblique triplication, and 9.72% for double triplication. No TI with strength less than the critical can display triplications. On the other hand, high values of the strength of anisotropy do not guarantee the existence of triplications. Hence, observations of triplications on the wave surface cannot serve as a unique criterion defining strong TI.

I. INTRODUCTION

Triplications and cusp edges on the wavefront can significantly complicate modelling of wavefields. They produce energy focusing (Kravtsov & Orlov 1990; Wolfe 1998) and phase shifting of signals (Bakker 1998; Červený 2001). In homogeneous transverse isotropy, the triplications can occur for the SV wave only and can be classified into four different types (see Fig. 1): (1) off-axis triplication, (2) on-axis triplication near the symmetry plane, (3) on-axis triplication near the symmetry axis and (4) double on-axis triplication. The existence of triplications is conditioned by the existence of concave or saddle-shaped areas on the slowness surface (see Fig. 2). These areas are separated from the convex area by parabolic lines, formed by points of the zero Gaussian curvature (Grechka & Obolentseva 1993; Vavryčuk 2003a). Generally, the stronger the anisotropy, the larger part of the slowness surface is convex or concave and the more distinct the triplications are. If anisotropy is decreased, the concave or saddle-shaped areas are reduced and triplications are localized to narrower interval of angles. If

two cusps defining the triplication coalesce into one (the width of triplication reduces to only one direction), then we speak of the "incipient" triplication. In this case, the parabolic line separates two convex areas on the slowness surface and the medium represents a borderline between media with and without a triplication (Helbig 1994).

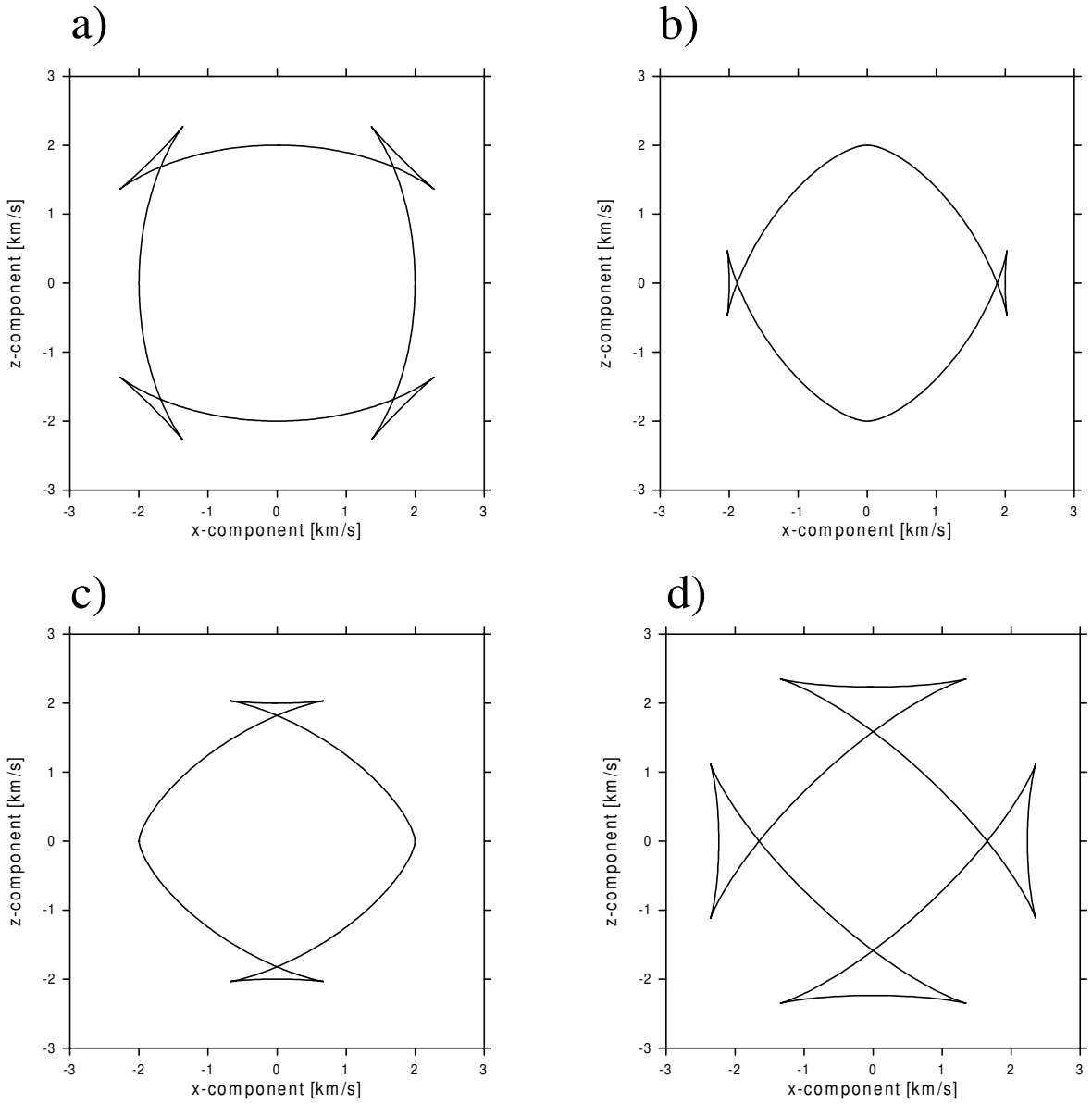

Figure 1. Types of triplications in transverse isotropy. Vertical sections of wave surfaces are shown for (a) oblique, (b) basal, (c) axial, and (d) double triplications.

