Slowness vectors in viscoelastic anisotropic media. Part 1: Theory

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Summary

A method to compute the complex-valued slowness vector of a plane wave, propagating in a viscoelastic anisotropic medium in an arbitrarily specified direction $\mathbf{N}$, is proposed. The method is quite universal and can be applied to homogeneous and inhomogeneous plane waves, propagating in isotropic and anisotropic, perfectly elastic and viscoelastic media. This is, in general, impossible with the well-known conventional approaches, used in isotropic media. The method is based on the so-called mixed specification of the slowness vector, and leads to the solution of a complex-valued algebraic equation of the sixth degree. Standard methods can be used to solve this equation. Once the roots of the algebraic equation have been found, the phase velocities, exponential decays of amplitudes, attenuation angles, polarisation vectors, etc., of qP, qS1 and qS2 plane waves, propagating along and against $\mathbf{N}$, can be easily determined.

Although the method can be used for an unrestricted anisotropy, a special case of qP, qSV and SH plane waves, propagating in a plane of symmetry of a monoclinic (orthorhombic, hexagonal) viscoelastic medium is discussed in greater detail. In this case, the algebraic equation of the sixth degree is factorized into one quartic equation (qP and qSV waves) and one quadratic equation (SH waves). The analytical discussion of these waves offers a simple physical insight into the problem of inhomogeneous plane waves, and valuable comparisons of advantages and drawbacks of different approaches.

In this paper (Part 1), the qP, qSV and SH plane waves, propagating in the plane of symmetry of anisotropic viscoelastic media, are studied as functions of propagation direction $\mathbf{N}$ and the inhomogeneity parameter $D \in (-\infty, \infty)$. For inhomogeneous plane waves, $D \neq 0$, and for homogeneous plane waves, $D = 0$. The inhomogeneity parameter D offers many advantages in comparison with the conventionally used attenuation angle γ. In the $\mathbf{N}, D$ domain, all combinations of $\mathbf{N}$ and D are physically acceptable. This is, however, not the case in the $\mathbf{N}, \gamma$ domain, where certain combinations of $\mathbf{N}$ and γ yield non-physical slowness vectors. Another advantage of the used inhomogeneity parameter D is the simplicity and universality of the algorithms in the $\mathbf{N}$, D domain.

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Combined effects of attenuation and anisotropy, not known in viscoelastic isotropic media or purely elastic anisotropic media, are studied. It is shown that, in anisotropic viscoelastic media, the slowness vector and the related quantities are not symmetrical with respect to $D = 0$ as in isotropic viscoelastic media. The phase velocity of an inhomogeneous plane wave may be higher than the phase velocity of the relevant homogeneous plane wave, propagating in the same direction $\mathbf{N}$. Similarly, the modulus of the attenuation vector of an inhomogeneous plane wave may be lower than that for the relevant homogeneous plane wave. The amplitudes of inhomogeneous plane waves in anisotropic viscoelastic media may increase exponentially in the direction of propagation $\mathbf{N}$ for certain D. Attenuation angle γ cannot exceed its boundary value, γ^*. The boundary attenuation angle γ^* is, in general, different from 90^0, and depends both on the direction of propagation $\mathbf{N}$ and on the sign of inhomogeneity parameter D. The polarization of qP and qSV plane waves is, in general, elliptical, both for homogeneous and inhomogeneous waves. Simple quantitative expressions or estimates for all these effects (and for many others) are presented. The results of the numerical treatment are presented in Part 2, see Červený and Pšenčík (2004).

Keywords
Viscoelastic media, inhomogeneous and homogeneous plane waves, phase velocity, attenuation angle, amplitude decay, slowness vector, inhomogeneity parameter.

1 Introduction

The complex-valued slowness vector of a plane wave propagating in viscoelastic, isotropic or anisotropic media may be determined in several ways. Individual approaches to the solution of this problem differ in the way the slowness vector of the plane wave under consideration is specified.

We consider a time-harmonic plane wave

$$u_j(x_k, t) = U_j \exp[-\mathrm{i}\omega(t - p_n x_n)] \ , \tag{1}$$

where u_j, p_j and U_j are Cartesian components of the complex-valued displacement vector $\mathbf{u}$, slowness vector $\mathbf{p}$ and polarization vector $\mathbf{U}$, respectively. Moreover, t is time and ω is a fixed, positive circular frequency. Equation (1) represents a plane wave if, and only if, U_j and p_j are chosen in such a way that (1) satisfies the elastodynamic equation. This requirement yields the relations

$$\Gamma_{ik}(p_n) U_k = U_i \ , \qquad i = 1, 2, 3 \ . \tag{2}$$

The condition of solvability of the system of linear equations (2) for U_1, U_2, U_3 reads

$$\det[\Gamma_{ik}(p_n) - \delta_{ik}] = 0 \ . \tag{3}$$

The 3×3 complex-valued matrix $\Gamma_{ik}(p_n)$ is given by the relation

$$\Gamma_{ik}(p_n) = a_{ijkl} p_j p_l \ , \tag{4}$$

where a_{ijkl} are complex-valued, frequency dependent, density-normalized viscoelastic moduli. Matrix $\Gamma_{ik}(p_n)$ is here referred to as the *generalized Christoffel matrix*, in contrast to the well-known *Christoffel matrix*, given by the relation

$$\Gamma_{ik}(N_n) = a_{ijkl}N_jN_l \ , \tag{5}$$

where N_i are the Cartesian components of real-valued unit vector $\mathbf{N}$ (Musgrave, 1970; Helbig, 1994). Note that the generalized Christoffel matrix has been broadly used in the seismic ray method, where it is simply called the Christoffel matrix, see Červený (2001). Here, however, we shall strictly distinguish between (4) and (5).

Equations (2) and (3) are the **constraint relations** imposed on slowness vector $\mathbf{p}$ and polarization vector $\mathbf{U}$, which must be satisfied by any time-harmonic plane wave. Equation (3) plays a basic role in determining slowness vector $\mathbf{p}$, and Equation (2) in determining polarisation vector $\mathbf{U}$.

Alternatively to a_{ijkl}, we also use the complex-valued density-normalized viscoelastic moduli in the Voigt notation, $A_{\alpha\beta}$ $(\alpha, \beta = 1, 2, ..., 6)$. Throughout the paper, we assume that the 6×6 matrix Re$\mathbf{A}$ is positive definite and the 6×6 matrix Im$\mathbf{A}$ negative definite or zero.

We now introduce certain notations used for slowness vector $\mathbf{p}$. It is complex-valued:

$$\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A} \ . \tag{6}$$

Here $\mathbf{P}$ is the real-valued propagation vector (perpendicular to the plane of constant phase) oriented in the direction of the propagation of the wavefront (briefly the direction of propagation), $\mathbf{A}$ is the real-valued attenuation vector (perpendicular to the plane of constant amplitude), oriented in the direction of the maximum decay of amplitudes. We also introduce the real-valued unit vectors $\mathbf{N}$ and $\mathbf{M}$ in the directions of $\mathbf{P}$ and $\mathbf{A}$, phase velocity $\mathcal{C}$, attenuation-propagation ratio δ and attenuation (or inhomogeneity) angle γ as

$$\mathbf{N} = \mathbf{P}/|\mathbf{P}| \ , \qquad \mathbf{M} = \mathbf{A}/|\mathbf{A}| \ , \tag{7}$$

$$\mathcal{C} = 1/|\mathbf{P}| \ , \qquad \delta = |\mathbf{A}|/|\mathbf{P}| \ , \qquad \cos\gamma = \mathbf{N}\cdot\mathbf{M} \ . \tag{8}$$

The plane waves are called homogeneous for $\gamma = 0$, and inhomogeneous for $\gamma \neq 0$. For inhomogeneous plane waves, the plane specified by unit vectors $\mathbf{N}$ and $\mathbf{M}$ is called the propagation-attenuation plane, and denoted by $\Sigma^{\|}$.

Note that a notation analogous to (6) has often been used in the seismological literature for wave vector $\mathbf{k} = \omega\mathbf{p}$, $\mathbf{k} = \mathbf{P} + \mathrm{i}\mathbf{A}$. See, e.g., Aki and Richards (1980, p. 183), Borcherdt (1973, 1977), Krebes (1983), Krebes and Le (1994), Carcione and Cavallini (1995), Carcione (2001), etc. The constraint relation (3) for wave vector $\mathbf{k}$ reads $\det[a_{ijkl}k_jk_l - \omega^2\delta_{ik}] = 0$, and is usually referred to as the *dispersion relation.* As $\mathbf{k} = \omega\mathbf{p}$, the differences between $\mathbf{k}$ and $\mathbf{p}$ are only formal, if we consider time-harmonic plane waves with $\omega > 0$ and fixed. In this paper, we use slowness vector $\mathbf{p}$ and Equations (3), (6) systematically, and avoid wave vector $\mathbf{k}$ completely.

Now we introduce the three specifications of slowness vector $\mathbf{p}$, discussed in the paper.

In the **directional specification** of slowness vector $\mathbf{p}$, we express the slowness vector in terms of known real-valued unit vectors $\mathbf{N}$ and $\mathbf{M}$,

$$\mathbf{p} = \mathcal{C}^{-1}(\mathbf{N} + \mathrm{i}\delta\mathbf{M}) \; . \tag{9}$$

The unknown quantities $\mathcal{C}$ and δ must be determined by inserting (9) into (3). Note that attenuation angle γ is assumed to be known, as $\cos\gamma = \mathbf{N}\cdot\mathbf{M}$, see (8).

In the **componental specification**, slowness vector $\mathbf{p}$ is expressed in terms of a known real-valued unit vector $\mathbf{n}$ and a complex-valued vector $\mathbf{p}^{\Sigma}$ as follows:

$$\mathbf{p} = \sigma\mathbf{n} + \mathbf{p}^{\Sigma} \; , \qquad \text{with} \quad \mathbf{p}^{\Sigma}\cdot\mathbf{n} = 0 \; . \tag{10}$$

In fact, $\mathbf{p}^{\Sigma}$ represents a known vectorial component of slowness vector $\mathbf{p}$ in the plane Σ perpendicular to $\mathbf{n}$. The unknown complex-valued quantity σ must be determined by inserting (10) into (3).

