First-order ray tracing for qP waves in inhomogeneous weakly anisotropic media

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Summary

Approximate ray tracing and dynamic ray tracing equations for a qP wave propagating in a smooth inhomogeneous weakly anisotropic medium are proposed. The equations are derived using the perturbation theory. The ray tracing and dynamic ray tracing equations are of the first order, travel time correction calculated along the rays to incerase accuracy of travel times is of the second order. The equations are expressed in terms of weak anisotropy parameters, which represent more natural parameterization of weakly anisotropic media than standard elastic parameters. Only 15 weak anisotropy parameters describing qP-wave propagation are used. Equations are thus considerably simpler than exact ray tracing equations. The equations further substantially reduce if media of higher symmetry are to be considered. For isotropic media, they reduce to standard, exact ray tracing equations for isotropic media.

Keywords
Ray tracing, dynamic ray tracing, second-order travel-time correction, qP wave, inhomogeneous weakly anisotropic media

1 Introduction

Most natural specification of weakly anisotropic media is by weak anisotropy (WA) parameters, see e.g., Farra and Pšenčík (2003). Individual seismic waves depend on only selected sets of these parameters. Specifically, the qP-wave propagation is controlled by 15 weak anisotropy parameters. Standard ray tracing equations do not respect these facts. Standard ray tracing equations are unique for all three waves propagating in anisotropic media. The type of the wave has to be specified by the initial (at the source) or the boundary (at an interface) conditions. The RHS of the standard ray tracing equations are quite complicated. They consist of up-to 81 terms. Very often, many of these terms are zero or mutually dependent. Thus calculation of the RHS' of ray tracing equations requires often unnecessary operations.

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We use the perturbation theory to derive approximate ray tracing equations for qP waves, which depend on the above-mentioned 15 qP-wave WA parameters, and can be used for tracing qP-wave rays only. Similarly, approximate ray tracing equations can be derived for qS waves, see Farra (2004b). The approximate equations are of the first order, we therefore speak about the first-order ray tracing (FORT). The FORT provides first-order travel times. In order to increase their accuracy, we propose a simple second-order travel-time correction. It can be evaluated by numerical quadratures along the FORT ray.

The FORT is very simple and transparent. It allows further substantial and straightforward reduction when anisotropy of higher symmetry is considered.

After a brief review of standard (exact) ray tracing equations in Sec.2, we derive the FORT in Sec.3, and present its specifications for most common types of anisotropic media of higher symmetry in Sec.4. In Sec.5, the second-order travel-time correction to be evaluated along the first-order rays is derived. Sec.6 contains brief description how to derive the first-order dynamic ray tracing (FODRT) equations.

2 Formulae for the exact ray tracing

We start from the basic formulae for the exact ray tracing in inhomogeneous anisotropic media

$$\frac{dx_i}{d\tau} = \frac{1}{2}\frac{\partial G}{\partial p_i}, \qquad \frac{dp_i}{d\tau} = -\frac{1}{2}\frac{\partial G}{\partial x_i}, \tag{1}$$

see, e.g., Červený (2001). Here x_i are coordinates of the trajectory of the ray, p_i are components of the slowness vector at corresponding points of the ray, $p_i = n_i/c$. The vector n_i is the unit normal to the wavefront (unit vector parallel to the slowness vector p_i), and $c = c(x_m, n_m)$ is the phase velocity. Let us mention that the group velocity is given by the first equation in (1). The variable τ in Eq.(1) is a parameter along the ray, which has meaning of propagation time. The symbol $G = G(x_m, p_m)$ denotes one of the three eigenvalues of the generalized Christoffel matrix

$$\Gamma_{jk}(x_m, p_m) = a_{ijkl}(x_m)p_i p_l. \tag{2}$$

The matrix $\Gamma_{jk}(x_m, p_m)$ is called the generalized Christoffel matrix in contrast to the standard Christoffel matrix $\Gamma_{jk}(x_m, n_m) = a_{ijkl}(x_m)n_i n_l$, in which the slowness vector p_m is substituted by the wave normal n_m, Here a_{ijkl} is a tensor of density-normalized elastic parameters. At each point of a ray, the slowness vector determined from Eq.(1) has to satisfy the eikonal equation

$$G(x_m, p_m) = \Gamma_{jk}(x_m, p_m)g_j g_k = 1. \tag{3}$$

Here the unit vectors g_i are the polarization vectors. They are identical with the eigenvectors of the generalized Christoffel matrix (2).