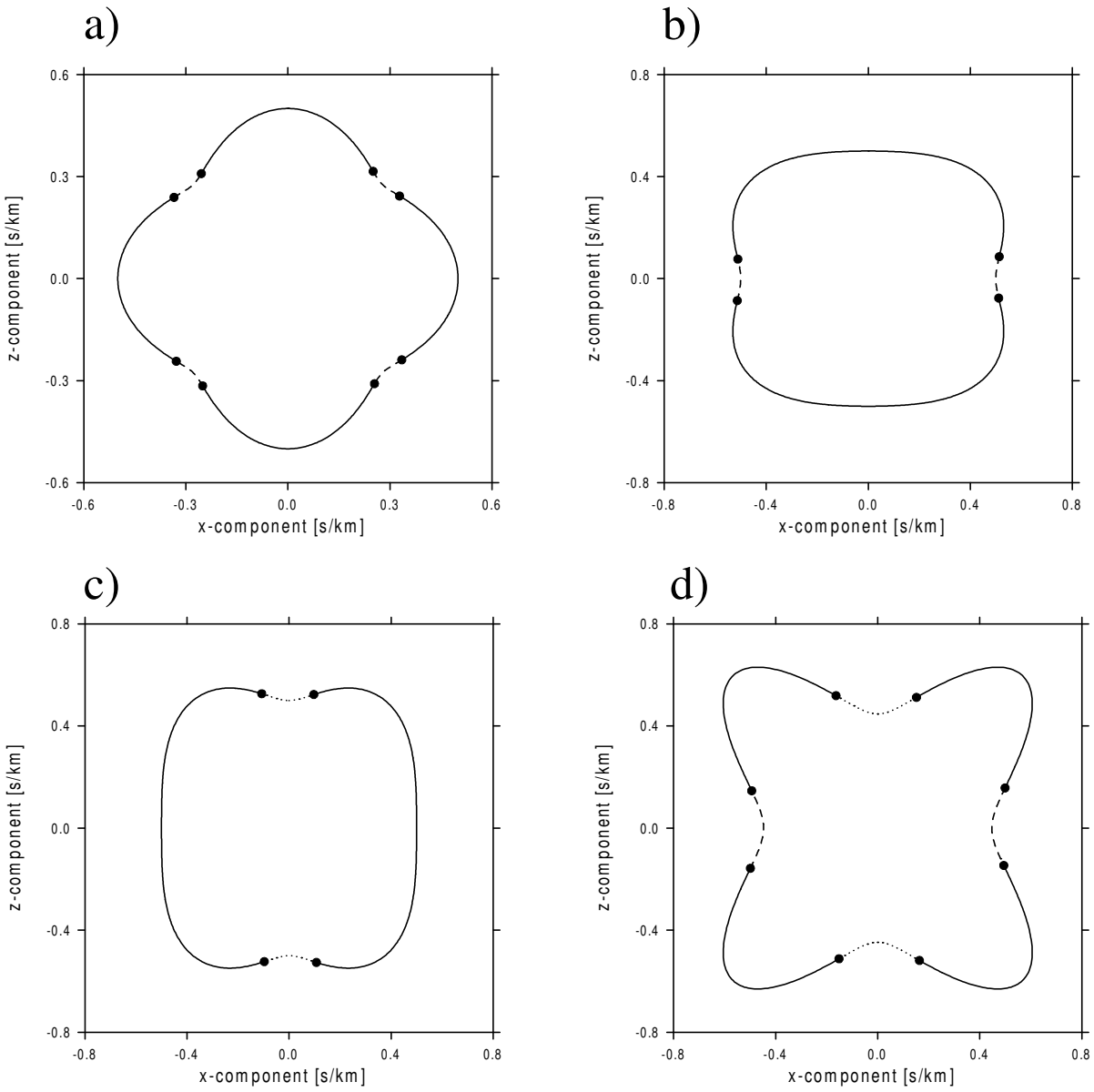

Figure 2. Vertical sections of slowness surfaces generating different types of triplications. For details, see the caption of Fig. 1. Parabolic points (marked by dots) separate convex (solid line), concave (dotted line) and saddle-shaped (dashed line) areas.

The conditions under which the triplications occur in transverse isotropy are well known (Helbig 1958; Musgrave 1970; Payton 1983; Musgrave & Payton 1984; Alshits & Chadwick 1997; Thomsen & Dellinger 2003; Vavryčuk 2003b). They yield inequalities, with which we can uniquely decide if transverse isotropy under study triplicates or not. The inequalities are, however, rather complicated and difficult to understand. Hence, we are able to classify any specific transverse isotropy, but we are not able to establish simple generalizations. We know that no triplication can occur in a sufficiently weak transverse isotropy (Vavryčuk 2003a), but we have no understanding how strong the anisotropy must be the triplications to be generated. The aim of this paper is to study which combinations of elastic parameters of transversely isotropic media generate triplications and to find the strength of anisotropy generating triplications. Finally, I shall address the point whether the occurrence of a triplication can be used as a criterion for distinguishing weak from strong transverse isotropy.

II. TRIPLICATION CONDITIONS

We consider TI media, which satisfy the "stability" conditions (Backus 1962, Helbig 1994),

$$a_{33}>0,\ a_{44}>0,\ a_{66}>0,\ a_{11}-a_{66}>0,\ a_{33}(a_{11}-a_{66})>a_{13}^2, \tag{1}$$

and the conditions, which prevent the P and SV slowness or phase-velocity surfaces to intersect one another:

$$a_{11}-a_{44}>0,\ a_{33}-a_{44}>0, \tag{2}$$

$$a_{13}+a_{44}>0, \tag{3}$$

where a_{kl} are the density normalized elastic parameters in the Voigt notation. Eqs (2) and (3) prevent the on-axis and off-axis P-SV intersections, respectively. Eq. (3) is also the condition for the so-called "normal polarization" of P and SV waves (Helbig & Schoenberg 1987). For analysis under less restrictive conditions, see Payton (1983) and Alshits & Chadwick (1997).

The following equation conditions the axial triplications (see Musgrave 1970, Eq. 8.3.4),

$$(a_{13}+a_{44})^2-a_{11}(a_{33}-a_{44})\le 0, \tag{4}$$

the basal triplications (see Musgrave[17], Eq. 8.3.5),

$$(a_{13}+a_{44})^2-a_{33}(a_{11}-a_{44})\le 0\,, \tag{5}$$

and the oblique triplications (Dellinger 1991, Eq. 2.19; Thomsen & Dellinger 2003, Eq. 9):

$$3a_{44}^2-(a_{13}+a_{44})^2-a_{44}(a_{33}+a_{11})+3a_{11}a_{33}-2\sqrt{(a_{33}-a_{44})(a_{11}-a_{44})}\,\frac{a_{11}a_{33}-a_{44}^2}{a_{13}+a_{44}}\le 0\ , \tag{6}$$

where the equality sign stands for the incipient triplication. The slowness angle θ_i of the incipient triplication is $\theta_i=0°$ for the axial triplication, $\theta_i=90°$ for the basal triplication and

$$\sin^2\theta_i=\frac{a_{33}-a_{44}}{a_{11}+a_{33}-2a_{44}}\,,\quad \cos^2\theta_i=\frac{a_{11}-a_{44}}{a_{11}+a_{33}-2a_{44}}\ , \tag{7}$$

for the oblique triplication.

