Simplified construction of 4×4 ray propagator matrices in ray-centered coordinates

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Summary

The 4×4 ray propagator matrices in ray-centered coordinates offer a large number of important applications, both in numerical modeling of seismic wave fields in isotropic and anisotropic inhomogeneous layered structures by asymptotic high-frequency methods, and in practical interpretational problems of seismic exploration. They have been traditionally calculated by dynamic ray tracing in ray-centered coordinates. The dynamic ray tracing in ray-centered coordinates, however, is not so simple and numerically efficient as the dynamic ray tracing in Cartesian coordinates. A simplified construction of 4×4 ray propagator matrix in ray-centered coordinates is proposed. It avoids the dynamic ray tracing in ray-centered coordinates, and relies fully upon the dynamic ray tracing in Cartesian coordinates. This increases considerably the numerical efficiency of its computation, and also its flexibility and transparency of results. Two approaches are proposed here. The first approach simply transforms the 6×6 ray propagator matrix in Cartesian coordinates to 4×4 ray propagator matrix in ray-centered coordinates at the initial and end points of the ray (or at any point of the ray where we need to know it). In the second approach, four solutions of the dynamic ray tracing in Cartesian coordinates, with strictly specified initial conditions, are exploited.

Keywords: Dynamic ray tracing, ray propagator matrices, seismic anisotropy, inhomogeneity, structural interfaces.

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1 Introduction

At present, the main principles of kinematic and dynamic ray tracing in inhomogeneous anisotropic layered media are well known. Conventional ray tracing can be used to compute the ray as a space trajectory, together with the travel time T, the slowness vector $\mathbf{p}$, the group velocity vector $\boldsymbol{\mathcal{U}}$, and the polarization vector at any point of the ray. The dynamic ray tracing (also called paraxial ray tracing) consists in a solution of several additional linear ordinary differential equations of the first order along the ray, and can be used to determine the geometrical spreading, amplitude and wavefront curvature along the ray. Moreover, it provides kinematic quantities not only along the ray, but also in some vicinity of the ray. We call this vicinity of the ray the paraxial vicinity of the ray, and speak of paraxial rays, paraxial travel times, paraxial slowness vectors, and so on. Next, the dynamic ray tracing can be used to compute the second-order and higher-order spatial derivatives of travel times, to investigate the chaotic rays and evaluate Lyapunov exponents, and to study the reciprocity relations along the ray. Finally, the dynamic ray tracing is essential in various extensions of the ray method, like the method of Gaussian beams and Gaussian packets, Maslov-Chapman method, Kirchhoff-Helmholtz method, ray perturbation theories, and more. For more details on kinematic and dynamic ray tracing in inhomogeneous anisotropic media we refer to Červený (1972, 2001), Hanyga (1982), Gajewski and Pšenčík (1987, 1990), Norris (1987), Shearer and Chapman (1989), Farra (1989), Kendall, Guest and Thomson (1992), Klimeš (1994, 2003), Farra and LeBégat (1995), Bakker (1996), Hanyga et al. (2001), Gjoystdal et al. (2002), Iversen (2004), Moser (2004), and Chapman (2004), where many other references can be found.

The dynamic ray tracing system can be expressed in many forms and in various coordinate systems. The most common coordinate systems are the global Cartesian coordinate system and the ray-centered coordinate system, which is connected to a suitable reference ray. Since the dynamic ray tracing system consists of linear ordinary differential equations, it can be conveniently solved in terms of ray propagator matrices. In ray-centered coordinates these matrices are 4×4, for global Cartesian coordinates they are 6×6 - corresponding to systems of four or six differential equations, respectively. In a smooth medium without interfaces, the propagators in both coordinate systems satisfy the important symplectic property. Symplecticity imposes an algebraic structure on the ray system, as a result of which several quantities can be calculated analytically. At intersections with structural interfaces, interface propagator matrices are needed to ensure continuity and preserve symplecticity (Červený, 2001; Moser, 2004).

Both global Cartesian and ray-centered coordinates have certain advantages and disadvantages. The main advantage of dynamic ray tracing in Cartesian coordinates is that the expressions have a general and simple structure, and that they are equally valid in isotropic and anisotropic media. However, from the six independent solutions two are redundant: the so-called ray-tangent and non-eikonal solutions. These redundant solutions complicate the solution of various boundary-value problems of general four-parameteric system of rays. Dynamic ray tracing in ray-centered coordinates, on the other hand, does not involve the redundant solutions, but for anisotropic media the dynamic ray tracing system is more complicated and cumbersome. Moreover, it requires to solve an additional differential equation along the ray to determine the basis vectors of the ray-centered coordinate system (see Klimeš, 1994, 2003; Bakker, 1996; Červený, 2001). The 4×4 propagator matrix in ray-centered coordinates, however, plays a natural role in the investigation of general four-parameteric systems of rays, as the size of the matrix fully corresponds to the number of free parameters. In applications, it is mostly more useful than the 6×6 ray propagator matrix in Cartesian coordinates.

In this paper, we show that the 4×4 ray propagator matrix in ray-centered coordinates can be computed by dynamic ray tracing in Cartesian coordinates. The cumbersome dynamic ray tracing in ray-centered coordinates is not required at all. Moreover, the differential equation for the basis vectors of the ray-centered coordinate system along the ray need not be solved. Two basic approaches are proposed. In the first approach, the ray propagator matrix in ray-centered coordinates is obtained by some transformation of the ray propagator matrix in Cartesian coordinates at the initial and end points of the ray. If needed, the ray propagator matrix in ray-centered coordinates can be also computed in the same way at any intermediate point of the ray. To find the ray propagator matrix in Cartesian coordinates, the dynamic ray tracing in Cartesian coordinates must be performed six times, and six linearly independent solutions (specified by 6x6 identity matrix at the initial point of the ray) must be found.

In the second approach, the ray propagator matrix in ray-centered coordinates is obtained from four solutions of the dynamic ray tracing system in Cartesian coordinates, with strictly specified initial conditions. Thus, only four solutions are needed in this approach, not six as in the previous case.