Using Eq.(3), the ray tracing system (1) can be rewritten as

$$\frac{dx_i}{d\tau} = a_{ijkl}p_l g_j g_k \quad , \quad \frac{dp_i}{d\tau} = -\frac{1}{2}a_{njkl,i}p_n p_l g_j g_k \quad . \tag{4}$$

The ray tracing system (4) looks very simple but it is only due to the used notation. In fact, the RHS of the first equation in Eq.(4) consists of 27 terms and the RHS of the second equation of 81 terms. In practical applications, many of these terms are zero or mutually dependent. Thus often evaluation of the RHS of Eqs.(4) requires much more operations than necessary.

Since the eigenvalue $G(x_m, p_m)$ of the generalized Christoffel matrix is a homogeneous function of the second degree, and $p_m = p^2 n_m = c^{-2} n_m$, we can write

$$G(x_m, p_m) = p^2 G(x_m, n_m) = c^{-2}(x_m, n_m) G(x_m, n_m). \tag{5}$$

Here $p = c^{-1}$ is the slowness ($p^2 = p_k p_k$). Combining Eq.(5) with the eikonal equation (3), we find that

$$G(x_m, n_m) = c^2(x_m, n_m). \tag{6}$$

The initial conditions for the ray tracing system (4) at the time $\tau = 0$, have the form

$$x_i(0) = x_i^0, \qquad p_i(0) = p_i^0, \tag{7}$$

where x_i^0 are the coordinates of the initial point of the ray and p_i^0 are the components of the slowness vector at the same point. They are specified as $p_i^0 = n_i/c_0$. The symbol c_0 denotes the phase velocity at the point x_i^0. For a given direction n_i, the phase velocity c_0 can be determined from Eq.(6).

3 First-order ray tracing formulae for qP wave

For the first set of the first-order ray tracing (FORT) equations we could use the expressions for the first-order group velocity vector derived by Pšenčík and Vavryčuk (2002) or Farra (2004a). Here, however, we start from the ray tracing equations (1), in which we substitute the exact eigenvalue $G(x_m, p_m)$ by its first-order approximation $G^{(1)}(x_m, p_m)$ for qP waves. It reads

$$G^{(1)}(x_m, p_m) = \Gamma_{jk}(x_m, p_m) n_j n_k, \tag{8}$$

see, for example, Farra (2001). In the same way, in which the exact eigenvalue G satisfies the exact eikonal equation (3), $G^{(1)}(x_m, p_m)$ satisfies the first-order eikonal equation

$$G^{(1)}(x_m, p_m) = 1. \tag{9}$$

The slowness vector p_m in Eqs.(8) and (9) can be now expressed as $p_i = n_i/c^{(1)}$, where $c^{(1)} = c^{(1)}(x_m, n_m)$ represents the first-order phase velocity. Since $G^{(1)}(x_m, p_m)$ is a homogeneous function of the second degree, we have

$$G^{(1)}(x_m, p_m) = (p^{(1)})^2(x_m, n_m) G^{(1)}(x_m, n_m) = (c^{(1)})^{-2}(x_m, n_m) G^{(1)}(x_m, n_m). \tag{10}$$

Here $p^{(1)} = (c^{(1)})^{-1}$ is the first-order slowness. Combination of Eqs.(10) and Eq.(9) leads to the first-order analogue of Eq.(6),

$$G^{(1)}(x_m, n_m) = (c^{(1)})^2(x_m, n_m), \tag{11}$$

see also, e.g., Farra (2001). From now on we are going to deal with first-order quantities only. In order to simplify the notation, we are, therefore, going to omit subscripts indicating that the quantities are of the first order. We shall return to this notation only when a confusion could arise. Under this specification, the FORT is formally of exactly the same form as Eq.(1). The FORT equations, however, generate only approximate (first-order) values of coordinates x_i and slowness vectors p_i, different from the exact values generated by the ray tracing equations (1) with the exact eigenvalue G.