III. ANISOTROPY PARAMETERS

Triplication conditions (4)-(6) can also be expressed in terms of parameters ε, γ, σ, κ, and a_{44}, which represent an alternative parameterization of TI (Thomsen 1986; Tsvankin & Thomsen 1994):

$$\varepsilon=\frac{a_{11}-a_{33}}{2a_{33}}\,, \tag{8}$$

$$\gamma=\frac{a_{66}-a_{44}}{2a_{44}}\ , \tag{9}$$

$$\sigma=\frac{1}{2a_{44}}\left(a_{11}-a_{44}-\frac{(a_{13}+a_{44})^2}{a_{33}-a_{44}}\right), \tag{10}$$

$$\kappa = a_{33} / a_{44} \ . \tag{11}$$

Parameters ε, γ, σ are called the anisotropy parameters and are frequently used for describing weak TI. For example, they control the angular variations of phase velocities in weak TI:

$$\left(c^{P}\right)^2 = \kappa a_{44}\left(1 + 2\varepsilon \sin^2\theta - 2\frac{\sigma}{\kappa}\sin^2\theta\cos^2\theta\right), \tag{12}$$

$$\left(c^{SV}\right)^2 = a_{44}\left(1 + 2\sigma \sin^2\theta\cos^2\theta\right), \tag{13}$$

$$\left(c^{SH}\right)^2 = a_{44}\left(1 + 2\gamma \sin^2\theta\right), \tag{14}$$

where θ is the angle between the slowness vector and the axis of symmetry. The anisotropy parameters become zero in isotropy and can thus serve as a measure of strength of TI. Therefore, expressing the triplication conditions using these parameters, provides a better understanding of how strong the anisotropy must be for the triplication to occur.

Eqs. (1)-(3) limit the values of κ, ε, γ, and σ. Eq. (2) yields for κ and ε,

$$2\varepsilon + 1 - \frac{1}{\kappa} > 0\,,\ \kappa > 1\ , \tag{15}$$

Eqs. (1), (2) and (10) yield for σ,

$$\frac{1}{2a_{44}}\left(a_{11} - a_{44} - \frac{\left(\sqrt{a_{11}a_{33}} + a_{44}\right)^2}{a_{33} - a_{44}}\right) < \sigma < \frac{1}{2a_{44}}\left(a_{11} - a_{44}\right), \tag{16}$$

and using Eq. (1), we obtain for γ

$$-\frac{1}{2} < \gamma < \frac{1}{2}\left(\frac{a_{11}a_{33} - a_{44}^2}{a_{33}a_{44}} - 1\right). \tag{17}$$

IV. TRIPLICATION CONDITIONS EXPRESSED USING ANISOTROPY PARAMETERS

Following Thomsen and Dellinger (2003), the conditions for the axial and basal triplications can be expressed as follows:

$$\sigma \le \sigma_c , \tag{18}$$

with σ_c expressed for the axial triplication ($\theta_i = 0°$) as

$$\sigma_c = -\frac{1}{2} , \tag{19}$$

and for the basal triplication ($\theta_i = 90°$) as

$$\sigma_c = -\frac{1}{2} - \varepsilon \frac{\kappa}{\kappa - 1} . \tag{20}$$

Parameter σ_c is the critical value of σ under which the incipient triplication occurs. The condition for oblique triplication (6) yields the following cubic equation:

$$\sigma_c^3 + A\sigma_c^2 + B\sigma_c + C = 0 , \tag{21}$$

where

$$A = \frac{1}{\kappa - 1}\left[\frac{3}{2}\kappa^2(2\varepsilon + 1) + \kappa(\varepsilon + 1) + \frac{3}{2}\right], \quad B = 2\frac{\kappa}{(\kappa - 1)^2}(\varepsilon + 1)\left[\kappa^2(2\varepsilon + 1) + 1\right],$$
$$C = -2\frac{\kappa^2}{(\kappa - 1)^2}\left[\kappa(2\varepsilon + 1)^2 - 2\varepsilon - 1\right]. \tag{22}$$

The solution of Eq. (21) yields the interval of values for the σ parameter under which the medium triplicates (for an equivalent solution in a different notation, see Payton 1983, Eq. 2.4.12):

$$\sigma \geq \sigma_c \,, \tag{23}$$

where

$$\sigma_c = u + v - \frac{1}{3} \,, \tag{24}$$

$$u = \left[-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} \right]^{\frac{1}{3}} \,, \quad v = \left[-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3} \right]^{\frac{1}{3}} \,, \tag{25}$$

$$p = B - \frac{1}{3}A^2 \;,\; q = -\frac{1}{3}AB + \frac{2}{27}A^3 + C \;. \tag{26}$$

Note that the triplication conditions do not depend on parameter a_{44}, which is only a scaling factor, and on parameter γ, which controls the propagation of the SH wave.

V. BEHAVIOR OF CRITICAL σ

We consider only the basal and oblique triplications, since the condition for the axial triplication is elementary. Figure 3 shows σ_c in dependence on ε and κ ranging in the intervals: $1 < \kappa < 10$ and $-0.2 < \varepsilon < 0.2$. The total interval for κ is subdivided into two intervals, $1 < \kappa < 2$ (lower plots) and $2 < \kappa < 10$ (upper plots), because the variation of σ for small κ is very strong. Fig. 3 indicates that σ_c attains negative values for the basal triplication, but positive values for the oblique triplication. Hence, the basal and oblique triplications cannot occur simultaneously under the same TI. The figure also shows that triplications can occur for σ very close to zero. If $\kappa \to 1$ and $\varepsilon = 0$, then $\sigma_c \to 0$. This means that no threshold value for σ_c exists for either triplication.