The **mixed specification** of slowness vector $\mathbf{p}$ is a special case of the componental specification (10), with $\mathbf{p}^{\Sigma}$ purely imaginary,

$$\mathbf{p}^{\Sigma} = \mathrm{i}D\mathbf{m} \; . \tag{11}$$

Here $\mathbf{m}$ is a real-valued unit vector, perpendicular to $\mathbf{n}$, and D is a scalar, real-valued quantity, here referred to as the *inhomogeneity parameter*, $D \in (-\infty, \infty)$. Its absolute value, $|D|$, measures the inhomogeneity strength of the plane wave. For $D = 0$, the plane wave is homogeneous, and for $D \neq 0$ inhomogeneous. Slowness vector $\mathbf{p}$ is then expressed as follows:

$$\mathbf{p} = \sigma\mathbf{n} + \mathrm{i}D\mathbf{m} \; , \qquad \text{with} \quad \mathbf{m}\cdot\mathbf{n} = 0 \; . \tag{12}$$

The unknown complex-valued quantity σ must be determined by inserting (12) into (3).

In the mixed specification, plane Σ represents the plane of constant phase (wavefront), as $\mathrm{Re}(\mathbf{p}^{\Sigma}) = \mathbf{0}$ along it. Consequently, $\mathbf{N} = \pm\mathbf{n}$. It is easy to see that $D\mathbf{m}$ represents the vectorial component of attenuation vector $\mathbf{A}$ in the wavefront, $D\mathbf{m} = \mathbf{n}\times(\mathbf{A}\times\mathbf{n})$. This shows that unit vectors $\mathbf{n}$ and $\mathbf{m}$, which are assumed to be known, define uniquely the propagation-attenuation plane $\Sigma^{\|}$, in which also propagation vector $\mathbf{P}$ and attenuation vector $\mathbf{A}$ are situated.

In this paper, we derive and discuss the expressions for complex-valued slowness vector $\mathbf{p}$ of plane waves, using all the above three specifications, (9), (10) and (12), of the slowness vector. In Section 2, we consider plane waves propagating in a general viscoelastic anisotropic medium, see also Červený (2004). In Section 3, we specify these equations for the SH, qP and qSV inhomogeneous and homogeneous plane waves propagating in the symmetry plane of a monoclinic viscoelastic medium, or in any anisotropic medium with higher symmetry than monoclinic. In both sections, the results obtained under different specifications of the slowness vector are mutually compared and discussed. The derived equations also offer certain interesting physical conclusions related to the properties of SH, qP and qSV inhomogeneous plane waves, see Sections 3.4 and 3.5. For a numerical treatment see Červený and Pšenčík (2004).

Throughout the paper, we consider plane waves propagating in homogeneous media. The conclusions of the paper concerning the behaviour of slowness vectors, polarisation vectors and other characteristics of wave propagation, however, also hold for local characteristics of waves propagating in general inhomogeneous viscoelastic anisotropic media.

2 Viscoelastic anisotropic media

In this section, we apply the three specifications of slowness vector $\mathbf{p}$ to the determination of the complex-valued slowness vector $\mathbf{p}$ of a plane wave propagating in an arbitrary anisotropic viscoelastic medium.

2.1 Directional specification of the slowness vector

Inserting the directional specification of slowness vector (9) into constraint relations (2) and (3) yields a system of linear equations for polarization vector $\mathbf{U}$,

$$a_{ijkl}(N_j + i\delta M_j)(N_l + i\delta M_l)U_k = \mathcal{C}^2 U_i \ , \quad i = 1, 2, 3 \ , \tag{13}$$

and the condition of solvability of the system (13):

$$\det[a_{ijkl}(N_j + i\delta M_j)(N_l + i\delta M_l) - \mathcal{C}^2\delta_{ik}] = 0 \ . \tag{14}$$

For known real-valued unit vectors $\mathbf{N}$ and $\mathbf{M}$, Equation (14) represents two real-valued coupled equations for δ and $\mathcal{C}^2$. These equations are coupled polynomials of the third degree in $\mathcal{C}^2$ and of the sixth degree in δ. See Romeo (1994). In the general case of $\mathbf{N} \neq \mathbf{M}$ and of complex-valued a_{ijkl}, the solution of (14) for δ and $\mathcal{C}^2$ is not simple. It simplifies only in exceptional cases. For example, for homogeneous waves ($\mathbf{N} = \mathbf{M}$), Equation (14) yields

$$\det[a_{ijkl}N_j N_l - (\mathcal{C}/(1 + \mathrm{i}\delta))^2\delta_{ik}] = 0 \ . \tag{15}$$

Thus, for homogeneous waves, $\mathcal{C}^2/(1 + \mathrm{i}\delta)^2$ are the eigenvalues of the 3×3 complex-valued Christoffel matrix $a_{ijkl}N_j N_l$, which can be determined by conventional methods. Quantities $\mathcal{C}^2$ and δ can then be computed from these complex-valued eigenvalues. The next two cases, which can be treated in a simpler way, correspond to inhomogeneous P and S plane waves propagating in isotropic viscoelastic media, and to inhomogeneous SH plane waves, propagating in a symmetry plane of an monoclinic (orthorhombic, hexagonal) viscoelastic medium, see Section 3.1.

In addition to the complicated solution of (14) for $\mathcal{C}^2$ and δ, Equation (14) has another great disadvantage. For certain combinations of $\mathbf{N}$ and $\mathbf{M}$, it may yield non-physical solutions, $\mathcal{C}^2 < 0$.

The system of two coupled equations for $\mathcal{C}^2$ and δ can be decoupled. Červený (2004) proposed a method, called semi-analytic, in which the equation for δ is fully separated from that for $\mathcal{C}^2$. Once δ has been found, $\mathcal{C}^2$ can be determined by an explicit formula. However, the price paid for decoupling is the more complicated, non-polynomial form of the equation for δ. The problems with the non-physical solutions remain.

2.2 Componental specification of the slowness vector

Inserting the componental specification (10) of slowness vector $\mathbf{p}$ into (2), we obtain a system of three linear equations for the components U_i of the complex-valued polarisation vector $\mathbf{U}$,

$$a_{ijkl}(\sigma n_j + p_j^\Sigma)(\sigma n_l + p_l^\Sigma)U_k = U_i \ , \quad i = 1,2,3 \ . \tag{16}$$

The condition of solvability of system (16) reads,

$$\det[a_{ijkl}(\sigma n_j + p_j^\Sigma)(\sigma n_l + p_l^\Sigma) - \delta_{ik}] = 0 \ . \tag{17}$$

Assuming a_{ijkl}, $\mathbf{n}$ and $\mathbf{p}^\Sigma$ known, Equation (17) represents an algebraic equation of the sixth degree in σ. The six roots σ of (17) correspond to qP, qS1, and qS2 inhomogeneous plane waves, propagating away from Σ, to both sides of it. One polarisation vector $\mathbf{U}$ corresponds to each root σ. For a non-degenerate case of a single root (not coinciding with any other), the relevant complex-valued polarisation vector $\mathbf{U}$ can be obtained from (16), supplemented by a suitable normalization condition, for example

$$U_i U_i^* = 1 \ , \qquad \text{or} \qquad U_i U_i = 1 \ . \tag{18}$$

Equation (17) is a basic equation for the determination of the slowness vectors of reflected/transmitted waves at structural interfaces, see, e. g., Fedorov (1968), Gajewski and Pšenčík (1987), Červený (2001, p. 527). These references, however, treat only real-valued quantities a_{ijkl} (perfectly elastic media).

Červený (2001, p. 532) proved that the roots σ of (17) can be alternatively determined as eigenvalues of the 6×6 complex-valued matrix $\mathbf{\Pi}$,

$$\mathbf{\Pi} = \begin{pmatrix} \mathbf{\Pi}_{11} & \mathbf{\Pi}_{12} \\ \mathbf{\Pi}_{21} & \mathbf{\Pi}_{22} \end{pmatrix} \ , \tag{19}$$

where the 3×3 complex-valued partition matrices $\mathbf{\Pi}_{IJ}$ read

$$\begin{aligned} \mathbf{\Pi}_{11} &= -\mathbf{C}^{(1)-1}\mathbf{C}^{(2)} = \mathbf{\Pi}_{22}^T \ , & \mathbf{\Pi}_{12} &= -\mathbf{C}^{(1)-1} \ , \\ \mathbf{\Pi}_{21} &= -\mathbf{I}_3 + \mathbf{C}^{(4)} - \mathbf{C}^{(3)}\mathbf{C}^{(1)-1}\mathbf{C}^{(2)} \ , & \mathbf{\Pi}_{22} &= -\mathbf{C}^{(3)}\mathbf{C}^{(1)-1} \ , \end{aligned} \tag{20}$$

with

$$\begin{aligned} C_{ik}^{(1)} &= a_{ijkl}n_j n_l \ , & C_{ik}^{(2)} &= a_{ijkl}n_j p_l^\Sigma \ , \\ C_{ik}^{(3)} &= a_{ijkl}p_j^\Sigma n_l \ , & C_{ik}^{(4)} &= a_{ijkl}p_j^\Sigma p_l^\Sigma \ . \end{aligned} \tag{21}$$

$\mathbf{I}_3$ in (20) is a 3×3 identity matrix. Eigenvalues σ are solutions of the characteristic equation

$$\det(\mathbf{\Pi} - \sigma \mathbf{I}_6) = 0 \ , \tag{22}$$

where $\mathbf{I}_6$ is a 6×6 identity matrix.

Thus, if a real-valued unit vector $\mathbf{n}$, perpendicular to plane Σ, and a complex-valued vector $\mathbf{p}^\Sigma$, situated in plane Σ, are given, the determination of slowness vector $\mathbf{p}$ reduces to the solution of an algebraic equation of the sixth degree, or to the solution of a conventional eigenvalue problem for a 6×6 complex-valued matrix $\mathbf{\Pi}$.