Briefly to the content of the paper. In Section 2, the kinematic and dynamic ray tracing and ray propagator matrices in Cartesian coordinates are reviewed. Section 3 introduces the ray-centered coordinate system and its basis vectors (both contravariant and covariant). Section 4 specifies the general transformation relation between the ray propagator matrices corresponding to two coordinate systems, derived in Červený (2001, eq. (4.3.38)), to Cartesian and ray-centered coordinate systems. Basic relation (23) with (29) and (30) is derived. In Section 5, an alternative equation (37) for the ray propagator matrix in ray-centered coordinates is discussed, which requires to determine only four solutions of the dynamic ray tracing equation in Cartesian coordinates. Section 6 shows how the 6×6 ray propagator matrices in ray-centered coordinates can be reduced to 4×4. Two basic relations are derived, namely (51) and (57). The relation (57) requires to know the complete 6×6 ray propagator matrix in Cartesian coordinates, and relation (51) only four solutions of the dynamic ray tracing system in Cartesian coordinates.

The notations in the paper are as follows: The 3×3 and 6×6 matrices are denoted by roman boldface uppercase symbols, the 2×2 and 4×4 matrices by italic boldface uppercase symbols. The vectors 3×1 are roman boldface, 2×1 italic boldface. The symbol T in the superscript of a matrix means "transpose", and the symbol -T in the superscript denotes "inverse transpose". The lowercase indices i, j, k, ... run over 1, 2, 3, and the uppercase indices over 1, 2. The Einstein summation convention is applied to repeated indices in the same product.

2 Dynamic ray tracing and ray-propagator matrices in Cartesian coordinates

Consider the eikonal equation for the travel-time field $T(\mathbf{x})$ in a Hamiltonian form

$$\mathcal{H}(\mathbf{x},\mathbf{p}) = 0 \ , \tag{1}$$

where $\mathcal{H}$ is Hamiltonian, $\mathbf{x}$ is the position vector, and $\mathbf{p}$ the slowness vector, $\mathbf{p} = \partial T/\partial \mathbf{x}$. We consider here the Hamiltonians which are homogeneous functions of the second degree in $\mathbf{p}$. Kinematic ray tracing equations then read

$$\mathrm{d}\mathbf{x}/\mathrm{d}\tau = \boldsymbol{\mathcal{U}} = \partial\mathcal{H}/\partial\mathbf{p} \ , \quad \mathrm{d}\mathbf{p}/\mathrm{d}\tau = \boldsymbol{\eta} = -\partial\mathcal{H}/\partial\mathbf{x} \ . \tag{2}$$

Here τ is a monotonically increasing parameter along the ray, which represents the travel time. Equations (2) also define the group velocity vector $\boldsymbol{\mathcal{U}}$ and the vector $\boldsymbol{\eta}$, which will be broadly used in this paper.

Dynamic ray tracing system is designed to compute along the ray six quantities

$$\mathbf{Q}^{(x)} = \partial\mathbf{x}/\partial\gamma \ , \quad \mathbf{P}^{(x)} = \partial\mathbf{p}/\partial\gamma \ , \tag{3}$$

where γ is an arbitrary ray parameter. Taking the partial derivative of (2) with respect to γ, we obtain the dynamic ray tracing system in Cartesian coordinates

$$\begin{aligned} \mathrm{d}\mathbf{Q}^{(x)}/\mathrm{d}\tau &= (\partial^2\mathcal{H}/\partial\mathbf{p}\partial\mathbf{x})\mathbf{Q}^{(x)} + (\partial^2\mathcal{H}/\partial\mathbf{p}\partial\mathbf{p})\mathbf{P}^{(x)} \ , \\ \mathrm{d}\mathbf{P}^{(x)}/\mathrm{d}\tau &= -(\partial^2\mathcal{H}/\partial\mathbf{x}\partial\mathbf{x})\mathbf{Q}^{(x)} - (\partial^2\mathcal{H}/\partial\mathbf{x}\partial\mathbf{p})\mathbf{P}^{(x)} \ . \end{aligned} \tag{4}$$

The solutions of the dynamic ray tracing equation must satisfy the constraint relation

$$\boldsymbol{\mathcal{U}}^T\mathbf{P}^{(x)} - \boldsymbol{\eta}^T\mathbf{Q}^{(x)} = 0 \ , \tag{5}$$

which follows from (1).

Dynamic ray tracing system (4) consists of six linear ordinary differential equations of the first order for $Q_i^{(x)}$ and $P_i^{(x)}$, and has six linearly independent solutions. Consequently, we can construct the 6×6 ray propagator matrix $\mathbf{\Pi}^{(x)}(\tau,\tau_0)$, which is defined as the fundamental matrix of (4), subject to initial conditions $\mathbf{\Pi}^{(x)}(\tau_0,\tau_0) = \mathbf{I}$, where $\mathbf{I}$ is 6×6 identity matrix. The propagator matrix $\mathbf{\Pi}^{(x)}(\tau,\tau_0)$ has a number of useful properties. The most important of them is the symplecticity:

$$\mathbf{\Pi}^{(x)T}\mathbf{J}\mathbf{\Pi}^{(x)} = \mathbf{J} \ , \tag{6}$$

where $\mathbf{J}$ is a 6×6 matrix given by the formula

$$\mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix} \ . \tag{7}$$

Here $\mathbf{0}$ is a 3×3 zero matrix, and $\mathbf{I}$ the 3×3 identity matrix. The continuation property of $\mathbf{\Pi}^{(x)}(\tau,\tau_0)$ reads

$$\begin{pmatrix} \mathbf{Q}^{(x)}(\tau) \\ \mathbf{P}^{(x)}(\tau) \end{pmatrix} = \mathbf{\Pi}^{(x)}(\tau,\tau_0) \begin{pmatrix} \mathbf{Q}^{(x)}(\tau_0) \\ \mathbf{P}^{(x)}(\tau_0) \end{pmatrix} \ . \tag{8}$$