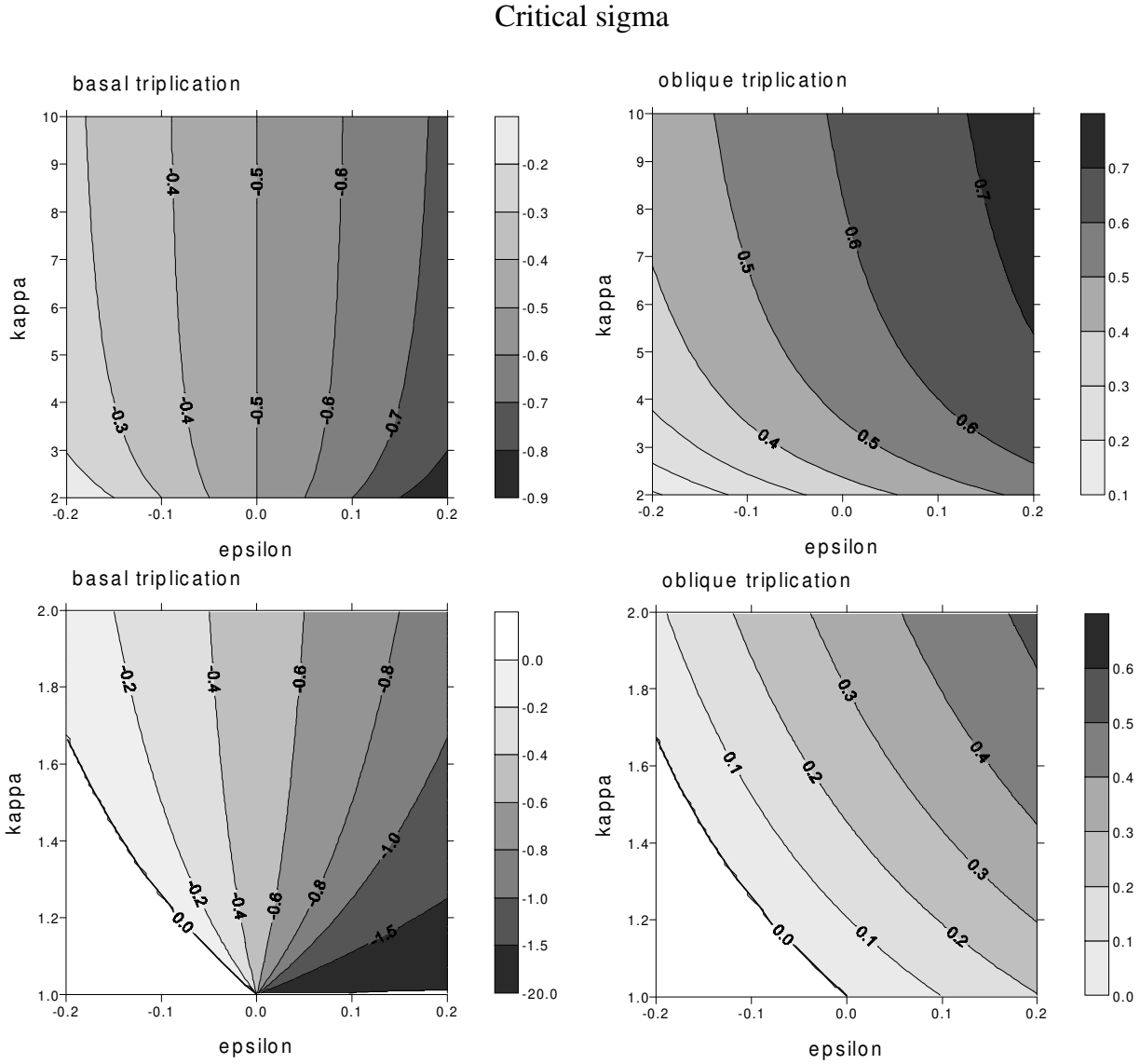

Figure 3. Behavior of the critical value of σ. The blank area in the left corner of the bottom plots is the area, where Eq. (15) is not satisfied.

VI. CRITICAL STRENGTH OF ANISOTROPY

We now pose the question: How strong transverse isotropy generates triplications? Figs 4, 5 and 6 show the critical values of the P-, SV- and SH-wave anisotropy for the occurrence of the axial, basal and oblique triplications as a function of parameters κ and ε. The strength of anisotropy is defined as

$$a^W = 2\frac{c^W_{\max} - c^W_{\min}}{c^W_{\max} + c^W_{\min}} \cdot 100\% \ , \tag{27}$$

where W denotes the type of wave (P, SV or SH), and $c_{\max}$ and $c_{\min}$ denote the maximum and minimum phase velocities of the P, SV or SH waves. The total strength of anisotropy sums the strength of anisotropy of all waves in the following way

$$a = \sqrt{\left(a^P\right)^2 + \left(a^{SV}\right)^2 + \left(a^{SH}\right)^2} \ . \tag{28}$$

Interestingly, the pattern of the critical strength of the P-wave anisotropy is very similar for all types of triplications. The strength is sensitive to ε, but almost insensitive to κ. For ε close to zero, the P-wave anisotropy is very small irrespective of κ. For $\kappa \to 1$ and $\varepsilon = 0$, the P-wave anisotropy tends to zero. On the contrary, the pattern of the SV-wave anisotropy depends on the type of the triplication. But the critical strength of the SV-anisotropy can also be zero. It occurs for $\kappa \to 1$ and $\varepsilon = 0$ independently of the type of triplication. For basal and oblique triplications, zero SV anisotropy is observed also for κ and ε close to the borderline delimiting the area of their permissible values. Hence, we observe that P and SV anisotropy can simultaneously attain values very close to zero for $\kappa \to 1$ and $\varepsilon = 0$. This means that the triplications can occur even under infinitesimally weak P and SV anisotropy and no P and SV anisotropy threshold exists for the occurrence of triplications in TI. However, when P and SV anisotropy simultaneously attain values close to zero, the SH anisotropy should be non-zero (see Figs. 4, 5 and 6, right-hand plots). This follows from the stability conditions (17) which constrain the values of the SH anisotropy. Remind that the triplication conditions yield no other constraints on the SH anisotropy.