The eigenvalue problem (22) is well-known for perfectly elastic media. It has been traditionally used in seismology in the solution of reflection/transmission problems and in displacement-stress (or velocity-stress) propagator matrix computations. For more details, see Woodhouse (1974), Kennett (1983, 2001), Frazer and Fryer (1989), Chapman (1994), Thomson (1996a,b), and Červený (2001, Section 5.4.7), where many other references can be found. A fully analogous approach has also been proposed by Stroh (1962), and is now usually referred to as the Stroh formalism. The most detailed description of the Stroh formalism can be found in Ting (1996). The Stroh formalism has now also been applied to studying homogeneous and inhomogeneous plane waves propagating in various types of perfectly elastic, viscoelastic and thermoviscoelastic anisotropic media, see Shuvalov and Scott (1999, 2000), Shuvalov (2001), and to the reflection/transmission problem of viscoelastic anisotropic media (see Caviglia and Morro, 1999).

In the componental specification, directions $\mathbf{N}$ of propagation vectors $\mathbf{P}$ of the individual waves are not known in advance and are not the same for all waves; they are obtained as a result of computation. If we wish to study the inhomogeneous plane waves propagating in the direction of an à priori specified unit vector $\mathbf{N}$, we have to choose p_i^Σ in a special way; see the next section.

Once a value of σ has been found, we can determine the relevant real-valued propagation vector $\mathbf{P}$, attenuation vector $\mathbf{A}$, unit vectors $\mathbf{N}$ and $\mathbf{M}$, phase velocity $\mathcal{C}$, attenuation-propagation ratio δ, and the attenuation angle γ. They are given by relations,

$$\begin{aligned}
&\mathbf{P} = \mathbf{n}(\mathrm{Re}\sigma) + \mathrm{Re}\mathbf{p}^\Sigma \ , && \mathbf{A} = \mathbf{n}(\mathrm{Im}\sigma) + \mathrm{Im}\mathbf{p}^\Sigma \ , \\
&|\mathbf{P}| = [(\mathrm{Re}\sigma)^2 + (\mathrm{Re}\mathbf{p}^\Sigma)(\mathrm{Re}\mathbf{p}^\Sigma)]^{1/2} \ , && |\mathbf{A}| = [(\mathrm{Im}\sigma)^2 + (\mathrm{Im}\mathbf{p}^\Sigma)(\mathrm{Im}\mathbf{p}^\Sigma)]^{1/2} \ , \\
&\mathbf{N} = \mathbf{P}/|\mathbf{P}| \ , && \mathbf{M} = \mathbf{A}/|\mathbf{A}| \ , \\
&\mathcal{C} = 1/|\mathbf{P}| \ , && \delta = |\mathbf{A}|/|\mathbf{P}| \ , \\
&\cos\gamma = [(\mathrm{Re}\mathbf{p}^\Sigma)(\mathrm{Im}\mathbf{p}^\Sigma) + (\mathrm{Re}\sigma)(\mathrm{Im}\sigma)]/|\mathbf{P}||\mathbf{A}| \ . &&
\end{aligned} \tag{23}$$

Inserting σ into (16) and solving system (16) with one of the normalisation conditions (18), we can also find the relevant polarisation vector $\mathbf{U}$.

2.3 Mixed specification of the slowness vector

If we wish to study inhomogeneous plane waves with the wavefront propagating in the direction of a known unit vector $\mathbf{N}$, it is suitable to use the componental specification with plane Σ perpendicular to $\mathbf{N}$. In this way, vector $\mathbf{N}$ is specified by unit vector $\mathbf{n}$ (or $-\mathbf{n}$), perpendicular to Σ. As plane Σ represents a wavefront in this case, we have $\mathrm{Re}(p_i^\Sigma) = 0$. Consequently, vector $\mathbf{p}^\Sigma$ is purely imaginary, $\mathbf{p}^\Sigma = \mathrm{i}D\mathbf{m}$, see (11). Here $\mathbf{m}$ is a real-valued unit vector, parallel to the wavefront (thus $\mathbf{n}\cdot\mathbf{m} = 0$), and D is an inhomogeneity parameter, $D \in (-\infty, \infty)$.

Using (11) in Equations (16) and (17), we obtain the basic relations for σ and $\mathbf{U}$ of the mixed specification of the slowness vector. For σ, we obtain the algebraic equation of the sixth degree,

$$\det[a_{ijkl}(\sigma n_j + \mathrm{i}Dm_j)(\sigma n_l + \mathrm{i}Dm_l) - \delta_{ik}] = 0 \ , \tag{24}$$

and for polarisation vector $\mathbf{U}$ the system of linear equations

$$a_{ijkl}(\sigma n_j + \mathrm{i}Dm_j)(\sigma n_l + \mathrm{i}Dm_l)U_k = U_i \ , \qquad i = 1,2,3 \ . \tag{25}$$

To determine polarisation vector $\mathbf{U}$ from (25), a suitable normalisation condition for $\mathbf{U}$ should be used, see (18).

Let us again emphasize that a_{ijkl}, U_j, p_j and σ are generally complex-valued, whereas $\mathbf{n}$ and $\mathbf{m}$ are real-valued, mutually perpendicular, unit vectors, and D is a real-valued scalar, $D \in (-\infty, \infty)$.

Equation (24) for σ simplifies considerably for homogeneous plane waves ($D = 0$). It yields

$$\det[a_{ijkl}n_j n_l - (1/\sigma)^2\delta_{ik}] = 0 \ .$$

This immediately provides

$$(1/\sigma)^2 = G \ , \tag{26}$$

where G is an eigenvalue of the 3×3 complex-valued Christoffel matrix $a_{ijkl}n_j n_l$. Equation (26) is valid for homogeneous plane waves only, but otherwise it is quite universal.

Alternatively to (24), σ can be found as an eigenvalue of the 6×6 complex-valued matrix $\mathbf{\Pi}$. The equations for matrix $\mathbf{\Pi}$ remain the same as in the componental specification, only $\mathbf{p}^{\Sigma} = \mathrm{i}D\mathbf{m}$ is used. Consequently, Equations (19), and(20) are not changed, but Equations (21) read

$$\begin{aligned} C_{ik}^{(1)} &= a_{ijkl}n_j n_l \ , \qquad & C_{ik}^{(2)} &= \mathrm{i}Da_{ijkl}n_j m_l \ , \\ C_{ik}^{(3)} &= \mathrm{i}Da_{ijkl}m_j n_l \ , & C_{ik}^{(4)} &= -D^2 a_{ijkl}m_j m_l \ . \end{aligned} \tag{27}$$

Once quantity σ has been found, we can determine the relevant real-valued propagation vector $\mathbf{P}$, attenuation vector $\mathbf{A}$, unit vectors $\mathbf{N}$ and $\mathbf{M}$, phase velocity $\mathcal{C}$, attenuation-propagation ratio δ and attenuation angle γ:

$$\begin{aligned} &\mathbf{P} = \mathbf{n}\,\mathrm{Re}\sigma \ , & &\mathbf{A} = \mathbf{n}\,\mathrm{Im}\sigma + D\mathbf{m} \ , \\ &|\mathbf{P}| = |\mathrm{Re}\sigma| \ , & &|\mathbf{A}| = [(\mathrm{Im}\sigma)^2 + D^2]^{1/2} \ , \\ &\mathbf{N} = \mathbf{P}/|\mathbf{P}| = \epsilon\mathbf{n} \ , & &\mathbf{M} = \mathbf{A}/|\mathbf{A}| \ , \\ &\mathcal{C} = 1/|\mathrm{Re}\sigma| \ , & &\delta = |\mathbf{A}|/|\mathbf{P}| \ , \\ &\cos\gamma = \epsilon\,\mathrm{Im}\sigma/[(\mathrm{Im}\sigma)^2 + D^2]^{1/2} \ , & & \end{aligned} \tag{28}$$

where

$$\epsilon = \mathrm{Re}\sigma/|\mathrm{Re}\sigma| = \pm 1 \ . \tag{29}$$

The mixed specification offers a simple and general algorithm to study inhomogeneous or homogeneous plane waves propagating in unbounded, anisotropic or isotropic, viscoelastic or perfectly elastic media. Assume that two real-valued, mutually perpendicular, unit vectors $\mathbf{n}$ and $\mathbf{m}$ are known ($\mathbf{n}$ perpendicular to Σ and $\mathbf{m}$ situated in Σ), and that the real-valued inhomogeneity parameter D is given. The determination of slowness vector $\mathbf{p}$ is then reduced to the solution of algebraic equation (24) of the sixth degree, or, alternatively, to the solution of a conventional eigenvalue problem for the 6×6 complex-valued matrix $\mathbf{\Pi}$, specified by (19), (20) and (27). The six roots σ_i ($i = 1,2,...,6$) of

(24), or, alternatively, the six eigenvalues σ_i of $\mathbf{\Pi}$, correspond to qP, qS1, and qS2 inhomogeneous plane waves, propagating to both sides of Σ. Three of them propagate in the direction of $\mathbf{N} = \mathbf{n}$, and other three in the opposite direction, $\mathbf{N} = -\mathbf{n}$. See Červený and Pšenčík (2003) for numerical examples, and a detailed numerical treatment based on Equation (24) in Červený and Pšenčík (2004).

The most important advantages of the mixed specification over the directional specification in the determination of the slowness vector of an inhomogeneous plane wave are as follows: 1) Conventional numerical algorithms can be used in the mixed specifications. Contrary to this, the directional specification requires the solution of a generally complicated coupled system of two equations. 2) The mixed specification yields the slowness vector not only for $\mathbf{N} = \mathbf{n}$, but also for $\mathbf{N} = -\mathbf{n}$. 3) The mixed specification yields a physically acceptable solution for any choice of real-valued, mutually perpendicular, unit vectors $\mathbf{n}$ and $\mathbf{m}$ and any choice of real-valued inhomogeneity parameter $D \in (-\infty, \infty)$. It never yields non-physical solutions, as the computations in the $\mathbf{N}, \mathbf{M}$ domain do.

It should be emphasized that the assumed mutual perpendicularity of $\mathbf{n}$ and $\mathbf{m}$ does not restrict the generality and universality of (12). A complete system of all possible homogeneous and inhomogeneous plane waves propagating in directions $\mathbf{N} = \mathbf{n}$ and $\mathbf{N} = -\mathbf{n}$ is obtained by choosing all possible real-valued unit vectors $\mathbf{m}$, perpendicular to $\mathbf{n}$, and all possible inhomogeneity parameters D, $0 \leq D < \infty$ (or $-\infty < D \leq 0$).