Let us now concentrate on the quantity $G(x_m, n_m)$ in Eq.(10). Combination of Eq.(15) and of the last equation of Eq.(A4) of Farra and Pšenčík (2003) yields an explicit expression for $G(x_m, n_m)$,

$$G(x_m,n_m) = c^2(x_m,n_m) = \alpha^2\Big(1 + 2[2n_3^3(\epsilon_{34}n_2 + \epsilon_{35}n_1) + n_3^2(\eta_x n_1^2 + \eta_y n_2^2 + 2\chi_z n_1 n_2 + \epsilon_z)$$

$$+2n_3(\chi_x n_1^2 n_2 + \chi_y n_1 n_2^2 + \epsilon_{15}n_1^3 + \epsilon_{24}n_2^3) + \epsilon_x n_1^2 + \epsilon_y n_2^2 + \eta_z n_1^2 n_2^2 + 2\epsilon_{16}n_1^3 n_2 + 2\epsilon_{26}n_1 n_2^3]\Big), \quad (12a)$$

where

$$\eta_x = \delta_x - \epsilon_x - \epsilon_z, \quad \eta_y = \delta_y - \epsilon_y - \epsilon_z, \quad \eta_z = \delta_z - \epsilon_x - \epsilon_y. \quad (12b)$$

The quantity α in Eq.(12) is a reference velocity used in the definition of weak anisotropy (WA) parameters, see Eq.(A1). It can be easily shown that Eq.(12) is *independent* of α. The choice of the reference velocity α is thus quite arbitrary. We shall assume α to be a constant.

We can now use Eqs.(10) and (12) in the FORT version of Eq.(1), and derive explicit formulae for the FORT. When doing so, we must carefully distinguish, which quantities depend on spatial coordinates x_i and which on the components p_i of the slowness vector. In Eq.(10), the only spatially dependent quantities are the elastic parameters a_{ijkl}. All remaining parameters are functions of p_i. In the following derivations, we also use $n_k = p^{-1}p_k = cp_k$. The resulting FORT equations read

$$\frac{dx_1}{d\tau} = \alpha^2 p_1 + 2\alpha^2\epsilon_x p_1 + 2\alpha^2(p_k p_k)^{-2}\Big[(p_2^2 + p_3^2)(\eta_x p_3^2 + 2\chi_x p_2 p_3 + \eta_z p_2^2)p_1$$

$$+(p_2^2 + p_3^2 - p_1^2)(\epsilon_{35}p_3^3 + \chi_z p_2 p_3^2 + \chi_y p_3 p_2^2 + \epsilon_{26}p_2^3) + (p_1^2 + 3(p_2^2 + p_3^2))(\epsilon_{15}p_3 + \epsilon_{16}p_2)p_1^2$$

$$-(2\epsilon_{34}p_3^2 + \eta_y p_2 p_3 + 2\epsilon_{24}p_2^2)p_1 p_2 p_3\Big],$$

$$\frac{dx_2}{d\tau} = \alpha^2 p_2 + 2\alpha^2\epsilon_y p_2 + 2\alpha^2(p_k p_k)^{-2}\Big[(p_1^2 + p_3^2)(\eta_y p_3^2 + 2\chi_y p_1 p_3 + \eta_z p_1^2)p_2$$

$$+(p_1^2 + p_3^2 - p_2^2)(\epsilon_{34}p_3^3 + \chi_z p_1 p_3^2 + \chi_x p_3 p_1^2 + \epsilon_{16}p_1^3) + (3(p_1^2 + p_3^2) + p_2^2)\Big(\epsilon_{24}p_3 + \epsilon_{26}p_1\Big)p_2^2$$

$$-(2\epsilon_{35}p_3^2 + \eta_x p_1 p_3 + 2\epsilon_{15}p_1^2)p_1 p_2 p_3\Big],$$

$$\frac{dx_3}{d\tau} = \alpha^2 p_3 + 2\alpha^2\epsilon_z p_3 + 2\alpha^2(p_k p_k)^{-2}\Big[(p_1^2 + p_2^2)(\eta_x p_1^2 + 2\chi_z p_1 p_2 + \eta_y p_2^2)p_3$$

$$+(p_1^2 + p_2^2 - p_3^2)(\epsilon_{15}p_1^3 + \chi_x p_1^2 p_2 + \chi_y p_1 p_2^2 + \epsilon_{24}p_2^3) + (3(p_1^2 + p_2^2) + p_3^2)(\epsilon_{34}p_2 + \epsilon_{35}p_1)p_3^2$$