Axial triplication

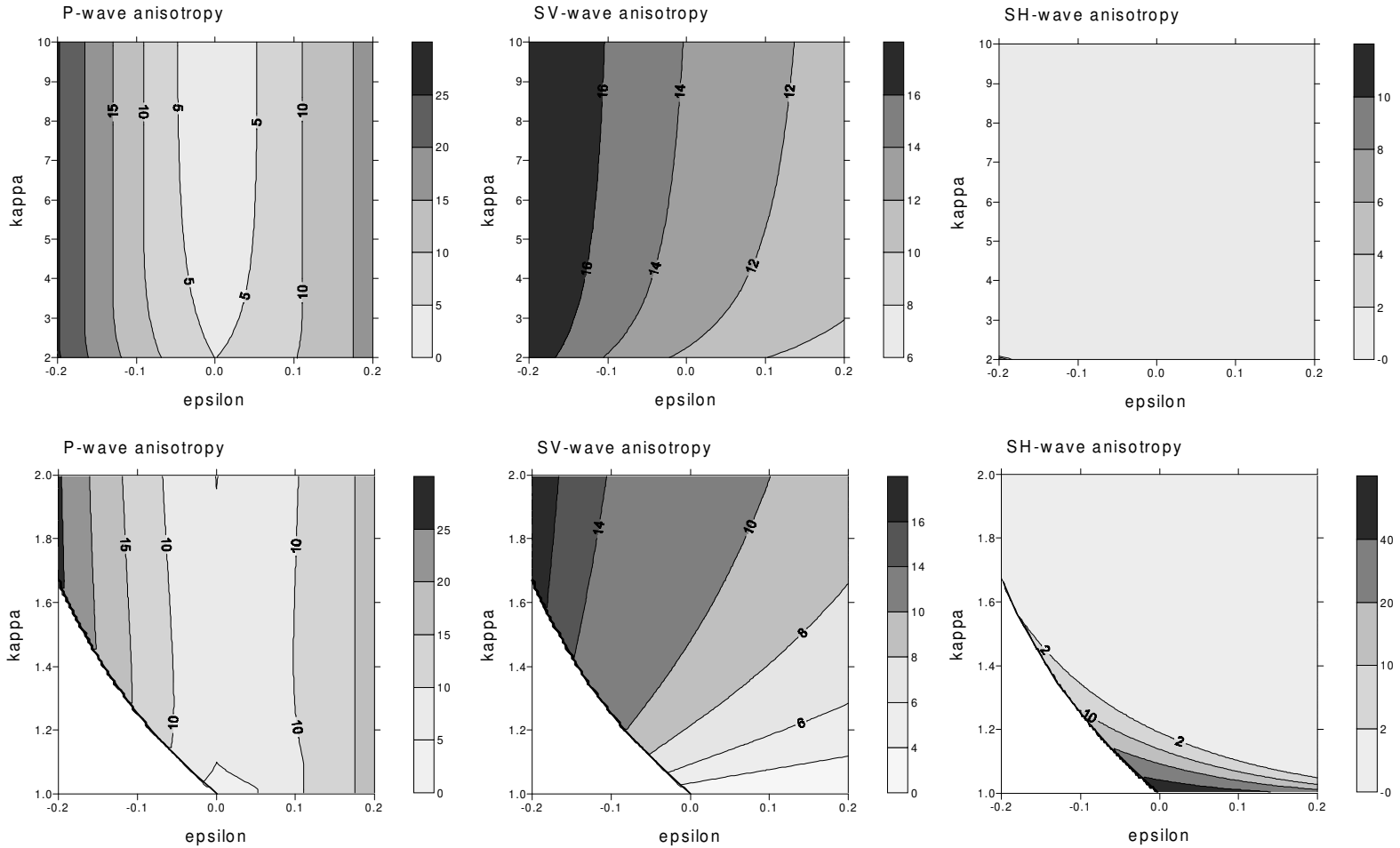

Figure 4. Minimum values of the P-, SV- and SH-wave anisotropy as a function of parameters κ and ε for the incipient axial triplication.

Basal triplication

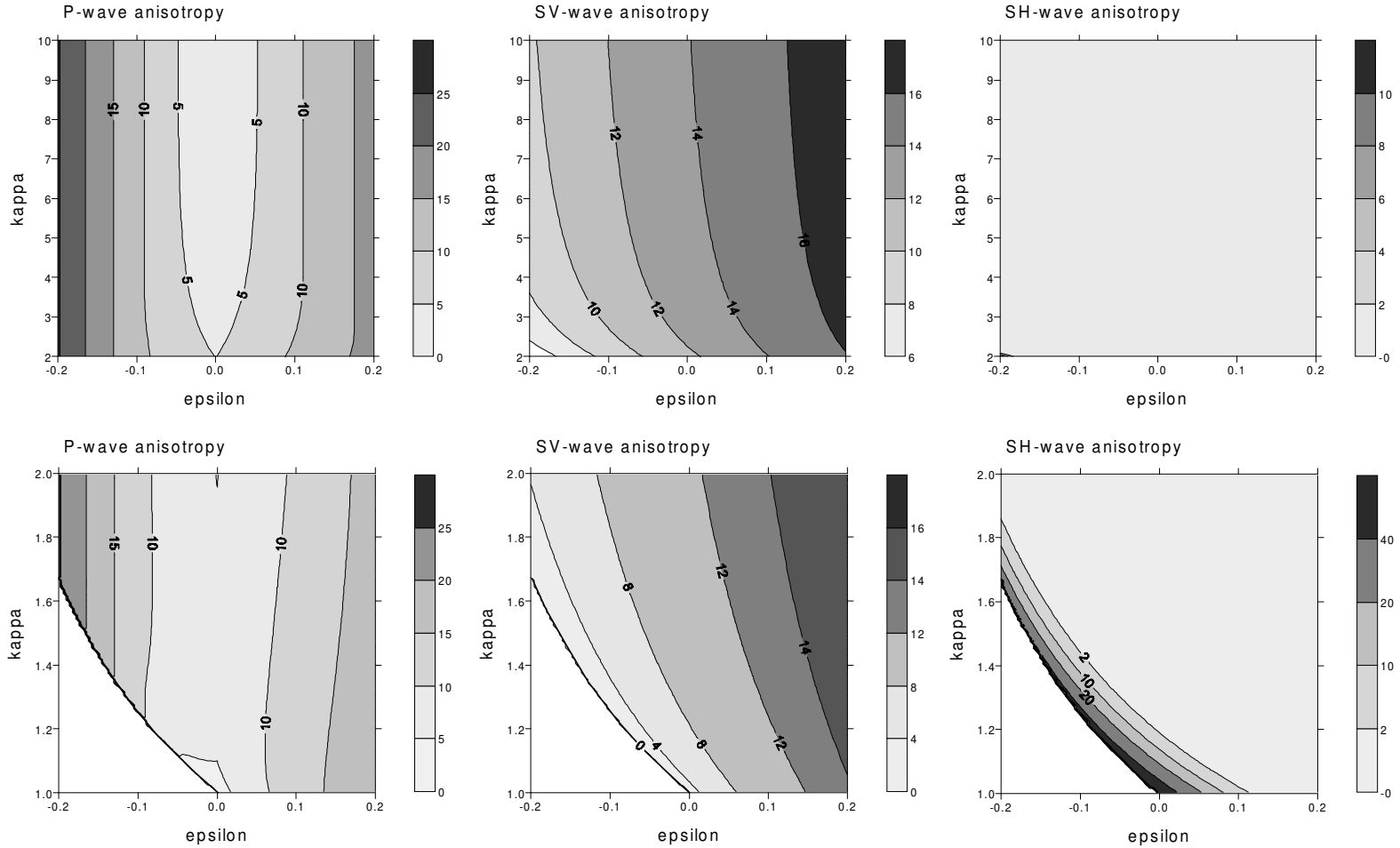

Figure 5. Minimum values of the P-, SV- and SH-wave anisotropy as a function of parameters κ and ε for the incipient basal triplication.