3 Plane of symmetry of an anisotropic viscoelastic medium of a monoclinic symmetry

In this section, we consider inhomogeneous plane waves propagating in the plane of symmetry Σ^S of a monoclinic viscoelastic medium. We choose the Cartesian coordinate system x_i in such a way that the plane of symmetry Σ^S corresponds to coordinate plane x_1x_3. In the Voigt notation, the density normalized elastic moduli $A_{\mu\nu}$ $(\mu, \nu = 1, 2, ..., 6)$ are then given by the matrix

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & 0 & A_{15} & 0 \\ A_{12} & A_{22} & A_{23} & 0 & A_{25} & 0 \\ A_{13} & A_{23} & A_{33} & 0 & A_{35} & 0 \\ 0 & 0 & 0 & A_{44} & 0 & A_{46} \\ A_{15} & A_{25} & A_{35} & 0 & A_{55} & 0 \\ 0 & 0 & 0 & A_{46} & 0 & A_{66} \end{pmatrix} . \tag{30}$$

See Carcione (2001, Equation (1.37)). For a viscoelastic monoclinic medium, the matrix remains the same, only $A_{\mu\nu}$ are complex-valued. As special cases of (30), we can consider an orthorhombic viscoelastic medium (with $A_{15} = A_{25} = A_{35} = A_{46} = 0$), a hexagonal viscoelastic medium with a vertical axis of symmetry ($A_{15} = A_{25} = A_{35} = A_{46} = 0$, and $A_{11} = A_{22}, A_{44} = A_{55}, A_{13} = A_{23}$, $A_{12} = A_{11} - 2A_{66}$), etc. We assume that matrix $\mathrm{Re}A_{\alpha\beta}$ is positive definite, and matrix $\mathrm{Im}A_{\alpha\beta}$ negative definite, or zero.

We concentrate on plane waves propagating in symmetry plane Σ^S, and consider

$$p_2 = 0 \ . \tag{31}$$

Thus, in this case the plane of symmetry Σ^S corresponds to propagation-attenuation plane $\Sigma^{\|}$. The elements of the 3×3 generalized Christoffel matrix $\Gamma_{ik} = a_{ijkl}p_j p_l$ are given by relations

$$\begin{aligned}
\Gamma_{11}(p_n) &= A_{11}p_1^2 + A_{55}p_3^2 + 2A_{15}p_1p_3 \ , \\
\Gamma_{22}(p_n) &= A_{66}p_1^2 + A_{44}p_3^2 + 2A_{46}p_1p_3 \ , \\
\Gamma_{33}(p_n) &= A_{55}p_1^2 + A_{33}p_3^2 + 2A_{35}p_1p_3 \ , \\
\Gamma_{13}(p_n) &= \Gamma_{31}(p_n) = A_{15}p_1^2 + A_{35}p_3^2 + (A_{13} + A_{55})p_1p_3 \ , \\
\Gamma_{12}(p_n) &= \Gamma_{21}(p_n) = \Gamma_{23}(p_n) = \Gamma_{32}(p_n) = 0 \ .
\end{aligned} \tag{32}$$

Constraint relation (3) factorizes:

$$\det[\Gamma_{ik}(p_n) - \delta_{ik}] = (\Gamma_{22}(p_n) - 1) \det \begin{pmatrix} \Gamma_{11}(p_n) - 1 & \Gamma_{13}(p_n) \\ \Gamma_{13}(p_n) & \Gamma_{33}(p_n) - 1 \end{pmatrix} = 0 \ . \tag{33}$$

Equation (33) can be satisfied in two cases:

1) **Case of SH plane waves**. The constraint relation reads

$$\Gamma_{22}(p_n) - 1 = 0 \ . \tag{34}$$

It follows from (2) that polarisation vector $\mathbf{U}$ is perpendicular to the plane of symmetry Σ^S,

$$\mathbf{U}(p_n) = (0, U_2(p_n), 0)^T \ . \tag{35}$$

It is common to call the relevant plane wave the SH plane wave.

2) **Case of qP and qSV waves**. The constraint relation reads

$$\det \begin{pmatrix} \Gamma_{11}(p_n) - 1 & \Gamma_{13}(p_n) \\ \Gamma_{13}(p_n) & \Gamma_{33}(p_n) - 1 \end{pmatrix} = 0 \ . \tag{36}$$

The polarisation vectors $\mathbf{U}$ of the relevant plane waves are situated in the plane of symmetry Σ^S,

$$\mathbf{U}(p_n) = (U_1(p_n) \ , \ 0 \ , \ U_3(p_n))^T \ , \tag{37}$$

where $U_1(p_n)$ and $U_3(p_n)$ satisfy the system of two complex-valued linear equations resulting from (2),

$$\begin{aligned}
(\Gamma_{11}(p_n) - 1)U_1 + \Gamma_{13}(p_n)U_3 &= 0 \ , \\
\Gamma_{13}(p_n)U_1 + (\Gamma_{33}(p_n) - 1)U_3 &= 0 \ .
\end{aligned} \tag{38}$$

Equation (38) should be supplemented by a proper normalization condition, see (18).

In this section, we derive the complete equations for the SH, qP and qSV homogeneous and inhomogeneous plane waves, propagating in symmetry plane Σ^S.

3.1 Directional specification of the slowness vector

We first treat the SH inhomogeneous plane waves, and after this briefly comment the qP and qSV waves.

a) **Case of SH waves**. For SH waves, constraint relation (34), with (32) for $\Gamma_{22}(p_n)$, yields

$$A_{66}p_1^2 + A_{44}p_3^2 + 2A_{46}p_1p_3 = 1 \ . \tag{39}$$

Inserting directional specification $p_i = \mathcal{C}^{-1}(N_i + \delta M_i)$, and separating the real and imaginary parts, we obtain two equations for δ and $\mathcal{C}^2$,

$$a_1 + \delta a_2 - \delta^2 a_3 = \mathcal{C}^2 \ , \tag{40}$$

$$-a_4 + \delta a_5 + \delta^2 a_6 = 0 \ , \tag{41}$$

where

$$\begin{aligned}
a_1 &= A_{66}^R N_1^2 + A_{44}^R N_3^2 + 2A_{46}^R N_1 N_3 \ , \\
a_2 &= -2[A_{66}^I N_1 M_1 + A_{44}^I N_3 M_3 + A_{46}^I (M_1 N_3 + N_1 M_3)] \ , \\
a_3 &= A_{66}^R M_1^2 + A_{44}^R M_3^2 + 2A_{46}^R M_1 M_3 \ , \\
a_4 &= -[A_{66}^I N_1^2 + A_{44}^I N_3^2 + 2A_{46}^I N_1 N_3] \ , \\
a_5 &= 2[A_{66}^R N_1 M_1 + A_{44}^R N_3 M_3 + A_{46}^R (N_1 M_3 + M_1 N_3)] \ , \\
a_6 &= -[A_{66}^I M_1^2 + A_{44}^I M_3^2 + 2A_{46}^I M_1 M_3] \ .
\end{aligned} \tag{42}$$

Here we have used the notation $A_{ij}^R = \mathrm{Re}(A_{ij})$, $A_{ij}^I = \mathrm{Im}(A_{ij})$.

For a given model of a viscoelastic monoclinic medium (30), and for given unit vectors $\mathbf{N} \equiv (N_1, 0, N_3)$ and $\mathbf{M} \equiv (M_1, 0, M_3)$, quantities a_i, $i = 1, 2, ..., 6$ in (42) are known. Thus, Equations (40) and (41) can be used to determine δ and $\mathcal{C}^2$.

An analogous system of equations for an SH inhomogeneous plane wave propagating in a viscoelastic transversely isotropic medium was first derived and numerically studied by Krebes and Le (1994). For SH inhomogeneous plane waves propagating in the symmetry plane of a monoclinic viscoelastic medium see Carcione and Cavallini (1995), Carcione (2001).

A brief note on the coefficients a_i in (42). We have assumed that matrix $\mathrm{Re}A_{\alpha\beta}$ is positive definite, and matrix $\mathrm{Im}A_{\alpha\beta}$ negative definite or zero, where $A_{\alpha\beta}$ is given by (30). For SH waves, the 6×6 matrix $A_{\alpha\beta}$ reduces to the 2×2 matrix:

$$\begin{pmatrix} A_{66} & A_{46} \\ A_{46} & A_{44} \end{pmatrix} \ . \tag{43}$$

Then a_1 and a_3 are positive, a_4 and a_6 non-negative, but a_2 and a_5 may be positive, zero, or negative.

Equation (41) for δ is quadratic, and for $a_6 \neq 0$ yields the solutions,

$$\delta_{1,2} = -a_5/2a_6 \pm \sqrt{(a_5/2a_6)^2 + a_4/a_6} \ . \tag{44}$$

As the imaginary part of matrix (43) is negative definite, or zero, the expression under square root in (44) is non-negative, so that δ_1 and δ_2 are real-valued. As δ must be non-negative, we have to use the "+" sign in (44),

$$\delta = -a_5/2a_6 + \sqrt{(a_5/2a_6)^2 + a_4/a_6}\ . \tag{45}$$

Inserting this δ into (40) yields the square of phase velocity, $\mathcal{C}^2$.

Thus, for given directions $\mathbf{N}$ and $\mathbf{M}$ and for $a_6 \neq 0$, (45) and (40) represent the final equations for the determination of phase velocity $\mathcal{C}$ and attenuation-propagation ratio δ. Slowness vector $\mathbf{p}$ is then given by (9), and attenuation angle γ by relation (8), $\cos\gamma = \mathbf{N} \cdot \mathbf{M}$.

If $a_6 = 0$, also $a_4 = 0$. This means that the medium is perfectly elastic. Equation (41) then yields $a_5\delta = 0$. This is satisfied in two cases: a) $\delta = 0$, a_5 arbitrary: Note that $\delta = 0$ indicates the real-valued slowness vector. Thus, a regular plane wave propagating in a perfectly elastic medium is obtained. Its phase velocity is $\mathcal{C} = \sqrt{a_1}$, see (40) for $\delta = 0$. b) $a_5 = 0$, δ arbitrary: This corresponds to an inhomogeneous plane wave propagating in a perfectly elastic medium. Condition $a_5 = 0$, however, may be then satisfied only for a special choice of $\mathbf{N}$ and $\mathbf{M}$. In isotropic media ($A_{44} = A_{66}$, $A_{46} = 0$), condition $a_5 = 0$ leads to the well-known condition of perpendicularity of $\mathbf{N}$ and $\mathbf{M}$. In anisotropic media, condition $a_5 = 0$ is satisfied for $\mathbf{N}$ and $\mathbf{M}$ making an angle different from 90^0 (but strictly defined).