Equation (8) can be also expressed in the form

$$\begin{pmatrix} \delta\mathbf{x}(\tau) \\ \delta\mathbf{p}(\tau) \end{pmatrix} = \mathbf{\Pi}^{(x)}(\tau,\tau_0) \begin{pmatrix} \delta\mathbf{x}(\tau_0) \\ \delta\mathbf{p}(\tau_0) \end{pmatrix} , \tag{9}$$

where $\delta\mathbf{x}$ and $\delta\mathbf{p}$ are small perturbations of $\mathbf{x}$ and $\mathbf{p}$ on the ray. Equation (9) shows well the physical meaning of the ray propagator matrix $\mathbf{\Pi}^{(x)}(\tau,\tau_0)$ in Cartesian coordinates. The chain rule states $\mathbf{\Pi}^{(x)}(\tau,\tau_0) = \mathbf{\Pi}^{(x)}(\tau,\tau_1)\mathbf{\Pi}^{(x)}(\tau_1,\tau_0)$ for any τ_1 situated on the ray, and the Liouville's theorem gives $\det(\mathbf{\Pi}^{(x)}(\tau,\tau_0)) = 1$ for any τ. Finally, the symplectic property gives a simple relation for the inverse of $\mathbf{\Pi}^{(x)}(\tau,\tau_0)$,

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}^{-1} = \begin{pmatrix} \mathbf{D}^T & -\mathbf{B}^T \\ -\mathbf{C}^T & \mathbf{A}^T \end{pmatrix} . \tag{10}$$

As the complete system of rays is four-parameteric, two solutions in $\mathbf{\Pi}^{(x)}(\tau,\tau_0)$ are redundant. The expressions for these two solutions can be found analytically. The first of them is the so-called ray-tangent solution:

$$\mathbf{Q}^{(x)} = \boldsymbol{\mathcal{U}} , \qquad \mathbf{P}^{(x)} = \boldsymbol{\eta} , \tag{11}$$

and the latter is the non-eikonal solution,

$$\mathbf{Q}^{(x)} = (\tau-\tau_0)\boldsymbol{\mathcal{U}} , \quad \mathbf{P}^{(x)} = \mathbf{p} + (\tau-\tau_0)\boldsymbol{\eta} . \tag{12}$$

For more details on dynamic ray tracing and ray propagator matrices in Cartesian coordinates in inhomogeneous anisotropic media and on the ray-tangent and non-eikonal solutions see Červený (2001).

All the above relations are valid for smooth media. The ray tracing equations (2), dynamic ray tracing equations (4) and the expression for the ray propagator matrix should be modified when the ray crosses a structural interface. The ray propagator matrix should be chained, and the so-called interface propagator matrix should be included in the chain at the point of incidence, to take into account the effects of the interface. The interface propagator matrix should be symplectic and must satisfy the constraint equation. For a more thorough treatment and useful relations for the interface propagator matrix in Cartesian coordinates see Moser (2004).

3 Ray-centered coordinate system

We consider a reference ray Ω and introduce the ray-centered coordinates q_1, q_2, q_3 along the ray, using the relation

$$x_i(q_j) = x_{i0}(q_3) + H^{(q)}_{iM}(q_3)q_M . \tag{13}$$

The reference ray Ω represents the q_3-coordinate line of the ray-tracing coordinate system, and $q_3 = \tau$, the travel-time along the ray. The coordinates q_1 and q_2 specify uniquely the position of a point in the plane tangent to the wavefront intersecting the reference ray Ω for a fixed q_3. For $q_1 = q_2 = 0$, the point is situated directly on the reference ray Ω.

We introduce the transformation matrices from Cartesian coordinates x_i to ray-centered coordinates q_m and back,

$$H^{(q)}_{im} = \frac{\partial x_i}{\partial q_m} \ , \quad \bar{H}^{(q)}_{mi} = \frac{\partial q_m}{\partial x_i} \ . \tag{14}$$

We have

$$H^{(q)}_{im} \bar{H}^{(q)}_{mj} = \delta_{ij} \ . \tag{15}$$

Contravariant basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are given by the relations

$$e_{Mi} = H^{(q)}_{iM} \ , \quad e_{3i} = H^{(q)}_{i3} = \mathcal{U}^{(x)}_i \ . \tag{16}$$

$\mathbf{e}_1$ and $\mathbf{e}_2$ are situated in the wavefront tangent plane, $\mathbf{e}_3$ coincides with the group velocity vector $\boldsymbol{\mathcal{U}}$, tangent to the ray.

Covariant basis vectors $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ are given by the relation

$$f_{Mi} = \bar{H}^{(q)}_{Mi} \ , \quad f_{3i} = \bar{H}^{(q)}_{3i} = p^{(x)}_i \ . \tag{17}$$

$\mathbf{f}_1$ and $\mathbf{f}_2$ are perpendicular to the ray, $\mathbf{f}_3$ equals the slowness vector $\mathbf{p}$. The basis vectors $\mathbf{e}_i$ and $\mathbf{f}_i$ satisfy the relation, see (15):

$$\mathbf{e}^T_k \mathbf{f}_l = \delta_{kl} \ . \tag{18}$$

We can choose $\mathbf{e}_1$ and $\mathbf{e}_2$ as mutually orthogonal unit vectors, perpendicular to the slowness vector. Then $\mathbf{f}_i$ follow from (18) and are determined by the relation

$$\mathbf{f}_1 = \mathcal{C}^{-1}(\mathbf{e}_2 \times \boldsymbol{\mathcal{U}}) \ , \quad \mathbf{f}_2 = \mathcal{C}^{-1}(\boldsymbol{\mathcal{U}} \times \mathbf{e}_1) \ , \quad \mathbf{f}_3 = \mathbf{p} \ . \tag{19}$$

Note that $\mathbf{f}_1$ and $\mathbf{f}_2$ are not, in general, mutually perpendicular unit vectors in this case.

We can alternatively choose $\mathbf{f}_1$ and $\mathbf{f}_2$ as mutually orthogonal unit vectors, perpendicular to the ray. Then

$$\mathbf{e}_1 = \mathcal{U}(\mathbf{f}_2 \times \mathbf{p}) \ , \quad \mathbf{e}_2 = \mathcal{U}(\mathbf{p} \times \mathbf{f}_1) \ , \quad \mathbf{e}_3 = \boldsymbol{\mathcal{U}} \ . \tag{20}$$

The basis vectors $\mathbf{e}_1, \mathbf{e}_2$ are not mutually perpendicular and are not unit on this case.