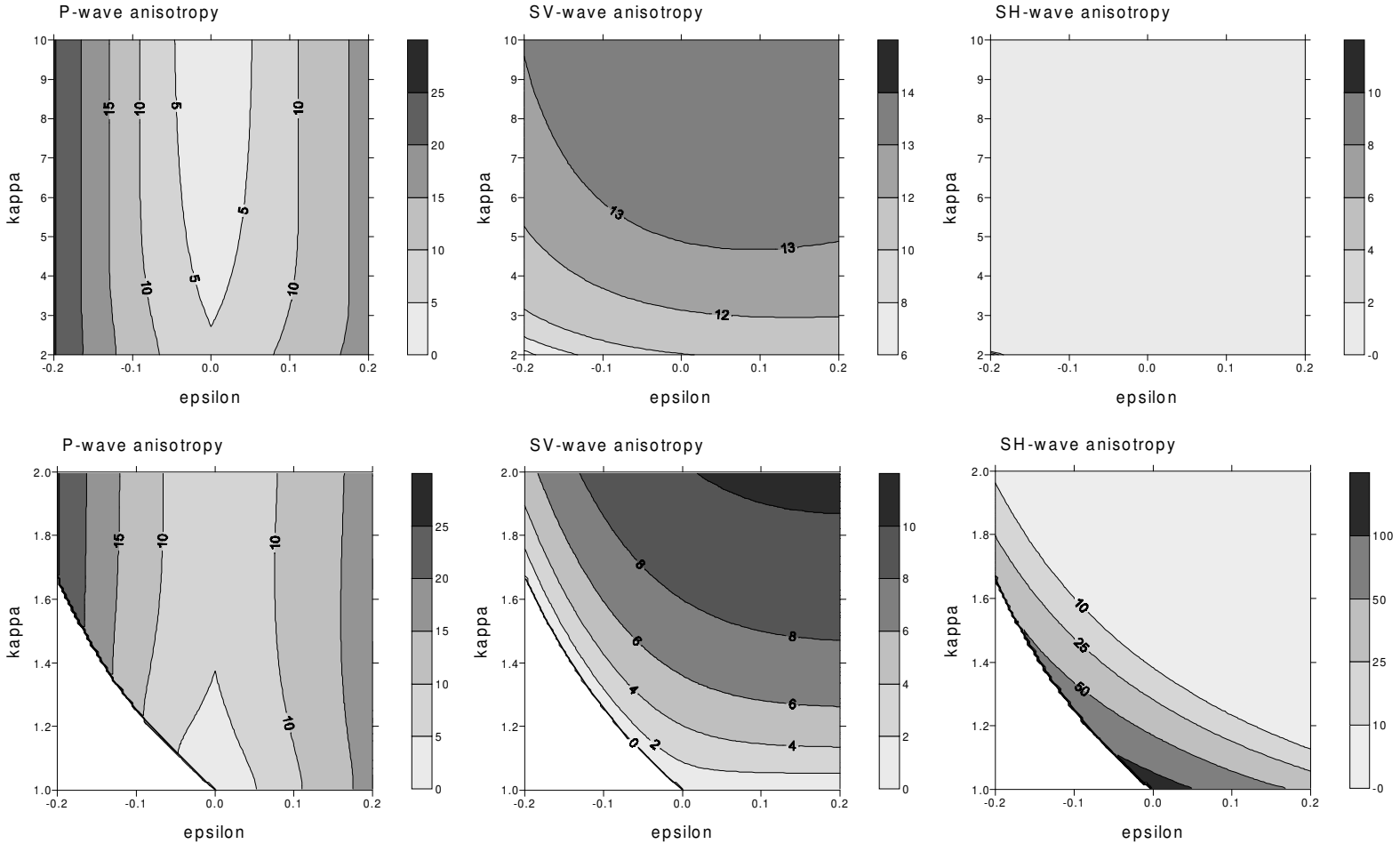

Figure 6. Minimum values of the P-, SV- and SH-wave anisotropy as a function of parameters κ and ε for the incipient oblique triplication.

Figure 7 shows the total strength of anisotropy, which sums the strength of all waves. The figure shows that the minimum strength of anisotropy for the occurrence of a triplication is close to 10%. This applies to all types of triplications. Specifically, the minimum strength of anisotropy is 9.50% for the axial triplication ($\varepsilon = 0.04$ and $\kappa = 1.16$), 9.71% for the basal triplication ($\varepsilon = -0.01$ and $\kappa = 1.22$), 8.86% for the oblique triplication ($\varepsilon = 0$ and $\kappa = 1.46$), and 9.72% for the double triplication ($\varepsilon = 0$ and $\kappa = 1.20$). These values represent the global minima of the critical strength of anisotropy. No triplication can occur in TI whose total strength of anisotropy is less than these minima. The minima are, however, rather shallow, because the strength of anisotropy increases very slightly with increasing κ. On the contrary, if κ decreases ($\kappa \to 1$), the strength of anisotropy rapidly increases.

A closed view of the strength of anisotropy as a function of κ is shown in Fig. 8. Parameter ε is fixed at zero. The critical P-wave anisotropy steeply increases from zero for $\kappa \to 1$ reaching its maximum at 6% for $\kappa = 1.4$ (double triplication) or for $\kappa = 1.8$ (oblique triplication). For higher values of κ, the P anisotropy decreases being less than 2% for $\kappa = 10$. Also the critical SV anisotropy steeply increases from zero ($\kappa \to 1$) to 12% for $\kappa = 3$. Thereafter the SV anisotropy increases very slowly being less than 14% for $\kappa = 10$. On the contrary, the SH anisotropy is high for $\kappa \to 1$, but rapidly decreases with increasing κ. The SH anisotropy is zero for $\kappa > 1.2$ (double triplication) or for $\kappa > 1.5$ (oblique triplication).