There is, however, a problem in the determination of $\mathcal{C}^2$ and δ using (40) and (45). For certain combinations of $\mathbf{N}$ and $\mathbf{M}$, Equations (45) with (40) can yield $\mathcal{C}^2 < 0$. This interesting result was discovered independently by Krebes and Le (1994) and by Carcione and Cavallini (1995), see also Carcione (2001). Let us consider a fixed attenuation angle γ ($\cos\gamma = \mathbf{N} \cdot \mathbf{M}$), and unit propagation vector $\mathbf{N}$ varying. Carcione (2001, Section 4.4.6) calls $\mathbf{N}$ for which $\mathcal{C}^2 \leq 0$ **forbidden directions of propagation**. Carcione and Cavallini (1995) also speak of "stop bands", where there is no wave propagation.

Let us denote by γ^* the attenuation angle γ for which phase velocity $\mathcal{C}$ vanishes ($\mathcal{C} = 0$), and call it the *boundary attenuation angle*. Direction $\mathbf{N}$ is a forbidden direction if the relevant attenuation angle γ satisfies the relation $\cos\gamma = \mathbf{N} \cdot \mathbf{M} \leq \cos\gamma^*$.

In isotropic viscoelastic media, the situation is simple, as the boundary attenuation angle γ^* always equals 90^0. In anisotropic viscoelastic media, however, the boundary attenuation angle γ^* is, in general, different from 90^0, and also depends on other parameters. For a more detailed discussion, see Section 3.4 and numerical examples in Červený and Pšenčík (2004).

Equations (45) and (40) do not allow the boundary attenuation angle γ^* to be computed in advance. If we wish to determine it, we must seek numerically such $\mathbf{N}$ and $\mathbf{M}$ from (40) and (41), for which $\mathcal{C} = 0$. This is the disadvantage of the directional specification of the slowness vector. If we use the mixed specification of the slowness vector, see Sections 3.3 and 3.4, the boundary attenuation angle γ^* can be determined explicitly.

If directions $\mathbf{N}$ and $\mathbf{M}$ are chosen in such a way that $\cos\gamma = \mathbf{N} \cdot \mathbf{M} > \cos\gamma^*$, Equations (45), (40) and (9) represent the final solution of the problem for SH inhomogeneous plane waves.

b) For **qP and qSV inhomogeneous plane waves**, the constraint relation is given by (36), with $\Gamma_{11}(p_n)$, $\Gamma_{33}(p_n)$ and $\Gamma_{13}(p_n)$ given by (32). The two relevant coupled equations for δ and $\mathcal{C}^2$ are not simple to solve. Moreover, also the non-physical solutions ($\mathcal{C}^2 \leq 0$) may be obtained. For this reason, we do not discuss them here. The method based on the mixed specification of the slowness vector, see Section 3.3, yields a simpler algorithm and avoids the problem of non-physical solutions completely.

3.2 Componental specification of the slowness vector

We now apply componental specification (10) to inhomogeneous and homogeneous plane waves propagating in the plane of symmetry Σ^S of monoclinic viscoelastic media, see (30). We assume that unit normal $\mathbf{n}$ to plane Σ is situated in the plane of symmetry Σ^S, so that $n_2 = 0$. We also assume $p_2^\Sigma = 0$.

We consider first SH plane waves, and then qSV and qP plane waves. The results may find applications in the solution of reflection/transmission problems of inhomogeneous plane waves, and in the initial-value problem, in which the initial values of slowness vector $\mathbf{p}^\Sigma$ are given along an initial plane Σ.

a) **Case of SH waves**. Inserting the componental specification $p_i = \sigma n_i + p_i^\Sigma$ into the constraint relation (34) for SH waves yields

$$A_{44}(\sigma n_3 + p_3^\Sigma)^2 + A_{66}(\sigma n_1 + p_1^\Sigma)^2 + 2A_{46}(\sigma n_1 + p_1^\Sigma)(\sigma n_3 + p_3^\Sigma) = 1 \ . \qquad (46)$$

This is a quadratic equation in σ,

$$\sigma^2\Gamma_{22} + 2\sigma E_{22} + F_{22} - 1 = 0 \ , \qquad (47)$$

where Γ_{22}, E_{22} and F_{22} are given by relations, see (32),

$$\begin{aligned} \Gamma_{22} &= A_{66}n_1^2 + A_{44}n_3^2 + 2A_{46}n_1n_3 \ , \\ E_{22} &= A_{66}n_1p_1^\Sigma + A_{44}n_3p_3^\Sigma + A_{46}(n_1p_3^\Sigma + n_3p_1^\Sigma) \ , \\ F_{22} &= A_{66}p_1^{\Sigma 2} + A_{44}p_3^{\Sigma 2} + 2A_{46}p_1^\Sigma p_3^\Sigma \ . \end{aligned} \qquad (48)$$

The two solutions σ are as follows,

$$\sigma_{1,2} = \left[-E_{22} \pm \sqrt{\Gamma_{22} + E_{22}^2 - \Gamma_{22}F_{22}}\right]/\Gamma_{22} \ . \qquad (49)$$

Using (48) and (32), Equation (49) can be expressed in a simpler form,

$$\sigma_{1,2} = -(E_{22}/\Gamma_{22}) \pm \sqrt{1/\Gamma_{22} - (p_1^\Sigma n_3 - n_1p_3^\Sigma)^2\Delta/\Gamma_{22}^2} \ , \qquad (50)$$

where quantity Δ is given by the relation

$$\Delta = A_{44}A_{66} - A_{46}^2 \ . \qquad (51)$$

b) **Case of qSV and qP waves**. Inserting $p_i^\Sigma = \sigma n_i + p_i^\Sigma$ into constraint relation (36) yields an algebraic equation of the fourth degree in σ,

$$\sigma^4 a_1 + 2\sigma^3 b_1 + \sigma^2(4a_2 + b_2 - c_1) + 2\sigma(b_3 - c_2) + a_3 - c_3 + 1 = 0 \ , \qquad (52)$$

where

$$\begin{array}{lll} a_1 = \Gamma_{11}\Gamma_{33} - \Gamma_{13}^2 \ , & b_1 = \Gamma_{11}E_{33} + F_{33}E_{11} - 2\Gamma_{13}E_{13} \ , & c_1 = \Gamma_{11} + \Gamma_{33} \ , \\ a_2 = E_{11}E_{33} - E_{13}^2 \ , & b_2 = \Gamma_{11}F_{33} + \Gamma_{33}F_{11} - 2\Gamma_{13}F_{13} \ , & c_2 = E_{11} + E_{33} \ , \\ a_3 = F_{11}F_{33} - F_{13}^2 \ , & b_3 = F_{11}E_{33} + F_{33}E_{11} - 2F_{13}E_{13} \ , & c_3 = F_{11} + F_{33} \ . \end{array} \qquad (53)$$

We have also used the notation

$$\Gamma_{ij} = \Gamma_{ij}(\mathbf{n}) \ , \qquad F_{ij} = \Gamma_{ij}(\mathbf{p}^\Sigma) \ , \qquad (54)$$

where $\Gamma_{ij}(\mathbf{n})$ and $\Gamma_{ij}(\mathbf{p}^\Sigma)$ are given by (32), with $\mathbf{p}$ substituted by $\mathbf{n}$ and $\mathbf{p}^\Sigma$. Finally, quantities E_{ij} are given by relations

$$\begin{array}{lll} E_{11} & = & A_{11}n_1p_1^\Sigma + A_{55}n_3p_3^\Sigma + A_{15}(n_1p_3^\Sigma + n_3p_1^\Sigma) \ , \\ E_{33} & = & A_{55}n_1p_1^\Sigma + A_{33}n_3p_3^\Sigma + A_{35}(n_1p_3^\Sigma + n_3p_1^\Sigma) \ , \\ E_{13} & = & A_{15}n_1p_1^\Sigma + A_{35}n_3p_3^\Sigma + \frac{1}{2}(A_{13} + A_{55})(n_1p_3^\Sigma + n_3p_1^\Sigma) \ . \end{array} \qquad (55)$$

If the density normalized viscoelastic moduli $A_{11}, A_{13}, A_{15}, A_{33}, A_{35}$ and A_{55}, unit vector $\mathbf{n}$ and vector $\mathbf{p}^\Sigma$ are known, all coefficients of equation (52) can be determined, and (52) can be solved for σ.

Inserting solution of (47) or (52) into (23), we obtain the relevant equations for propagation vector $\mathbf{P}$, attenuation vector $\mathbf{A}$, unit vectors $\mathbf{N}$ and $\mathbf{M}$, phase velocity $\mathcal{C}$, attenuation-propagation ratio δ, and attenuation angle γ.

3.3 Mixed specification of the slowness vector

The mixed specification may be treated as a special case of the componental specification discussed in Section 3.2, if we put $\mathbf{p}^\Sigma = \mathrm{i}D\mathbf{m}$, see (11). Consequently, for SH, qP and qSV plane waves propagating in the plane of symmetry Σ^S of a monoclinic viscoelastic medium, we can use the equations of the previous section, where we insert $\mathbf{p}^\Sigma = \mathrm{i}D\mathbf{m}$. Here $\mathbf{m}$ is a unit vector perpendicular to $\mathbf{n}$, and D is the inhomogeneity parameter, $D \in (-\infty, \infty)$.