$$-(2\epsilon_{16}p_1^2 + \eta_z p_1 p_2 + 2\epsilon_{26}p_2^2)p_1p_2p_3\Big], \tag{13a}$$

$$\frac{dp_i}{d\tau} = -\alpha^2(p_kp_k)^{-1}[2p_3^2(\epsilon_{34,i}p_2 + \epsilon_{35,i}p_1) + p_3^2(\eta_{x,i}p_1^2 + \eta_{y,i}p_2^2 + 2\chi_{z,i}p_1p_2)$$
$$+2p_3(\chi_{x,i}p_1^2p_2 + \chi_{y,i}p_1p_2^2 + \epsilon_{15,i}p_1^3 + \epsilon_{24,i}p_2^3) + (p_kp_k)(\epsilon_{x,i}p_1^2 + \epsilon_{y,i}p_2^2 + \epsilon_{z,i}p_3^2)$$
$$+\eta_{z,i}p_1^2p_2^2 + 2\epsilon_{16,i}p_1^3p_2 + 2\epsilon_{26,i}p_1p_2^3]. \tag{13b}$$

The subscript i following comma indicates differentiation with respect to x_i. The symbols η_x, η_y and η_z are defined in Eq.(12b).

The equations (13) represent the FORT equations for inhomogeneous anisotropic media of arbitrary symmetry in the explicit form. We can see that the RHS' contain maximum 16 terms (in contrast to 27 and 81 in the case of exact ray tracing). Moreover, in the FORT equations (13), we can see immediately how each of the WA parameters specifying the medium affects the ray. At each point of the ray, the slowness vector p_i determined from the FORT equations (13) satisfies the first-order eikonal equation (9).

The initial conditions at the source, for $\tau = 0$, have the form

$$x_i(0) = x_i^0, \qquad p_i(0) = p_i^0, \tag{14}$$

where x_i^0 are the coordinates of the source point, p_i^0 are the components of the slowness vector, which is specified as $p_i^0 = n_i^0/c_0$. The symbol c_0 denotes the first-order approximation of the phase velocity at the source point. It is given by square root of the expression given in Eq.(12) specified for x_i^0 and n_i^0.

4 FORT for anisotropic media of higher symmetry

The FORT equations (13) simplify considerably in media with higher symmetry anisotropy. We present FORT equations for several most common types of anisotropic media.

4.1 FORT for orthorhombic media

If the planes of symmetry of an orthorhombic medium coincide with coordinate planes, we have

$$\chi_x = \chi_y = \chi_z = \epsilon_{15} = \epsilon_{16} = \epsilon_{24} = \epsilon_{26} = \epsilon_{34} = \epsilon_{35} = 0. \tag{15}$$

Taking this into account in Eqs.(13), we arrive at the FORT for orthorhombic media:

$$\frac{dx_1}{d\tau} = \alpha^2 p_1\Big[1 + 2\epsilon_x + 2(p_kp_k)^{-2}[(p_2^2 + p_3^2)(\eta_x p_3^2 + \eta_z p_2^2) - \eta_y p_2^2 p_3^2]\Big],$$

$$\frac{dx_2}{d\tau} = \alpha^2 p_2\Big[1 + 2\epsilon_y + 2(p_kp_k)^{-2}[(p_1^2 + p_3^2)(\eta_y p_3^2 + \eta_z p_1^2) - \eta_x p_1^2 p_3^2]\Big],$$

$$\frac{dx_3}{d\tau} = \alpha^2 p_3 \Big[1 + 2\epsilon_z + 2(p_k p_k)^{-2}[(p_1^2 + p_2^2)(\eta_x p_1^2 + \eta_y p_2^2) - \eta_z p_1^2 p_2^2]\Big],$$

$$\frac{dp_i}{d\tau} = -\alpha^2 (p_k p_k)^{-1} [\eta_{x,i} p_1^2 p_3^2 + \eta_{y,i} p_2^2 p_3^2 + \eta_{z,i} p_1^2 p_2^2 + (p_k p_k)(\epsilon_{x,i} p_1^2 + \epsilon_{y,i} p_2^2 + \epsilon_{z,i} p_3^2)]. \quad (16)$$