Finally, we introduce the slowness vector $\mathbf{p}^{(q)}$ in ray-centered coordinate system q_i by its ray-centered components

$$p^{(q)}_i = \partial T / \partial q_i \ . \tag{21}$$

As q_3 represents the travel-time along the reference ray,

$$p^{(q)}_3 = 1 \ . \tag{22}$$

Let us emphasize the difference between the ray-centered coordinate system introduced here, and the ray-centered coordinate system, which has been traditionally used for isotropic inhomogeneous media. In ray-centered coordinate system for isotropic media, $q_3 =$

s was mostly used, where s represents the arclength measured along reference ray. See Popov and Pšenčík (1978a,b), Červený, Klimeš and Pšenčík (1988), Červený (2001, section 4.1), where many other references can be found. It was shown by Klimeš (1994), that in anisotropic media it is more useful to consider the travel time (not the arclength) as the coordinate q_3. The ray-centered coordinate system introduced here (with $q_3 = \tau$) can be, of course, used even for isotropic inhomogeneous media.

4 Relation between propagator matrices $\mathbf{\Pi}^{(x)}(R,S)$ and $\mathbf{\Pi}^{(q)}(R,S)$

A general relation between 6×6 propagator matrices, computed along the same reference ray Ω from S to R, related to two arbitrary coordinate systems, was found in Červený (2001), see Eq. (4.3.38). In the equation, it is assumed that the parameter τ along the ray is the same in both coordinate systems, and represents the travel time along the ray. The dynamic ray tracing is not required in the application of this equation, it is expressed fully in terms of propagator matrices.

We apply the equation (4.3.38) to the 6×6 propagator matrix $\mathbf{\Pi}^{(x)}(R,S)$ in Cartesian coordinates, and to the 6×6 propagator matrix $\mathbf{\Pi}^{(q)}(R,S)$ in ray centered coordinates. We obtain

$$\mathbf{\Pi}^{(q)}(R,S) = \mathbf{\Psi}_1(R)\mathbf{\Pi}^{(x)}(R,S)\mathbf{\Psi}_2(S) \ , \tag{23}$$

where $\mathbf{\Psi}_1(R)$ and $\mathbf{\Psi}_2(S)$ are 6×6 matrices

$$\mathbf{\Psi}_1(R) = \begin{pmatrix} \bar{\mathbf{H}}^{(q)}(R) & \mathbf{0} \\ \mathbf{F}(R)\bar{\mathbf{H}}^{(q)}(R) & \mathbf{H}^{(q)T}(R) \end{pmatrix} \ , \tag{24}$$

$$\mathbf{\Psi}_2(S) = \begin{pmatrix} \mathbf{H}^{(q)}(S) & \mathbf{0} \\ -\bar{\mathbf{H}}^{(q)T}(S)\mathbf{F}(S) & \bar{\mathbf{H}}^{(q)T}(S) \end{pmatrix} \ . \tag{25}$$

Here $\mathbf{0}$ is a 3×3 zero matrix, and $\mathbf{F}$ a 3×3 matrix with Cartesian components $F_{mn} = p_i^{(x)}\partial^2 x_i/\partial q_m \partial q_n$. It is not difficult to obtain

$$F_{MN} = 0 \ , \qquad F_{m3} = F_{3m} = -H_{im}^{(q)}\eta_i^{(x)} = -\mathbf{e}_m^T\boldsymbol{\eta} \ , \tag{26}$$

see Červený (2001, eq. (4.2.76)).

Using (15), (16), (17) and (26), we obtain

$$(\mathbf{F}\bar{\mathbf{H}}^{(q)})_{Ij} = -(\mathbf{e}_I^T\boldsymbol{\eta})p_j^{(x)} \ , \quad (\mathbf{F}\bar{\mathbf{H}}^{(q)})_{3j} = -\eta_j^{(x)} \ . \tag{27}$$

Analogously,

$$-(\bar{\mathbf{H}}^{(q)T}\mathbf{F})_{iJ} = p_i^{(x)}(\mathbf{e}_J^T\boldsymbol{\eta}) \ , \quad -(\bar{\mathbf{H}}^{(q)T}\mathbf{F})_{i3} = \eta_i^{(x)} \ . \tag{28}$$

Inserting (27) and (28) into (24) and (25) yields:

$$\mathbf{\Psi}_1(R) \;=\; \begin{pmatrix} \mathbf{f}_1^T & \mathbf{0}^T \\ \mathbf{f}_2^T & \mathbf{0}^T \\ \mathbf{f}_3^T & \mathbf{0}^T \\ -(\mathbf{e}_1^T\boldsymbol{\eta})\mathbf{p}^T & \mathbf{e}_1^T \\ -(\mathbf{e}_2^T\boldsymbol{\eta})\mathbf{p}^T & \mathbf{e}_2^T \\ -\boldsymbol{\eta}^T & \mathbf{e}_3^T \end{pmatrix} , \tag{29}$$

$$\mathbf{\Psi}_2(S) \;=\; \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ (\mathbf{e}_1^T\boldsymbol{\eta})\mathbf{p} & (\mathbf{e}_2^T\boldsymbol{\eta})\mathbf{p} & \boldsymbol{\eta} & \mathbf{f}_1 & \mathbf{f}_2 & \mathbf{f}_3 \end{pmatrix} . \tag{30}$$

Here $\mathbf{0} = (0,0,0)^T$, $\mathbf{0}^T = (0,0,0)$.

It is simple to show that all the matrices $\mathbf{\Pi}^{(q)}(R,S)$, $\mathbf{\Pi}^{(x)}(R,S)$, $\mathbf{\Psi}_1(R)$ and $\mathbf{\Psi}_2(R)$ are symplectic, so that their inverses are given by (10). Using (10), we also immediatelly see that

$$\mathbf{\Psi}_2 = \mathbf{\Psi}_1^{-1} \; . \tag{31}$$

Equation (23) represents the final equation for the 6×6 propagator matrix $\mathbf{\Pi}^{(q)}(R,S)$ in ray-centered coordinates. It expresses the propagator matrix $\mathbf{\Pi}^{(q)}(R,S)$ in terms of the 6×6 propagator matrix $\mathbf{\Pi}^{(x)}(R,S)$ in Cartesian coordinates.