As $\mathbf{m}$ is assumed to be perpendicular to $\mathbf{n} = (n_1, 0, n_3)$ and situated in the symmetry plane $\Sigma^\parallel$, we use $\mathbf{m}$ in the following form:

$$\mathbf{m} = (n_3, 0, -n_1)^T \ . \qquad (56)$$

a) **Case of SH waves**. Inserting (11) and (56) into (50), we obtain simple explicit formulae for σ

$$\sigma_{1,2} = -\mathrm{i}D\Lambda/\Gamma_{22} \pm [1/\Gamma_{22} + D^2\Delta/\Gamma_{22}^2]^{1/2} \ , \qquad (57)$$

where

$$\begin{aligned}
\Gamma_{22} &= A_{66}n_1^2 + A_{44}n_3^2 + 2A_{46}n_1n_3 \ , \\
\Lambda &= (A_{66} - A_{44})n_1n_3 + A_{46}(n_3^2 - n_1^2) \ ,
\end{aligned} \tag{58}$$

and where Δ is given by (51).

b) **Case of qP and qSV waves**. In this case, we obtain a quartic equation for σ. Inserting (11) and (56) into (52) yields

$$\sigma^4 A_1 + 2\mathrm{i}D\sigma^3 B_1 - \sigma^2[D^2(4A_2+B_2)+C_1] + 2\mathrm{i}\sigma(D^3B_3+DC_2) + D^4A_3 + D^2C_3 + 1 = 0 \ . \tag{59}$$

Here A_i, B_i and C_i $(i = 1, 2, 3)$ are given by the same expressions as a_i, b_i, c_i, see (53), only $\mathbf{p}^\Sigma$ in E_{ij} and F_{ij} is substituted by $\mathbf{m}$. Standard methods for solving quartic equations can be used to determine the four roots σ of (59).

Once the values of σ have been found, we can use (28) and determine the relevant real-valued propagation vector $\mathbf{P}$, attenuation vector $\mathbf{A}$, unit vectors $\mathbf{N}$ and $\mathbf{M}$, phase velocity $\mathcal{C}$, attenuation-propagation ratio δ and attenuation angle γ, related to the SH, qSV and qSH inhomogeneous plane under consideration,

$$\begin{array}{ll}
P_1 = n_1(\mathrm{Re}\sigma) \ , & A_1 = n_1(\mathrm{Im}\sigma) + n_3D \ , \\
P_3 = n_3(\mathrm{Re}\sigma) \ , & A_3 = n_3(\mathrm{Im}\sigma) - n_1D \ , \\
|\mathbf{P}| = |\mathrm{Re}\sigma| \ , & |\mathbf{A}| = [(\mathrm{Im}\sigma)^2 + D^2]^{1/2} \ , \\
N_1 = \epsilon n_1 \ , & M_1 = [n_1(\mathrm{Im}\sigma) + n_3D]/[(\mathrm{Im}\sigma)^2 + D^2]^{1/2} \ , \\
N_3 = \epsilon n_3 \ , & M_3 = [n_3(\mathrm{Im}\sigma) - n_1D]/[(\mathrm{Im}\sigma)^2 + D^2]^{1/2} \ , \\
\mathcal{C} = 1/|\mathrm{Re}\sigma| \ , & \delta = [(\mathrm{Im}\sigma)^2 + D^2]^{1/2}/|\mathrm{Re}\sigma| \ , \\
\cos\gamma = \epsilon\,\mathrm{Im}\sigma/[(\mathrm{Im}\sigma)^2 + D^2]^{1/2} \ . &
\end{array} \tag{60}$$

Here ϵ is given by (29). We also get $P_2 = 0$, $A_2 = 0$, $N_2 = 0$, $M_2 = 0$.

3.4 Some properties of plane SH waves

The mixed specification of slowness vector $\mathbf{p}$, particularly Equations (57), (58) and (60), offers a simple possibility to investigate the properties of homogeneous and inhomogeneous plane SH waves, propagating in the symmetry planes of viscoelastic anisotropic media. Actually, this is the only case of inhomogeneous plane waves propagating in anisotropic viscoelastic media, which can be investigated analytically. We present here only the results which follow from simple analytic considerations. For a detailed numerical treatment, see Červený and Pšenčík (2004).

a) **Direction of propagation N**. As we can see from (60), $\mathbf{N} = \epsilon\,\mathbf{n}$, where ϵ equals either $+1$, or -1. Thus, the unit vector $\mathbf{N}$ specifying the direction of propagation vector $\mathbf{P}$ (perpendicular to the wavefront) equals $\mathbf{n}$, or is opposite to it. In the directional specification, unit vector $\mathbf{N}$ is specified in advance, and we seek plane waves, whose wavefront propagates only in the direction of $\mathbf{N}$. In the mixed specification, however, unit vector $\mathbf{n}$ is specified in advance, and we seek *all plane waves*, whose wavefronts propagate in the directions of $\mathbf{N} = +\mathbf{n}$ and $\mathbf{N} = -\mathbf{n}$. The actual orientation of $\mathbf{N}$ depends on the sign of $\mathrm{Re}\sigma$. For $\mathrm{Re}\sigma > 0$, we obtain $\mathbf{N} = \mathbf{n}$, and for $\mathrm{Re}\sigma < 0$ $\mathbf{N} = -\mathbf{n}$.

b) **Attenuation angle** γ. It is introduced here as a positive angle between unit vectors $\mathbf{N}$ and $\mathbf{M}$, $0^0 \leq \gamma \leq 180^0$, in such a way that $\cos\gamma = \mathbf{N} \cdot \mathbf{M}$. In the planar case that we are treating, it has been common to use an "oriented attenuation angle γ" in the seismological literature (although the word "oriented" has been considered only tacitly). The oriented attenuation angle γ would then be uniquely defined in the range $-180^0 < \gamma < 180^0$. Simply speaking, the sign of γ specifies, to which side of propagation vector $\mathbf{P}$ attenuation vector $\mathbf{A}$ is pointing. Such a definition of γ has been particularly useful in the directional specification of the slowness vector, where γ is used as an input parameter of the inhomogeneous plane wave under consideration. In the mixed specification, however, we use D as the input parameter, and the correct orientation of $\mathbf{A}$ is fully specified by the sign of D. Thus, we do not need to consider an oriented attenuation angle γ at all. We always define γ by (60), in the range $0 \leq \gamma \leq 180^0$. For $0 \leq \cos\gamma \leq 1$, we obtain $0^0 \leq \gamma \leq 90^0$, and for $-1 \leq \cos\gamma \leq 0$, we get $90^0 \leq \gamma \leq 180^0$.

c) **Roots** σ_1 and σ_2. It is reasonable to assume that the following relations are valid for any D:

$$\mathrm{Re}\sigma_1 > 0 \ , \qquad \mathrm{Re}\sigma_2 < 0 \ . \tag{61}$$

For homogeneous plane waves ($D = 0$), the validity of (61) is obvious from (57). In general, if relations (61) were not satisfied everywhere, then $\mathrm{Re}\sigma$ would have to be zero and phase velocity $\mathcal{C}$ would be infinite at least for one D. This would not be physically acceptable. Consequently, root $\sigma = \sigma_1$ corresponds to $\mathbf{N} = \mathbf{n}$ (the wavefront propagating along $\mathbf{n}$), and root $\sigma = \sigma_2$ corresponds to $\mathbf{N} = -\mathbf{n}$ (the wavefront propagating against $\mathbf{n}$).

d) **Exponential decay of amplitudes**. **Attenuation** $|\mathbf{A}|$. The maximum exponential decay of amplitudes occurs along attenuation vector $\mathbf{A}$. In our treatment, however, we decompose attenuation vector $\mathbf{A}$ into its two Cartesian components. The first component is along the direction of propagation $\mathbf{N}$, and the second along the wavefront. The exponential decay along $\mathbf{N}$ is controlled by quantity $\mathrm{Im}\sigma$, and the exponential decay along the wavefront by D. For homogeneous plane waves, the exponential decay is only along $\mathbf{N}$. Contrary to it, the decay is only along the wavefront if $\mathrm{Im}\sigma = 0$. The complete attenuation effect is given by the modulus $|\mathbf{A}|$ of attenuation vector $\mathbf{A}$, $|\mathbf{A}| = [(\mathrm{Im}\sigma)^2 + D^2]^{1/2}$, see (60). $|\mathbf{A}|$ is briefly referred to as the *attenuation*.

e) **Homogeneous SH plane waves**. For homogeneous plane waves, $\mathbf{N} \equiv \mathbf{M}$, and inhomogeneity parameter D vanishes. Then (57) yields,

$$\sigma_{1,2} = \pm\sqrt{1/\Gamma_{22}} \ , \tag{62}$$

where Γ_{22} is given by (58). Equations (62), (58) and (60) yield simple expressions for phase velocity $\mathcal{C}$, attenuation $|\mathbf{A}|$ and attenuation-propagation ratio δ,

$$\mathcal{C} = 1/|\mathrm{Re}\sqrt{1/\Gamma_{22}}| \ , \quad |\mathbf{A}| = |\mathrm{Im}\sqrt{1/\Gamma_{22}}| \ , \quad \delta = |\mathrm{Im}\sqrt{1/\Gamma_{22}}|/|\mathrm{Re}\sqrt{1/\Gamma_{22}}| \ . \tag{63}$$

f) **Isotropic viscoelastic media**. The properties of inhomogeneous plane waves propagating in isotropic viscoelastic media have been extensively studied in the seismological literature. See, e.g., Buchen (1971), Borcherdt (1973, 1977), Auld (1973), Aki and Richards (1980), Krebes (1983), Borcherdt and Wennerberg (1986), Borcherdt, Glassmoyer and Wennerberg (1986), Caviglia and Morro (1992), Brokešová and Červený (1998), Carcione (2001), etc. Most of these studies, however, use attenuation angle γ as the basic parameter of the inhomogeneous plane wave. It is interesting to see how the results change, if the mixed specification of the slowness vector with inhomogeneity parameter D were used.

For isotropic viscoelastic media, the values of σ follow immediately from (57), if we use $A_{44} = A_{66}$ and $A_{46} = 0$. We then obtain a surprisingly simple result,

$$\sigma_{1,2} = \pm[1/A_{44} + D^2]^{1/2} . \tag{64}$$

Quantity σ is applicable to inhomogeneous plane S waves propagating in any 3-D medium, in any direction. It is invariant with respect to unit vectors $\mathbf{n}$ and $\mathbf{m}$. If we use a standard notation $A_{44} = V_S^2(1 - \mathrm{i}/Q_S)$, where V_S is the real-valued velocity and Q_S the real-valued quality factor of S waves, we obtain,

$$\sigma_{1,2} = \pm[1/V_S^2(1 - \mathrm{i}/Q_S) + D^2]^{1/2} . \tag{65}$$

Let us summarize the properties of homogeneous and inhomogeneous plane S waves propagating in isotropic viscoelastic media, which follow from Equations (64) and (65): i) The values of σ and of all consequent quantities are direction independent, and do not depend on the sign of D (they are symmetrical with respect to $D = 0$). ii) The maximum phase velocity is always obtained for homogeneous plane waves ($D = 0$), and equals $1/|\mathrm{Re}\sqrt{1/A_{44}}|$. With increasing $|D|$ the phase velocity decreases as $1/|\mathrm{Re}\sqrt{1/A_{44} + D^2}|$, and for $|D| \to \infty$, it approaches zero. iii) The minimum attenuation is always obtained for homogeneous plane waves ($D = 0$). With increasing $|D|$, attenuation $|\mathbf{A}|$ grows. iv) The attenuation angle γ is zero for homogeneous plane waves ($D = 0$), and increases to 90^0 with $|D|$ increasing. Boundary attenuation angle γ^* is always 90^0.