VII. DISCUSSION

The simplest triplication condition in TI is the condition for axial triplication. This condition requires the σ parameter to be less than –0.5. No other parameters control the occurrence of this triplication. The condition for the basal triplication is more involved. This triplication is controlled by anisotropy parameters σ and ε, and by parameter κ. The triplication condition requires σ to be negative. Therefore, if σ is less than –0.5, then the double (axial and basal) triplication can occur. For $\varepsilon = 0$, the condition for the axial and basal triplications becomes identical, hence they cannot be observed separately but only as the double triplication. For $\varepsilon \neq 0$ and $\sigma > -0.5$, only the basal triplication can occur. The condition for this triplication yields

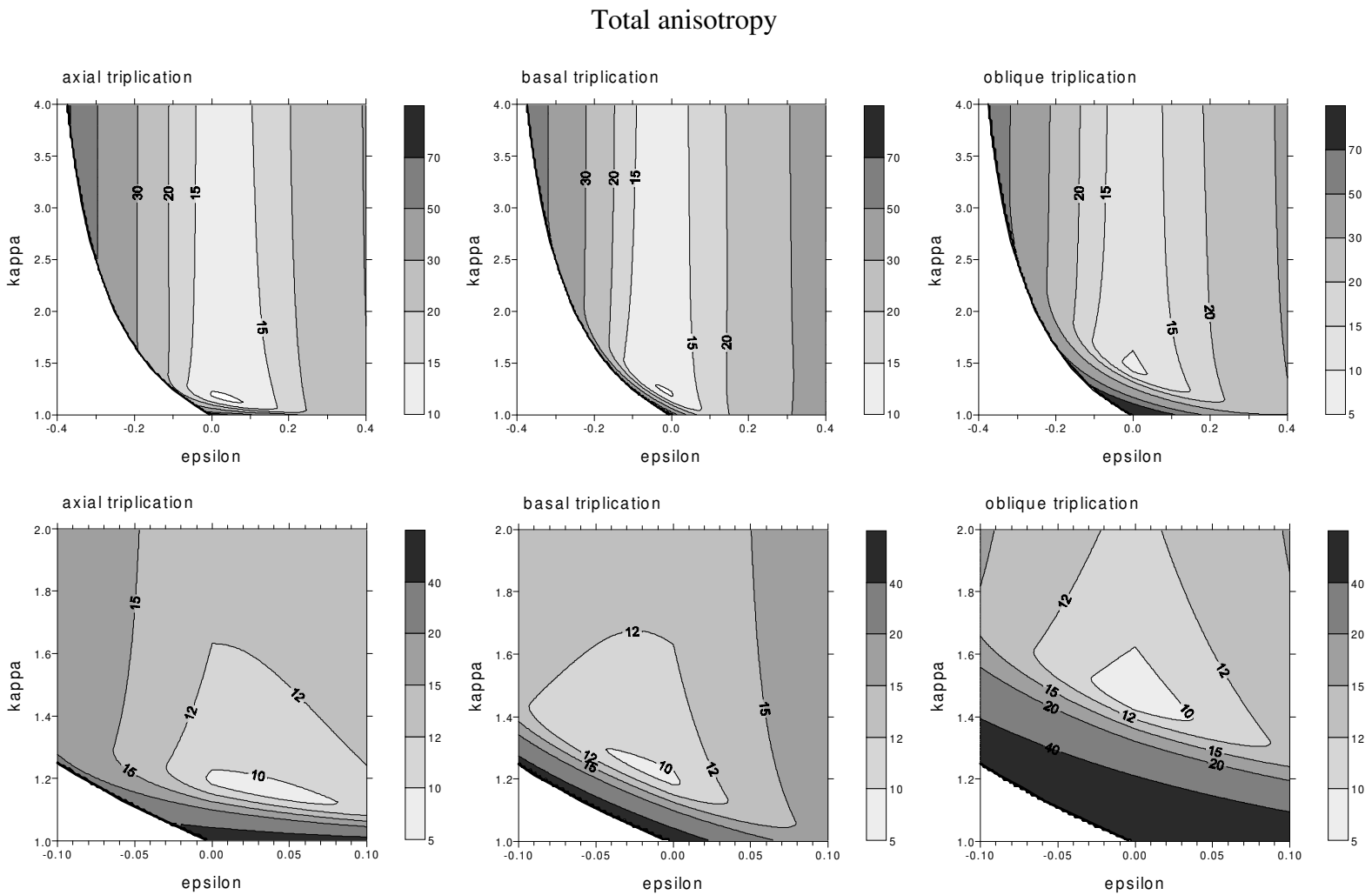

Figure 7. Minimum values of the total anisotropy as a function of parameters κ and ε for the incipient axial (left), basal (middle) and oblique (right) triplications. The lower plots show the detailed behavior of anisotropy near its minimum.