Once we know the 6×6 propagator matrix in ray-centered coordinates $\mathbf{\Pi}^{(q)}(r,S)$, we can write the relevant continuation relation (in analogy to (8)):

$$\begin{pmatrix} \mathbf{Q}^{(q)}(R) \\ \mathbf{P}^{(q)}(R) \end{pmatrix} = \mathbf{\Pi}^{(q)}(R,S) \begin{pmatrix} \mathbf{Q}^{(q)}(S) \\ \mathbf{P}^{(q)}(S) \end{pmatrix} , \tag{32}$$

where $\mathbf{Q}^{(q)}$ and $\mathbf{P}^{(q)}$ are given by relations,

$$\mathbf{Q}^{(q)} = \partial\mathbf{q}/\partial\gamma \; , \quad \mathbf{P}^{(q)} = \partial\mathbf{p}^{(q)}/\partial\gamma \; , \tag{33}$$

and γ is an arbitrary ray parameter. Due to (22),

$$P_3^{(q)} = 0 \; , \tag{34}$$

along the whole reference ray. Equation (34) represents a constraint relation in ray-centered coordinates, similarly as (5) represents a constraint relation in Cartesian coordinates. Further,

$$Q_3^{(q)} = \partial q_3/\partial\gamma = \partial\tau/\partial\gamma \; . \tag{35}$$

We can also express $\mathbf{Q}^{(q)}$ and $\mathbf{P}^{(q)}$ in terms of $\mathbf{Q}^{(x)}$ and $\mathbf{P}^{(x)}$. It is not difficult to show that

$$\begin{pmatrix} \mathbf{Q}^{(q)} \\ \mathbf{P}^{(q)} \end{pmatrix} = \mathbf{\Psi}_1 \begin{pmatrix} \mathbf{Q}^{(x)} \\ \mathbf{P}^{(x)} \end{pmatrix} , \quad \begin{pmatrix} \mathbf{Q}^{(x)} \\ \mathbf{P}^{(x)} \end{pmatrix} = \mathbf{\Psi}_2 \begin{pmatrix} \mathbf{Q}^{(q)} \\ \mathbf{P}^{(q)} \end{pmatrix} . \tag{36}$$

To determine $\mathbf{\Pi}^{(x)}(R,S)$, we must perform the dynamic ray tracing in Cartesian coordinates six times, with the initial conditions specified by the relation $\mathbf{\Pi}^{(x)}(S,S) = \mathbf{I}$.

5 Alternative expressions for $\mathbf{\Pi}^{(q)}(R, S)$

We now use an alternative approach to determine $\mathbf{\Pi}^{(q)}(R,S)$ from $\mathbf{\Pi}^{(x)}(R,S)$. The advantage of this approach is that it does not require six dynamic ray tracings in Cartesian coordinates, but only four, for some selected initial conditions. The remaining two solutions of the dynamic ray tracing system in Cartesian coordinates are known analytically (ray tangent solutions, non-eikonal solutions).

We can proceed in the following way. We express (23) as follows:

$$\mathbf{\Pi}^{(q)}(R,S) = \mathbf{\Psi}_1(R)\mathbf{Z}(R,S) \ , \tag{37}$$

where

$$\mathbf{Z}(R,S) = \mathbf{\Pi}^{(x)}(R,S)\mathbf{\Psi}_2(S) \ . \tag{38}$$

The 6×6 matrix $\mathbf{Z}(R,S)$ is not a propagator matrix, as $\mathbf{Z}(S,S) = \mathbf{\Psi}_2(S)$ is different from the identity matrix $\mathbf{I}$. The six columns of $\mathbf{Z}(R,S)$ are the solutions of the dynamic ray tracing system in Cartesian coordinates, with initial conditions given by $\mathbf{\Psi}_2(S)$.

Two of the solutions of the dynamic ray tracing system in Cartesian coordinates, however, are known analytically. This is the ray-tangent solution, see (11), and the non-eikonal solution (12). Conventionally, we store the ray tangent solution (11) in the third column of $\mathbf{Z}(R,S)$, and the non-eikonal solution (12) in its sixth column. The remaining columns represent standard paraxial solutions of the DRT in Cartesian coordinates, with the initial condition given in the relevant column of $\mathbf{\Psi}_2(S)$. The standard paraxial solutions in $\mathbf{Z}(R,S)$ must satisfy the constraint equation (5). Moreover, $\mathbf{Q}^{(x)}$ must satisfy the relation

$$\mathbf{p}^T\mathbf{Q}^{(x)} = 0 \ , \tag{39}$$

expressing the fact that the vector $\mathbf{Q}^{(x)}$ is situated in the wavefront.

Now we shall compute $\mathbf{\Pi}^{(q)}(R,S)$ using Eq. (37). First we compute the product of $\mathbf{\Psi}_1$, given by (29), with the third and sixth columns of $\mathbf{Z}(R,S)$, representing the ray-tangent and non-eikonal solutions (11) and (12). For ray-tangent solutions, (29) and (11) give

$$\mathbf{\Psi}_1 \begin{pmatrix} \boldsymbol{\mathcal{U}} \\ \boldsymbol{\eta} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1^T\boldsymbol{\mathcal{U}} \\ \mathbf{f}_2^T\boldsymbol{\mathcal{U}} \\ \mathbf{f}_3^T\boldsymbol{\mathcal{U}} \\ -(\mathbf{e}_1^T\boldsymbol{\eta})(\mathbf{p}^T\boldsymbol{\mathcal{U}}) + \mathbf{e}_1^T\boldsymbol{\eta} \\ -(\mathbf{e}_2^T\boldsymbol{\eta})(\mathbf{p}^T\boldsymbol{\mathcal{U}}) + \mathbf{e}_2^T\boldsymbol{\eta} \\ -\boldsymbol{\eta}^T\boldsymbol{\mathcal{U}} + \mathbf{e}_3^T\boldsymbol{\eta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{40}$$