Equations (64) and (65) remain valid even for inhomogeneous plane S waves propagating in perfectly elastic isotropic media; we only insert $1/Q_S = 0$ or $\mathrm{Im}(A_{44}) = 0$. Then $\mathcal{C}_{1,2} = 1/\sqrt{1/V_S^2 + D^2}$, and $\mathrm{Im}\sigma = 0$. Attenuation angle γ depends neither on direction, nor on D, and equals 90^0. For $D = 0$, the expression (60) for $\cos\gamma$ is indefinite, but both limits ($D \to 0$ and $-D \to 0$) yield $\gamma = 90^0$.

g) **Weakly inhomogeneous plane waves**. For small D, the square root in (57) may be expanded in terms of D. Then we obtain approximately:

$$\mathrm{Re}\sigma_{1,2} \doteq \pm a + bD \pm cD^2 , \qquad \mathrm{Im}\sigma_{1,2} \doteq \pm e - fD \pm gD^2 . \tag{66}$$

Here

$$\begin{aligned} a &= \mathrm{Re}(1/\Gamma_{22})^{1/2} , \quad b = \mathrm{Im}(\Lambda/\Gamma_{22}) , \quad c = \tfrac{1}{2}\mathrm{Re}(\Delta/\Gamma_{22}^{3/2}) , \\ e &= \mathrm{Im}(1/\Gamma_{22})^{1/2} , \quad f = \mathrm{Re}(\Lambda/\Gamma_{22}) , \quad g = \tfrac{1}{2}\mathrm{Im}(\Delta/\Gamma_{22}^{3/2}) . \end{aligned} \tag{67}$$

Coefficient g is usually very small. Thus, for small D, $\mathrm{Re}\sigma_{1,2}$ can be approximated by a quadratic parabola, and $\mathrm{Im}\sigma_{1,2}$ by a straight line. The minima (maxima) of these parabolas correspond to $D = D_{1,2}^M$, where

$$D_{1,2}^M \doteq \mp b/2c . \tag{68}$$

As phase velocity $\mathcal{C}$ is given by the relation $\mathcal{C} = 1/|\mathrm{Re}\sigma|$, the phase velocities $\mathcal{C}^M_{1,2}$ corresponding to $D^M_{1,2}$ are

$$\mathcal{C}^M_{1,2} \doteq 1/(a - b^2/4c) \ . \tag{69}$$

Thus, the maximum phase velocity (as a function of D) does not correspond to the homogeneous plane wave ($D = 0$), but to the inhomogeneous plane wave with inhomogeneity parameter $D = D^M$, given by (68).

Similarly, we can determine the value of $D = D_0$, at which $\mathrm{Im}\sigma$ equals zero:

$$D_0 \doteq \pm e/f \ . \tag{70}$$

For $D = D_0$, attenuation angle γ equals 90^0. Moreover, the value of D_0 is a boundary between the $D's$, for which amplitudes of inhomogeneous plane waves exponentially decay or grow along the direction of propagation $\mathbf{N}$.

Finally, attenuation $|\mathbf{A}|$ is minimum for the inhomogeneity parameter $D = D_{att}$, given by the relation

$$D_{att} \doteq \pm ef/(1 + f^2) \ . \tag{71}$$

The attenuation $|\mathbf{A}|$ at $D = D_{att}$ is given by the relation

$$A_{min} \doteq e/\sqrt{1 + f^2} \ , \tag{72}$$

and is here referred to as the minimum attenuation. Thus, the minimum attenuation (as a function of D) does not correspond to the homogeneous plane wave ($D = 0$), but to the inhomogeneous plane wave with the inhomogeneity parameter $D = D_{att}$, given by (71).

The inhomogeneity parameters D^M and D_{att} play an important role also in the investigation of the energy flux of weakly inhomogeneous SH plane waves propagating in symmetry planes of an anisotropic viscoelastic medium. Although we do not discuss the energy flux in this paper, we present here (without derivation) two interesting properties valid for $D = D^M$ and $D = D_{att}$. We introduce the complex valued Poynting vector $\mathbf{F}$, and the time-averaged, real-valued energy flux $\mathbf{S} = \mathrm{Re}\mathbf{F}$. These quantities can be simply computed, once the complex-valued slowness vector $\mathbf{p}$ is known. Then, for $D = D_{att}$, energy flux $\mathbf{S}$ and attenuation vector $\mathbf{A}$ are linearly dependent ($\mathbf{S}$ is parallel to $\mathbf{A}$). Similarly, for $D = D^M$, the real and imaginary parts of the Poynting vector, $\mathrm{Re}\mathbf{F}$ and $\mathrm{Im}\mathbf{F}$, are linearly dependent ($\mathrm{Re}\mathbf{F} \times \mathrm{Im}\mathbf{F} = \mathbf{0}$).

h) **Strongly inhomogeneous SH plane waves**. For large $|D|$, Equations (57) yield approximately,

$$\begin{aligned} \mathrm{Re}\sigma_{1,2} &\doteq D\,\mathrm{Im}(\Lambda/\Gamma_{22}) \pm |D|\mathrm{Re}(\Delta/\Gamma^2_{22})^{1/2} \ , \\ \mathrm{Im}\sigma_{1,2} &\doteq -D\,\mathrm{Re}(\Lambda/\Gamma_{22}) \pm |D|\mathrm{Im}(\Delta/\Gamma^2_{22})^{1/2} \ . \end{aligned} \tag{73}$$

We denote all quantities corresponding to the limiting case of infinite $|D|$ by asterisks. For $|D| \to \infty$, the first equation of (73) with (60) immediately yields

$$\mathcal{C}^*_{1,2} = 0 \ . \tag{74}$$

Of course, the zero phase velocity $\mathcal{C}^* = 0$ is not realistic; it corresponds to the limiting case of $|D| \to \infty$, which cannot, in fact, occur. This limiting case, however, plays an important role in the discussion of inhomogeneous plane waves. Similarly, the second equation yields the expressions for the limiting (boundary) attenuation angles $\gamma^*_{1,2}$:

$$\cos\,\gamma^*_{1,2} = \theta_{1,2}/[\theta^2_{1,2} + 1]^{1/2}\ , \tag{75}$$

where $\theta_{1,2}$ are given by the relation

$$\theta_{1,2} = \mathrm{Im}(\Delta/\Gamma^2_{22})^{1/2} \mp (D/|D|)\mathrm{Re}(\Lambda/\Gamma_{22})\ . \tag{76}$$

Thus, the boundary attenuation angle γ^* does not depend on the magnitude of D, but only on the sign of D.

i) **Perfectly elastic anisotropic media**. The density normalized moduli A_{44}, A_{66} and A_{46} are real-valued in this case. Consequently, also Γ_{22}, Λ and Δ are real-valued, and (57) yields

$$\mathrm{Re}\sigma_{1,2} = \pm[1 + D^2\Delta/\Gamma_{22}]^{1/2}/\Gamma^{1/2}_{22}\ , \quad \mathrm{Im}\sigma_{1,2} = -D\,\Lambda/\Gamma_{22}\ . \tag{77}$$

Consequently, phase velocity $\mathcal{C}$ and attenuation angle γ are given by the relations

$$\mathcal{C}_{1,2} = \Gamma^{1/2}_{22}/[1 + D^2\Delta/\Gamma_{22}]^{1/2}\ , \qquad \cos\,\gamma_{1,2} = \mp(D/|D|)\Lambda/[\Gamma^2_{22} + \Lambda^2]^{1/2}\ . \tag{78}$$

The expression for $\mathcal{C}_{1,2}$ shows how $\mathcal{C} \to 0$ for $|D| \to \infty$. Attenuation angle γ does not depend on D, but only on the sign of D. If the medium is anisotropic and $\Lambda \neq 0$, attenuation angle γ is different from 90^0.

i) **Polarization of SH waves**. Both homogeneous and inhomogeneous SH plane waves are always linearly polarized in the direction perpendicular to Σ^S, see (35).

3.5 Some properties of plane qP and qSV waves

The properties of the slowness vectors of plane qP and qSV waves propagating in the plane of symmetry of a monoclinic viscoelastic medium are very similar to those of SH inhomogeneous plane waves, discussed in the preceding section. It is, however, usually more difficult to find simple analytical estimates for the individual quantities, as the case of qP and qSV inhomogeneous plane waves requires the solution of a quartic equation. Mostly, the numerical treatment is more effective, see Červený and Pšenčík (2004).

In two cases, however, the quartic equation in σ reduces to a quadratic equation in σ^2. This applies to homogeneous qP and qSV waves, propagating in a symmetry plane of an anisotropic viscoelastic medium, and to P and SV inhomogeneous plane waves, propagating in an isotropic viscoelastic medium. We present explicit solutions for both cases, as perturbation methods may be used to extend them to qP and qSV weakly inhomogeneous plane waves (small D) propagating in viscoelastic anisotropic media, and to qP and qSV inhomogeneous plane waves propagating in anisotropic, weakly dissipative media. We also briefly discuss, more or less qualitatively, some more general cases, for example strongly inhomogeneous plane waves.