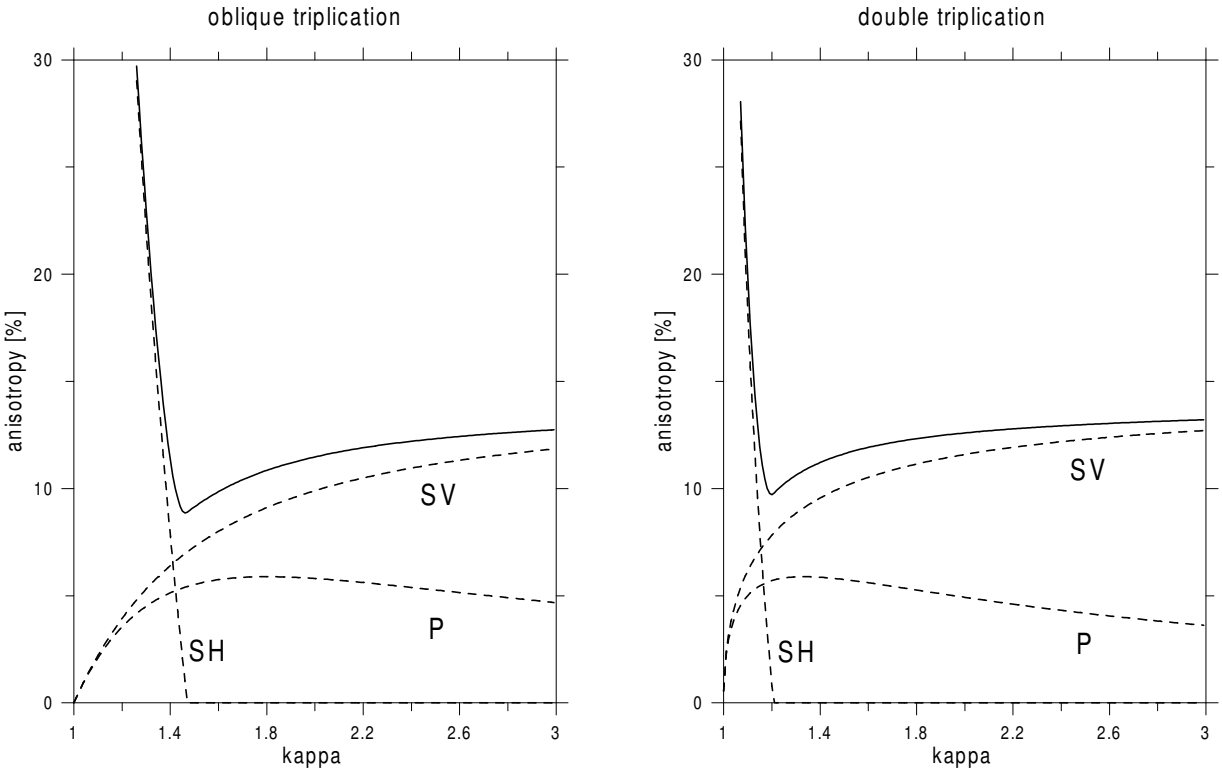

Figure 8. Minimum values of the P-, SV- and SH-wave anisotropy (dashed lines) together with the total anisotropy (solid line) as a function of parameter κ for the double (left-hand plots) and oblique (right-hand plots) triplications. Parameter ε equals zero.

$$\sigma_c = -\frac{1}{2}\frac{a_{11} - a_{44}}{a_{33} - a_{44}} \ . \tag{29}$$

Taking into account Eq. (2), one can readily see that σ_c can never be positive. The maximum value, which σ_c can attain, is zero.

The most complicated condition is established for the oblique triplication. This triplication is controlled by anisotropy parameters σ, ε and by parameter κ. The triplication requires σ to be positive. A more detailed analysis would show that TI conditioned by Eqs. (1)-(3) could never

produce the oblique triplication with negative values of σ. Hence, the basal and oblique triplications or axial and oblique triplications cannot occur simultaneously under the same TI (Payton 1983).

Analyzing the strength of anisotropy under which TI triplicates, we conclude that the critical strength of anisotropy is 9.50% for axial triplication, 9.71% for basal triplication, 8.86% for oblique triplication, and 9.72% for double triplication. No TI whose strength is less than the critical can display triplications. On the other hand, high values of the strength of anisotropy do not guarantee the existence of triplications. This can be illustrated on TI with elliptical angular dependence of phase velocities. Even an extremely high strength of such anisotropy produces no triplications. Hence, observations of triplications on the wave surface cannot serve as a unique criterion defining strong TI.

ACKNOWLEDGEMENTS

The study was supported by the Consortium Project "Seismic waves in complex 3-D structures", and by the Grant Agency of the Academy of Sciences of the Czech Republic, Grant No. A3012309.

References

Alshits, V.I. & Chadwick, P., 1997. Concavities on the zonal slowness section of a transversely isotropic material, *Wave Motion*, **25**, 347-359.

Backus, G.E., 1962. Long-wave anisotropy produced by horizontal layering, *J. Geophys. Res.*, **66**, 4427-4440.

Bakker, P.M., 1998. Phase shift at caustics along rays in anisotropic media, *Geophys. J. Int.*, **134**, 515-518.

Červený, V., 2001. *Seismic Ray Theory*, Cambridge University Press, Cambridge.

Dellinger, J., 1991. Anisotropic seismic wave propagation, PhD Thesis, Stanford University.

Grechka, V.Yu. & Obolentseva, I.R., 1993. Geometrical structure of shear wave surfaces near singularity directions in anisotropic media, *Geophys. J. Int.*, **115**, 609-616.

Helbig, K., 1958. Elastischen wellen in anisotropen medien, Teil I, *Gerlands Beiträge zur Geophysik*, **67**, 177-211.

Helbig, K., 1994. *Foundations of Anisotropy for Exploration Seismics*, Pergamon, N.Y.

Helbig, K. & Schoenberg, M., 1987. Anomalous polarization of elastic waves in transversely isotropic media, *J. Acoust. Soc. Am.*, **81**, 1235-1245.

Kravtsov, Yu.A. & Orlov, Yu.I., 1990. *Geometrical Optics of Inhomogeneous Media*, Springer-Verlag, New York.

Musgrave, M.J.P., 1970. *Crystal Acoustics*, Holden-Day, London.

Musgrave, M.J.P & Payton, R.G., 1984. Criteria for elastic waves in anisotropic media – a consolidation, *J. Elasticity*, **14**, 269-285.

Payton, R. G., 1983. *Elastic Wave Propagation in Transversely Isotropic Media*, Martinus Nijhoff Publishers, The Hague.

Thomsen, L., 1986. Weak elastic anisotropy, *Geophysics*, **51**, 1954-1966.

Thomsen, L. & Dellinger, J., 2003. On shear-wave triplication in transversely isotropic media, *J. Appl. Geophys*., accepted.

Tsvankin, I., & Thomsen, L., 1994. Nonhyperbolic reflection moveout in anisotropic media, *Geophysics*, **59**, 1290-1304.

Vavryčuk, V., 2003a. Parabolic lines and caustics in homogeneous weakly anisotropic solids, *Geophys. J. Int.*, in print.

Vavryčuk, V., 2003b. Approximate conditions for the off-axis triplication in transversely isotropic media, In: *Seismic Waves in Complex 3-D Structures*, Report **13**, pp. 91-105, Dept. of Geophysics, Charles University Prague.

Wolfe, J.P., 1998. *Imaging Phonons. Acoustic Wave Propagation in Solids*, Cambridge University Press, Cambridge.