Here we have taken into account that $\mathbf{f}_I^T\boldsymbol{\mathcal{U}} = 0$, $\mathbf{p}^T\boldsymbol{\mathcal{U}} = 1$ and $\mathbf{f}_3^T\boldsymbol{\mathcal{U}} = \mathbf{p}^T\boldsymbol{\mathcal{U}} = 1$. Similarly, for the non-eikonal solution, (29) and (12) yield

$$\mathbf{\Psi}_1 \begin{pmatrix} \tau\boldsymbol{\mathcal{U}} \\ \mathbf{p} + \tau\boldsymbol{\eta} \end{pmatrix} = \begin{pmatrix} \tau(\mathbf{f}_1^T\boldsymbol{\mathcal{U}}) \\ \tau(\mathbf{f}_2^T\boldsymbol{\mathcal{U}}) \\ \tau(\mathbf{f}_3^T\boldsymbol{\mathcal{U}}) \\ -\tau(\mathbf{e}_1^T\boldsymbol{\mathcal{U}})(\mathbf{p}^T\boldsymbol{\mathcal{U}}) + \mathbf{e}_1^T\mathbf{p} + \tau(\mathbf{e}_1^T\boldsymbol{\eta}) \\ -\tau(\mathbf{e}_2^T\boldsymbol{\mathcal{U}})(\mathbf{p}^T\boldsymbol{\mathcal{U}}) + \mathbf{e}_2^T\mathbf{p} + \tau(\mathbf{e}_2^T\boldsymbol{\eta}) \\ -\tau(\boldsymbol{\eta}^T\boldsymbol{\mathcal{U}}) + \mathbf{e}_3^T\mathbf{p} + \tau(\mathbf{e}_3^T\boldsymbol{\eta}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \tau \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{41}$$

Here $\tau = \tau(R) - \tau(S)$.

Thus, the third and sixth columns of $\mathbf{\Pi}^{(q)}(R,S)$ are given by simple expressions (40) and (41).

We shall now consider the standard paraxial solutions (the first, second, fourth and fifth columns) of $\mathbf{Z}(R,S)$. We denote any of them

$$\begin{pmatrix} \mathbf{Q}^{(x)} \\ \mathbf{P}^{(x)} \end{pmatrix} , \tag{42}$$

where $\mathbf{Q}^{(x)} \equiv (Q_1^{(x)}, Q_2^{(x)}, Q_3^{(x)})^T$ and $\mathbf{P}^{(x)} \equiv (P_1^{(x)}, P_2^{(x)}, P_3^{(x)})^T$. The product of the matrix $\boldsymbol{\Psi}_1$ with any of the solutions (42) reads

$$\boldsymbol{\Psi}_1 \begin{pmatrix} \mathbf{Q}^{(x)} \\ \mathbf{P}^{(x)} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1^T \mathbf{Q}^{(x)} \\ \mathbf{f}_2^T \mathbf{Q}^{(x)} \\ \mathbf{f}_3^T \mathbf{Q}^{(x)} \\ -(\mathbf{e}_1^T \boldsymbol{\eta})(\mathbf{p}^T \mathbf{Q}^{(x)}) + \mathbf{e}_1^T \mathbf{P}^{(x)} \\ -(\mathbf{e}_2^T \boldsymbol{\eta})(\mathbf{p}^T \mathbf{Q}^{(x)}) + \mathbf{e}_2^T \mathbf{P}^{(x)} \\ -\boldsymbol{\eta}^T \mathbf{Q}^{(x)} + \mathbf{e}_3^T \mathbf{P}^{(x)} \end{pmatrix} = \begin{pmatrix} \times \\ \times \\ 0 \\ \times \\ \times \\ 0 \end{pmatrix} . \tag{43}$$

The crosses $\times$ in the last matrix take the same values as in the preceding matrix. The zero in the third row follows immediately from (39), and the zero in the sixth row from the constraint relation (5).

Thus, the 6×6 matrix $\mathbf{\Pi}^{(q)}(R,S)$ takes a specific form: It has zero values in third and sixth columns and rows, with the exception of the position 33 and 66, where it is 1, and of the position 36, where it equals $\tau(R,S)$. It can be expressed in the following form:

$$\mathbf{\Pi}^{(q)}(R,S) = \begin{pmatrix} \boldsymbol{Q}_1^{(q)}(R,S) & \boldsymbol{0} & \boldsymbol{Q}_2^{(q)}(R,S) & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 & \boldsymbol{0}^T & \tau(R,S) \\ \boldsymbol{P}_1^{(q)}(R,S) & \boldsymbol{0} & \boldsymbol{P}_2^{(q)}(R,S) & \boldsymbol{0} \\ \boldsymbol{0}^T & 0 & \boldsymbol{0}^T & 1 \end{pmatrix} . \tag{44}$$

Here $\boldsymbol{Q}_I^{(q)}(R,S)$ and $\boldsymbol{P}_I^{(q)}(R,S)$ are 2×2 matrices, $\boldsymbol{0} = (0,0)^T$ and $\boldsymbol{0}^T = (0,0)$. See also Klimeš (1994), where the dynamic ray tracing system in ray-centered coordinates and relevant 6×6 ray propagator matrix (44) was first derived. Using the dynamic ray tracing in ray-centered coordinates, he also obtained an equation analogous to (37). For a brief exposition of the Klimeš's approach see Červený (2001, sections 4.2.4/3 and 4.3.6/4).

6 Reduction of 6×6 propagator matrix $\mathbf{\Pi}^{(q)}(R,S)$ to 4×4

Using (44), we can write the continuation relation for $\mathbf{Q}^{(q)}$ and $\mathbf{P}^{(q)}$ along the ray in the following form:

$$\begin{pmatrix} \mathbf{Q}^{(q)}(R) \\ \mathbf{P}^{(q)}(R) \end{pmatrix} = \mathbf{\Pi}^{(q)}(R,S) \begin{pmatrix} \mathbf{Q}^{(q)}(S) \\ \mathbf{P}^{(q)}(S) \end{pmatrix} . \tag{45}$$

Due to specific properties of $\mathbf{\Pi}^{(q)}(R,S)$, see (44), the 6×6 matrix equation (45) can be decomposed into two matrix equations, one of them in 2×2 matrices, and the latter in 4×4 matrices. The first of them reads:

$$\begin{pmatrix} Q_3^{(q)}(R) \\ P_3^{(q)}(R) \end{pmatrix} = \begin{pmatrix} 1 & \tau(R,S) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Q_3^{(q)}(S) \\ P_3^{(q)}(S) \end{pmatrix} . \tag{46}$$

This system can be expressed analytically,

$$\begin{aligned} Q_3^{(q)}(R) &= Q_3^{(q)}(S) + \tau(R,S)P_3^{(q)}(S) , \\ P_3^{(q)}(R) &= P_3^{(q)}(S) . \end{aligned} \tag{47}$$

For $P_3^{(q)}(S)=0$, the quantity $P_3^{(q)}$ vanishes along the whole ray. This corresponds to the constraint equation in ray-centered coordinates, $P_3^{(q)}=0$, see (34).