We do not discuss here again points (a)-(d) of Section 3.4, derived for SH waves. They remain valid even for qP and qSV waves.

a) **Homogeneous plane qP and qSV waves**. For $D = 0$, the quartic equation in σ (59) reduces to the quadratic equation in σ^2,

$$\sigma^4 A_1 - \sigma^2 C_1 + 1 = 0 \; , \tag{79}$$

where $A_1 = \Gamma_{11}\Gamma_{33} - \Gamma_{13}^2$ and $C_1 = \Gamma_{11} + \Gamma_{33}$ are complex-valued coefficients. Consequently, the roots σ^2 for qP and qSV waves can be expressed analytically,

$$\sigma^2 = \tfrac{1}{2}\left[\Gamma_{11} + \Gamma_{33} \mp \sqrt{(\Gamma_{11} - \Gamma_{33})^2 + 4\Gamma_{13}^2}\right] / (\Gamma_{11}\Gamma_{33} - \Gamma_{13}^2) \; . \tag{80}$$

Here the "$-$" sign is for qP waves, the "$+$" sign for qSV waves. Each σ^2 in (80) yields two solutions σ, one to each side of Σ, both for qP and qSV waves.

We can also verify that the general relation $\sigma^2 = 1/G$, given in (26), remains valid. Here G is an eigenvalue of the complex-valued Christoffel matrix, corresponding to qP or qSV wave, and satisfying the quadratic equation $G^2 - C_1 G + A_1 = 0$.

b) **Plane P and SV waves in isotropic viscoelastic media**. In isotropic media, B_1, C_2 and B_3 in (59) vanish. Thus, the quartic equation in σ reduces to a quadratic equation in σ^2. If we further use $A_1 = A_3 = A_{11}A_{55}$, $A_2 = C_1 = C_3 = A_{11} + A_{55}$ and $B_2 = A_{11}^2 + A_{55}^2$, valid for isotropic media, the quadratic equation reads

$$\sigma^4 A_{11}A_{55} - \sigma^2[2D^2 A_{11}A_{55} + A_{11} + A_{55}] + D^4 A_{11}A_{55} + D^2(A_{11} + A_{55}) + 1 = 0 \; . \tag{81}$$

Solving this equation, we obtain two simple solutions for σ^2

$$\begin{aligned} \sigma^2 &= 1/A_{11} + D^2 \; , \qquad && \text{for P waves} \; , \\ \sigma^2 &= 1/A_{55} + D^2 \; , \qquad && \text{for SV waves} \; . \end{aligned} \tag{82}$$

Thus, for isotropic viscoelastic media, the expressions for σ^2 for plane P and SV waves, have exactly the same form as for SH waves, see (64). Actually, this has been expected. For SH and SV waves, the expressions are identical, as $A_{44} = A_{55}$, and for P waves, we use A_{11} instead of A_{44} (or A_{55}). Equation (65) can also be used for all the three waves, only for P waves we replace V_S and Q_S by V_P and Q_P. The same conclusions (i)-(iv), given in Section 3.4/f for SH waves, also apply to P and SV waves. In addition, however, we obtain one interesting property of phase velocities of P and S waves. We take into account that $\sigma^2 \to D^2$ for $|D|$ increasing. This implies that the differences between the phase velocities of P and S waves decrease with increasing D.

c) **Weakly inhomogeneous plane qP and qSV waves**. Similarly as for SH waves, we can derive approximate relations for σ, assuming that the inhomogeneity parameter D is small. We put

$$\sigma \doteq \sigma_0 + \nu D \; , \tag{83}$$

where σ_0 is given by (80), and corresponds to homogeneous plane waves. Using (59), we obtain an approximate relation for ν, valid for D small,

$$\nu \doteq -\mathrm{i}(\sigma_0^2 B_1 - C_2)/(2\sigma_0^2 A_1 - C_1) \; . \tag{84}$$

It follows from (83) and (84) that $\nu \doteq 0$ and $\sigma(D) \doteq \sigma(-D)$ for isotropic viscoelastic media (where $B_1 = C_2 = 0$), but not for anisotropic viscoelastic media. For weakly dissipative anisotropic media, ν is approximately purely imaginary. Consequently, for small D, the differences between $\sigma(D)$ and $\sigma(-D)$ are particularly expressed in $\mathrm{Im}\sigma$ and in related quantities ($\cos\gamma$). For small D, the quantity $\mathrm{Im}\sigma$ as a function of D can be roughly approximated by an inclined straight line $\mathrm{Im}\sigma \doteq \mathrm{Im}\sigma_0 + D\,\mathrm{Im}\nu$, passing through $\mathrm{Im}\sigma_0$ for $D = 0$, and intersecting the axis $\mathrm{Im}\sigma = 0$ not at $D = 0$ as in isotropic media, but at $D = D_0$, where D_0 is given by the expression

$$D_0 \doteq -\mathrm{Im}\sigma_0/\mathrm{Im}\nu \ , \tag{85}$$

An important property of D_0 is that attenuation angle γ equals 90^0 for $D = D_0$. This D_0 also represents the boundary between the values of D, for which the amplitudes exponentially decay and exponentially grow along the unit propagation vector $\mathbf{N}$.

The behaviour of $\mathrm{Re}\sigma$ for D small is similar, $\mathrm{Re}\sigma \doteq \mathrm{Re}\sigma_0 + D\,\mathrm{Re}\nu$. The values of $\mathrm{Re}\nu$ are, however, considerably smaller in magnitude than those of $\mathrm{Im}\nu$, but still exist. Consequently, phase velocity $\mathcal{C}$ (as a function of D) of a weakly inhomogeneous qP or qSV wave is not maximum for homogeneous plane waves ($D = 0$), but for a certain weakly inhomogeneous plane wave ($D = D^M \neq 0$). The same applies to the minimum of attenuation $|\mathbf{A}|$. The derivation of the relevant approximate analytical estimates is straightforward, but is not given here.

d) **Strongly inhomogeneous qP and qSV waves**. Here we consider large $|D|$. We introduce a new variable w instead of σ by a simple relation

$$\sigma = wD \ . \tag{86}$$

Inserting (86) into (59), we obtain a quartic equation in w,

$$A_1w^4 + 2\mathrm{i}w^3B_1 - w^2[4A_2 + B_2 + D^{-2}C_1] - 2\mathrm{i}w(B_3 + D^{-2}C_2) + A_3 + D^{-2}C_3 + D^{-4} = 0 \ . \tag{87}$$

In the limiting case of $|D| \rightarrow \infty$, Equation (87) simplifies, but still remains quartic,

$$A_1w^4 + 2\mathrm{i}w^3B_1 - w^2(4A_2 + B_2) - 2\mathrm{i}wB_3 + A_3 = 0 \ . \tag{88}$$

For viscoelastic isotropic media, we have $B_1 = B_3 = 0$, and $A_1 = A_3 = \frac{1}{2}(4A_2 + B_2) = A_{11}A_{22}$. Equation (88) then simplifies to $w^4 - 2w^2 + 1 = 0$, which has a double root $w^2 = 1$, and yields $\sigma^2 = D^2$ for $|D| \rightarrow \infty$. For anisotropic viscoelastic media, the solution σ for qP and qSV waves may be expressed for $|D| \rightarrow \infty$ in the following form,

$$\sigma = Dw_\infty \ , \tag{89}$$

where w_∞ is a root of (88). This yields simple expressions for phase velocity $\mathcal{C}_\infty$ and $\cos\gamma_\infty$,

$$\mathcal{C}_\infty = 1/|D\mathrm{Re}w_\infty| \ , \quad \cos\gamma_\infty = \epsilon(D/|D|)\mathrm{Im}w_\infty/\sqrt{(\mathrm{Im}w_\infty)^2 + D^{-2}} \ , \tag{90}$$

where $\epsilon = \mathrm{Re}w_\infty/|\mathrm{Re}w_\infty| = \pm 1$.

Thus, phase velocity $\mathcal{C} \to 0$ for $|D| \to \infty$, both for qP and qSV waves. The behaviour of γ_∞ is more complex. Only for $\mathrm{Im}w_\infty = 0$ (e. g., for isotropic viscoelastic medium), $\gamma_\infty = 90^0$. For anisotropic viscoelastic media, however, $\mathrm{Im}w_\infty \neq 0$, and γ_∞ deviates from 90^0. The deviation is positive for one sign of D, and negative for the other sign of D.

e) **Polarization of qP and qSV waves**. It is not difficult to show from (38) that qP and qSV plane waves propagating in viscoelastic anisotropic or isotropic media are generally elliptically polarized. This is valid both for homogeneous and inhomogeneous plane waves.

4 Concluding remarks

The determination of slowness and polarization vectors of plane waves is a well-understood problem for perfectly elastic, isotropic or anisotropic, media. In this paper, universal and safe algorithms are proposed for homogeneous/inhomogeneous plane waves, propagating in isotropic/anisotropic, viscoelastic/perfectly elastic media. Usefulness of the viscoelastic anisotropic description of dissipative media in seismology and seismic exploration (for example for shallow consolidated sediments, reservoir rocks, zones of partial melting, mining areas, etc.) is indubitable.

The proposed algorithms could be extended even to rheologically more complicated media (poroviscoelasticity, Biot, etc.). In most of these extensions, the degree of the algebraic equation in σ, analogous to (24), increases from six to eight.

It was shown that attenuation angle γ is a quantity, which should be sought and not used as an à priori given parameter in the studies of qP, qSV and SH plane waves. Parameter γ, used as an input parameter, complicates computations considerably, and its use sometimes even leads to non-physical results. The fact that γ cannot be used quite freely is known already from studies of inhomogeneous plane waves propagating in perfectly elastic media, where γ can attain only a certain constant value. In case of anisotropic media, the value of γ is generally different from 90^0, in isotropic media it is exactly 90^0. Other values of γ are forbidden. On the other hand, the inhomogeneity parameter D introduced in the paper has all the necessary properties of a parameter of the problem.

Similarly, for qP, qS1 and qS2 waves, propagating in an arbitrary direction (generally outside symmetry planes) in a viscoelastic medium of unrestricted anisotropy, unit vector $\mathbf{M}$ in the direction of attenuation vector $\mathbf{A}$ should be sought and not assumed to be known. Instead of $\mathbf{M}$, the unit vector $\mathbf{m}$, perpendicular to $\mathbf{n}$, and inhomogeneity parameter D, should be specified. See Equation (12) and the text relevant to it.

We have only considered plane waves throughout the paper. The knowledge of plane wave algorithms, however, is a necessary prerequisite for the solution of more general problems of wave propagation (for example, point sources, Green functions) in inhomogeneous layered media (asymptotic high-frequency methods, ray methods).

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