The initial conditions are given again by Eq.(14). The formula for the square of the first-order phase velocity (12) reduces in this case to

$$c^2(x_m, n_m) = \alpha^2 \Big(1 + 2[(\eta_x n_1^2 + \eta_y n_2^2 + \epsilon_z) n_3^2 + \epsilon_x n_1^2 + \epsilon_y n_2^2 + \eta_z n_1^2 n_2^2]\Big). \quad (17)$$

4.2 FORT for HTI media

In the case of transverse isotropy with horizontal axis of symmetry (HTI), oriented along the x_1-axis, we have

$$\epsilon_y = \epsilon_z = \frac{1}{2}\delta_y, \qquad \delta_x = \delta_z, \qquad \eta_x = \eta_z, \qquad \eta_y = 0. \quad (18)$$

Inserting this into Eqs.(16), we get the FORT equations, which have the following form:

$$\frac{dx_1}{d\tau} = \alpha^2 p_1 \Big(1 + 2\epsilon_x + 2(p_k p_k)^{-2} \eta_x (p_2^2 + p_3^2)^2\Big),$$

$$\frac{dx_2}{d\tau} = \alpha^2 p_2 \Big(1 + 2\epsilon_z + 2(p_k p_k)^{-2} \eta_x p_1^4\Big),$$

$$\frac{dx_3}{d\tau} = \alpha^2 p_3 \Big(1 + 2\epsilon_z + 2(p_k p_k)^{-2} \eta_x p_1^4\Big),$$

$$\frac{dp_i}{d\tau} = -\alpha^2 (p_k p_k)^{-1} [\eta_{x,i} (p_2^2 + p_3^2) p_1^2 + (p_k p_k)\Big(\epsilon_{x,i} p_1^2 + \epsilon_{z,i} (p_2^2 + p_3^2)\Big)]. \quad (19)$$

The initial conditions are given by Eq.(14). Square of the first-order phase velocity has now the following form:

$$c^2(x_m, n_m) = \alpha^2 \Big(1 + 2[\eta_x n_1^2 (1 - n_1^2) + \epsilon_x n_1^2 + \epsilon_z (1 - n_1^2)]\Big). \quad (20)$$

4.3 FORT for VTI media

In the case of transverse isotropy with vertical axis of symmetry (VTI), oriented along the x_3-axis, we have

$$\epsilon_x = \epsilon_y = \frac{1}{2}\delta_z, \qquad \delta_x = \delta_y, \qquad \eta_x = \eta_y, \qquad \eta_z = 0. \quad (21)$$

Inserting this into Eqs.(16), we get the FORT equations which have the following simple form:

$$\frac{dx_1}{d\tau} = \alpha^2 p_1\Big(1 + 2\epsilon_x + 2(p_k p_k)^{-2}\eta_x p_3^4\Big),$$

$$\frac{dx_2}{d\tau} = \alpha^2 p_2\Big(1 + 2\epsilon_x + 2(p_k p_k)^{-2}\eta_x p_3^4\Big),$$

$$\frac{dx_3}{d\tau} = \alpha^2 p_3\Big(1 + 2\epsilon_z + 2(p_k p_k)^{-2}\eta_x (p_1^2 + p_2^2)^2\Big),$$

$$\frac{dp_i}{d\tau} = -\alpha^2 (p_k p_k)^{-1}[\eta_{x,i}(p_1^2 + p_2^2)p_3^2 + (p_k p_k)\Big(\epsilon_{x,i}(p_1^2 + p_2^2) + \epsilon_{z,i}p_3^2\Big)] \qquad (22)$$

The initial conditions are given by Eq.(14). Square of the phase velocity is given by

$$c^2(x_m, n_m) = \alpha^2\Big(1 + 2[\eta_x n_3^2(1 - n_3^2) + \epsilon_z n_3^2 + \epsilon_x(1 - n_3^2)]\Big). \qquad (23)$$

4.4 FORT for isotropic media media

In isotropic media we have

$$\epsilon_x = \epsilon_y = \epsilon_z = \frac{1}{2}\delta_x = \frac{1}{2}\delta_y = \frac{1}{2}\delta_z, \qquad \eta_x = \eta_y = \eta_z = 0. \qquad (24)$$

The FORT equations have then the following form:

$$\frac{dx_i}{d\tau} = 2\alpha^2 p_i(1 + 2\epsilon_x), \qquad \frac{dp_i}{d\tau} = -\alpha^2 (p_k p_k)\epsilon_{x,i}. \qquad (25)$$

Using the expression for ϵ_x from (A1) and denoting $A_{11} = c^2$, we can find that Eqs.(25) represent exact ray tracing equations for isotropic media. The FORT can thus be used universally for isotropic as well as anisotropic media. In isotropic media it yields exact rays, in anisotropic media their first-order approximation.