We are, however, not interested in the trivial solutions (47), but in the 4×4 system, which reads:

$$\begin{pmatrix} \boldsymbol{Q}^{(q)}(R) \\ \boldsymbol{P}^{(q)}(R) \end{pmatrix} = \boldsymbol{\Pi}^{(q)}(R,S) \begin{pmatrix} \boldsymbol{Q}^{(q)}(S) \\ \boldsymbol{P}^{(q)}(S) \end{pmatrix} . \tag{48}$$

The form of this equation is the same as (45), but the involved matrices have different sizes:

$$\boldsymbol{Q}^{(q)} \equiv (Q_1^{(q)}, Q_2^{(q)})^T , \quad \boldsymbol{P}^{(q)} \equiv (P_1^{(q)}, P_2^{(q)})^T , \tag{49}$$

and $\boldsymbol{\Pi}^{(q)}(R,S)$ is a 4×4 propagator matrix in ray-centered coordinates,

$$\boldsymbol{\Pi}^{(q)}(R,S) = \begin{pmatrix} \boldsymbol{Q}_1^{(q)}(R,S) & \boldsymbol{Q}_2^{(q)}(R,S) \\ \boldsymbol{P}_1^{(q)}(R,S) & \boldsymbol{P}_2^{(q)}(R,S) \end{pmatrix} . \tag{50}$$

Here $\boldsymbol{Q}_I^{(q)}(R,S)$ and $\boldsymbol{P}_I^{(q)}(R,S)$ are 2×2 matrices. The 4×4 ray propagator matrix $\boldsymbol{\Pi}^{(q)}(R,S)$ is symplectic, $\boldsymbol{\Pi}^{(q)}(S,S)=\mathbf{I}$, $\det \boldsymbol{\Pi}^{(q)}(R,S)=1$. The inverse of $\boldsymbol{\Pi}^{(q)}(R,S)$ is given by (10). Consequently, $\boldsymbol{\Pi}^{(q)}(R,S)$ satisfies a chain rule.

Let us emphasize again that the 4×4 propagator matrix $\boldsymbol{\Pi}^{(q)}(R,S)$ in ray-centered coordinates can be fully computed by dynamic ray tracing in Cartesian coordinates. We only use (37), remove the third and sixth rows from $\mathbf{\Psi}_1(R)$, and the third and sixth columns from $\mathbf{Z}(R,S)$. The final equation for the 4×4 ray propagator matrix $\boldsymbol{\Pi}^{(q)}(R,S)$ in ray-centered coordinates is

$$\boldsymbol{\Pi}^{(q)}(R,S) = \mathbf{\Psi}_1^r(R)\mathbf{Z}^r(R,S) , \tag{51}$$

where $\mathbf{\Psi}_1^r(R)$ ($\mathbf{\Psi}_1$-reduced) is a 4×6 matrix, given by equation

$$\mathbf{\Psi}_1^r(R) = \begin{pmatrix} \mathbf{f}_1^T & \mathbf{0}^T \\ \mathbf{f}_2^T & \mathbf{0}^T \\ -(\mathbf{e}_1^T\boldsymbol{\eta})\mathbf{p}^T & \mathbf{e}_1^T \\ -(\mathbf{e}_2^T\boldsymbol{\eta})\mathbf{p}^T & \mathbf{e}_2^T \end{pmatrix} , \tag{52}$$

see (29). The 6×4 matrix $\mathbf{Z}^r(R,S)$ ($\mathbf{Z}$ - reduced) contains four solutions of the dynamic ray tracing in Cartesian coordinates along the reference ray from S to R. These four solutions are specified by initial conditions given by the 6×4 matrix $\mathbf{\Psi}_2^{(r)}(S)$ ($\mathbf{\Psi}_2$-reduced), see (30),

$$\mathbf{\Psi}_2^r(S) = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{0} & \mathbf{0} \\ (\mathbf{e}_1^T\boldsymbol{\eta})\mathbf{p} & (\mathbf{e}_2^T\boldsymbol{\eta})\mathbf{p} & \mathbf{f}_1 & \mathbf{f}_2 \end{pmatrix} . \tag{53}$$

The first two columns in $\mathbf{\Psi}_2^r(S)$ correspond to the plane wavefront initial conditions

$$\mathbf{\Psi}^{pw} = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 \\ (\mathbf{e}_1^T\boldsymbol{\eta})\mathbf{p} & (\mathbf{e}_2^T\boldsymbol{\eta})\mathbf{p} \end{pmatrix} , \tag{54}$$

and the third and fourth column correspond to the point source initial condition

$$\mathbf{\Psi}^{ps} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{f}_1 & \mathbf{f}_2 \end{pmatrix} . \tag{55}$$

Exactly the same initial conditions as (54) and (55) were also derived in Červený (2001, eqs. (4.2.50) – (4.2.52)), in relation to wavefront orthonormal coordinates. To compare (55) for a point source with (4.2.51) of Červený (2001), we can take into account that $\mathbf{f}_I$, given by (19), can be also expressed in the form:

$$\mathbf{f}_I = \mathbf{e}_I - \mathbf{p}(\mathbf{e}_I^T\boldsymbol{\mathcal{U}}) . \tag{56}$$

Equations fully analogous to (55) for a point source initial conditions were also derived by Pšenčík and Teles (1996).

Equation (51) represents the final result. It expresses the 4×4 propagator matrix $\boldsymbol{\Pi}^{(q)}(R, S)$ in ray-centered coordinates, in terms of the 6×4 matrix $\mathbf{Z}^r(R, S)$, composed of four solutions of the dynamic ray tracing system in Cartesian coordinates, with initial conditions specified by matrix $\mathbf{\Psi}_2^r(S)$, given by (53). The dynamic ray tracing in ray-centered coordinates is not necessary.

In analogy to (51), we obtain an equation for the 4×4 ray propagator matrix $\boldsymbol{\Pi}^{(q)}(R, S)$ in ray-centered coordinates from (23):

$$\boldsymbol{\Pi}^{(q)}(R, S) = \mathbf{\Psi}_1^r(R)\mathbf{\Pi}^{(x)}(R, S)\mathbf{\Psi}_2^r(S) . \tag{57}$$

Here $\mathbf{\Pi}^{(x)}(R, S)$ is a conventional 6×6 ray-propagator matrix in Cartesian coordinates, $\mathbf{\Psi}_2^r(S)$ is the 6×4 matrix of initial conditions, given by (53), and $\mathbf{\Psi}_1^r(R)$ a 4×6 matrix given by (52).