5 Second-order travel time perturbation

Farra and Le Bégat (1995) or Červený (2001) show that an approximate travel time τ calculated along an approximate ray (x_i, p_i), specified by an approximate eigenvalue $G(x_m, p_m)$ of the generalized Christoffel matrix can be corrected to the first order by using the correction $\Delta\tau$

$$\Delta\tau = -\frac{1}{2}\int_{\Omega_0} \Delta G(x_m, p_m)d\tau. \qquad (26)$$

The formula (26) follows, for example, from Eqs.(3.9.6) and (3.9.11) of Červený (2001). Here Ω_0 denotes the mentioned approximate reference ray, and x_m and p_m are coordinates and components of the slowness vector along it. The symbol ΔG denotes the difference between the exact eigenvalue $G_{ex}(x_m, p_m)$ and its approximation $G(x_m, p_m)$.

We can apply Eq.(26) to the FORT. For a while we shall use again superscripts indicating the approximation order. In our case, the above approximate travel time τ is first-order travel time $\tau^{(1)}$ calculated along a FORT ray specified by $G^{(1)}(x_m, p_m)$. Note that the travel time $\tau^{(1)}$ is obtained automatically by solving any of the FORT systems presented in Sec.3,

$$\tau^{(1)} = \int_{\Omega_0} G^{(1)}(x_m, p_m) d\tau = \int_{\Omega_0} d\tau. \tag{27}$$

In (27), we took into account that $G^{(1)} = 1$, see Eq.(9). In our case, $\Delta G = G_{ex} - G^{(1)} = G_{ex} - 1$. It would thus make no sense to approximate the exact eigenvalue $G_{ex}(x_m, p_m)$ by its first-order approximation because it would bring no correction to the travel time. Our travel time is already of the first order. We can, therefore, approximate $G_{ex}(x_m, p_m)$ along the ray Ω_0 by $G^{(2)}(x_m, p_m^{(1)})$.

Farra and Pšenčík (2003) offer the expression for the $G^{(2)}(x_m, n_m)$. From their Eq.(14), one gets

$$G^{(2)}(x_m, n_m) = G^{(1)}(x_m, n_m) + \frac{B_{13}^2(x_m, n_m) + B_{23}^2(x_m, n_m)}{V_P^2 - V_S^2}. \tag{28}$$

Because $G^{(1)}(x_m, n_m)$ and $G^{(2)}(x_m, n_m)$ are homogeneous functions of the second degree, we can rewrite (28) in the following form

$$G^{(2)}(x_m, p_m^{(1)}) = 1 + (c^{(1)}(x_m, n_m))^{-2} \frac{B_{13}^2(x_m, n_m) + B_{23}^2(x_m, n_m)}{V_P^2 - V_S^2}. \tag{29}$$

We took again Eq.(9) into account. In Eq.(29), V_P and V_S are P- and S-wave velocities of a reference isotropic medium. The velocity V_P can be chosen different from α used for the definition of the WA parameters, se (A1). The quantities B_{13} and B_{23} are elements of the matrix

$$B_{mn}(x_m, n_m) = \Gamma_{jk}(x_m, n_m) e_j^m e_k^n, \tag{30}$$

see Farra and Pšenčík (2003). For its elements B_{13}, B_{23} and B_{33} see the Appendix. The symbols e_i^j denote three mutually perpendicular unit vectors e_i^1, e_i^2 and e_i^3. The vector e_i^3 is chosen so that $e_i^3 = n_i$. The vectors e_i^1 and e_i^2 are situated in the plane perpendicular to e_i^3. Let us note that $G^{(1)} = B_{33}$, see Farra and Pšenčík (2003). The matrix B_{mn} is independent of the choice of the reference isotropic medium. It is easy to show that B_{33} as well as $B_{13}^2 + B_{23}^2$ are also independent of the choice of the vectors e_i^1 and e_i^2. In Eq.(29), we can thus use Eq.(A2), which gives explicit expressions for the elements B_{13}, B_{23} and B_{33} for the choice, in which the vector e_i^2 is horizontal.

We can now return to Eq.(26). Inserting Eq.(29) into it, we get

$$\Delta\tau = \tau^{(2)} - \tau^{(1)} = -\frac{1}{2} \int_{\Omega_0} (c^{(1)}(x_m, n_m))^{-2} \frac{B_{13}^2(x_m, n_m) + B_{23}^2(x_m, n_m)}{V_P^2 - V_S^2} d\tau. \tag{31}$$

This is the second-order correction of the first-order ravel time $\tau^{(1)}$. It can be easily evaluated by numerical quadratures along the FORT ray.

6 First-order dynamic ray tracing formulae

The first-order dynamic ray tracing (FODRT) can be developed in exactly the same way as the exact DRT, see e.g., Gajewski and Pšenčík (1990), Červený (2001). If we denote a parameter specifying the FORT ray (e.g. one of its initial angles) by γ, we can introduce the following important quantities

$$X_j = (\frac{\partial x_j}{\partial \gamma})_{\tau=const}, \quad Y_j = (\frac{\partial p_j}{\partial \gamma})_{\tau=const}. \tag{32}$$

The quantities X_j and Y_j describe variations along the wavefront of the coordinates x_i and of the components p_i of the slowness vector due to the variations of the parameter γ. The values of X_j and Y_j can be found from a system of linear differential equations obtained by the differentiation of the FORT equations with respect to γ:

$$\frac{dX_i}{d\tau} = \frac{1}{2}\Big(\frac{\partial^2 G(x_m, p_m)}{\partial p_i \partial x_j} X_j + \frac{\partial^2 G(x_m, p_m)}{\partial p_i \partial p_j} Y_j\Big),$$

$$\frac{dY_i}{d\tau} = \frac{1}{2}\Big(\frac{\partial^2 G(x_m, p_m)}{\partial x_i \partial x_j} X_j + \frac{\partial^2 G(x_m, p_m)}{\partial x_i \partial p_j} Y_j\Big). \tag{33}$$

The second derivatives of the first-order eigenvalue $G(x_m, p_m)$ of the Christoffel matrix in Eq.(33) can be obtained in the same way, in which the first derivatives necessary for the evaluation of the RHS of the FORT equations were obtained. The solution of the FODRT has equally broad potential applications as its exact counterpart, the DRT, see e.g. Červený (2001).

7 Conclusion

Presented FORT and FODRT equations for qP waves propagating in inhomogeneous weakly anisotropic media have several important and interesting properties. They are expressed in terms of the WA parameters, which represent more natural description of a weakly anisotropic medium than standard elastic parameters. The qP-wave FORT equations have much simpler and more transparent form than the exact ray tracing equations. They depend on only 15 qP-wave WA parameters. They thus allow separate study of qP waves as in isotropic media. As indicated by preliminary tests, the use of the second-order travel-time correction along FORT rays will make travel-time errors effectively negligible. Since FODRT represents a system of linear differential equations, it can be solved using the propagator techniques, similarly as DRT.

The next step in this study is the use of FODRT in the computation of two-point rays. This will allow extensive testing of accuracy and speed of proposed formulae. Further, FODRT will be used for the evaluation of the geometrical spreading. This will allow calculation of synthetic seismograms using FORT and FODRT equations.

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APPENDIX: WA parameters

If we denote by α the velocity of a reference isotropic medium, we can introduce 15 qP-wave WA parameters in the following way:

$$\epsilon_x = \frac{A_{11} - \alpha^2}{2\alpha^2}, \quad \epsilon_y = \frac{A_{22} - \alpha^2}{2\alpha^2}, \quad \epsilon_z = \frac{A_{33} - \alpha^2}{2\alpha^2},$$

$$\delta_x = \frac{A_{13} + 2A_{55} - \alpha^2}{\alpha^2}, \quad \delta_y = \frac{A_{23} + 2A_{44} - \alpha^2}{\alpha^2}, \quad \delta_z = \frac{A_{12} + 2A_{66} - \alpha^2}{\alpha^2},$$

$$\chi_x = \frac{A_{14} + 2A_{56}}{\alpha^2}, \quad \chi_y = \frac{A_{25} + 2A_{46}}{\alpha^2}, \quad \chi_z = \frac{A_{36} + 2A_{45}}{\alpha^2},$$

$$\epsilon_{15} = \frac{A_{15}}{\alpha^2}, \quad \epsilon_{16} = \frac{A_{16}}{\alpha^2}, \quad \epsilon_{24} = \frac{A_{24}}{\alpha^2}, \quad \epsilon_{26} = \frac{A_{26}}{\alpha^2}, \quad \epsilon_{34} = \frac{A_{34}}{\alpha^2}, \quad \epsilon_{35} = \frac{A_{35}}{\alpha^2}, \qquad (A1)$$

$$\begin{aligned} B_{13} = \alpha^2 D^{-1}\{ & n_3^4(\epsilon_{34}n_2 + \epsilon_{35}n_1) + n_3^3[(\delta_x - \epsilon_x - \epsilon_z)n_1^2 + (\delta_y - \epsilon_y - \epsilon_z)n_2^2 + 2\chi_z n_1 n_2] \\ & + n_3^2[(4\chi_x - 3\epsilon_{34})n_1^2 n_2 + (4\chi_y - 3\epsilon_{35})n_1 n_2^2 + (4\epsilon_{15} - 3\epsilon_{35})n_1^3 + (4\epsilon_{24} - 3\epsilon_{34})n_2^3] \\ & + n_3[(2\delta_z - \delta_x - \delta_y - \epsilon_x - \epsilon_y + 2\epsilon_z)n_1^2 n_2^2 + 2(2\epsilon_{16} - \chi_z)n_1^3 n_2 + 2(2\epsilon_{26} - \chi_z)n_1 n_2^3 \\ & + (\epsilon_x + \epsilon_z - \delta_x)n_1^4 + (\epsilon_y + \epsilon_z - \delta_y)n_2^4 + (\epsilon_x - \epsilon_z)n_1^2 + (\epsilon_y - \epsilon_z)n_2^2] \\ & - \chi_x n_1^2 n_2 - \chi_y n_1 n_2^2 - \epsilon_{15} n_1^3 - \epsilon_{24} n_2^3\}, \end{aligned}$$

$$\begin{aligned} B_{23} = \alpha^2 D^{-1}\{ & n_3^3(\epsilon_{34}n_1 - \epsilon_{35}n_2) + n_3^2[(\delta_y - \delta_x - \epsilon_y + \epsilon_x)n_1 n_2 + \chi_z n_1^2 - \chi_z n_2^2] \\ & + n_3[(2\chi_y - 3\epsilon_{15})n_1^2 n_2 - (2\chi_x - 3\epsilon_{24})n_1 n_2^2 + \chi_x n_1^3 - \chi_y n_2^3] \\ & + (\delta_z - \epsilon_x - \epsilon_y)n_1^3 n_2 - (\delta_z - \epsilon_x - \epsilon_y)n_1 n_2^3 + 3(\epsilon_{26} - \epsilon_{16})n_1^2 n_2^2 + \epsilon_{16}n_1^4 - \epsilon_{26}n_2^4 + (\epsilon_y - \epsilon_x)n_1 n_2\}, \end{aligned}$$

$$\begin{aligned} B_{33} = \alpha^2 + 2\alpha^2[& 2n_3^3(\epsilon_{34}n_2 + \epsilon_{35}n_1) + n_3^2((\delta_x - \epsilon_x - \epsilon_z)n_1^2 + (\delta_y - \epsilon_y - \epsilon_z)n_2^2 + 2\chi_z n_1 n_2 + \epsilon_z) \\ & + 2n_3(\chi_x n_1^2 n_2 + \chi_y n_1 n_2^2 + \epsilon_{15}n_1^3 + \epsilon_{24}n_2^3) + \epsilon_x n_1^2 + \epsilon_y n_2^2 + (\delta_z - \epsilon_x - \epsilon_y)n_1^2 n_2^2 + 2\epsilon_{16}n_1^3 n_2 + 2\epsilon_{26}n_1 n_2^3]. \end{aligned} \qquad (A2)$$

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