7 Concluding remarks

We have proposed four methods how to calculate the 4×4 ray propagator matrix $\boldsymbol{\Pi}^{(q)}(R, S)$ in ray-centered coordinates along the reference ray Ω from S to R. None of these methods requires the cumbersome dynamic ray tracing in ray-centered coordinates; all are based fully on dynamic ray tracing in Cartesian coordinates.

1) The first method is based on the computation of the 6×6 ray propagator matrix $\mathbf{\Pi}^{(q)}(R,S)$ using (23). The 4×4 ray propagator matrix $\boldsymbol{\Pi}^{(q)}(R,S)$ is obtained from the 6×6 ray propagator matrix $\mathbf{\Pi}^{(q)}(R,S)$ by simply removing the third and sixth columns and rows from $\mathbf{\Pi}^{(q)}(R,S)$. The method requires to know the 6×6 ray propagator matrix $\mathbf{\Pi}^{(x)}(R,S)$ in Cartesian coordinates.

2) The second method is again based on the computation of the 6×6 ray propagator matrix $\mathbf{\Pi}^{(q)}(R,S)$, but using (37). The 4×4 ray propagator matrix $\boldsymbol{\Pi}^{(q)}(R,S)$ is again obtained from the 6×6 ray propagator matrix $\mathbf{\Pi}^{(q)}(R,S)$ by removing the third and sixth columns and rows. The method does not require to know the whole 6×6 ray propagator matrix $\mathbf{\Pi}^{(x)}(R,S)$ in Cartesian coordinates, but only four solutions of the dynamic ray tracing system in Cartesian coordinates, for strictly specified initial conditions.

3) The third method is based on Equation (57). This equation yields the 4×4 ray propagator matrix in ray-centered coordinates $\boldsymbol{\Pi}^{(q)}(R,S)$ directly.

4) Finally, the fourth method is based on Equation (51). Similarly as in the third method, Equation (51) yields the 4×4 ray propagator matrix $\boldsymbol{\Pi}^{(q)}(R,S)$ in ray-centered coordinates directly.

All the four methods should give the same 4×4 ray propagator matrix $\boldsymbol{\Pi}^{(q)}(R,S)$ in ray-centered coordinates; the differences are only in the numerical efficiency of computations. The third and fourth method are in general numerically more efficient than the first and second method, as they avoid the unnecessary computation of elements in the third and sixth columns and rows, which are later removed. Even the third and fourth method, however, differ in numerical efficiency. In general, the fourth method, based on Equation (51), is numerically most efficient, as it requires to compute only four solutions of the dynamic ray tracing in Cartesian coordinates.

In some exceptional cases, the third method may be more useful than the fourth. This applies to the case that we need to know to compute $\boldsymbol{\Pi}^{(q)}(R,S)$ for various initial conditions. In this case, it may be useful to compute the whole 6×6 propagator matrix $\mathbf{\Pi}^{(x)}(R,S)$, and vary only $\mathbf{\Psi}_2^r(S)$. We do not need to perform new dynamic ray tracing in Cartesian coordinates when we change the matrix of initial conditions $\mathbf{\Psi}_2^r(S)$.

Let us now consider any point Q situated on the reference ray Ω between S and R. Assuming the 6×6 propagator matrix $\mathbf{\Pi}^{(x)}(R,S)$ is known along the ray from S to R, including the point Q, we can evaluate the 4×4 propagator matrix $\boldsymbol{\Pi}^{(q)}(Q,S)$ at that point using (57), where we replace R by Q. Analogously, we can store $\mathbf{Z}^r(Q,S)$ and use (51), where we again replace R by Q. Actually, the only additional work is the computation of the 4×6 matrix $\mathbf{\Psi}_1^r(Q)$ at point Q.

A note to the choice of basis vectors $\mathbf{e}_I(S)$, $\mathbf{e}_I(R)$, $\mathbf{f}_I(S)$ and $\mathbf{f}_I(R)$. Assume first that $\mathbf{e}_I(S)$ are chosen as mutually perpendicular unit vectors, perpendicular to the slowness vector $\mathbf{p}(S)$. Actually, it is sufficient to specify only $\mathbf{e}_1(S)$, as $\mathbf{e}_2(S) = \mathcal{C}(S)\mathbf{p}(S) \times \mathbf{e}_1(S)$. The orientation of the unit vector $\mathbf{e}_1(S)$ (perpendicular to $\mathbf{p}(S)$), however, may be quite arbitrary in the plane perpendicular to the slowness vector $\mathbf{p}(S)$. The vectors $\mathbf{f}_I(S)$ must, of course, strictly correspond to this choice of $\mathbf{e}_I(S)$, see (19). The same is valid even at the point R: We can choose $\mathbf{e}_I(R)$ arbitrarily.

Analogous rules are valid even if we choose $\mathbf{f}_I(S)$ as mutually perpendicular unit vectors, perpendicular to the group velocity vector $\boldsymbol{\mathcal{U}}(S)$. It is again sufficient to specify the unit vector $\mathbf{f}_1(S)$ arbitrarily in the plane perpendicular to $\boldsymbol{\mathcal{U}}(S)$; the vector $\mathbf{f}_2(S) = (\boldsymbol{\mathcal{U}}(S) \times \mathbf{f}_1(S))/\mathcal{U}(S)$. The vectors $\mathbf{e}_I(S)$, however, must strictly correspond to the choice of $\mathbf{f}_I(S)$, see (20). The same is again valid at the point R: We can choose $\mathbf{f}_I(R)$ arbitrarily.

The important property of the approaches proposed here is that the variation of basis vectors $\mathbf{e}_I$ need not be calculated along the reference ray, like in the computation of $\boldsymbol{\Pi}^{(q)}(R, S)$ by dynamic ray tracing in ray-centered coordinates. In dynamic ray tracing in ray-centered coordinates, the basis vectors $\mathbf{e}_I$ must be calculated along the whole ray by solving a special ordinary differential equation of the first order. See Klimeš (1994, 2003), Červený (2001), etc. Nothing like this is needed here. The explanation is simple: The dynamic ray tracing in Cartesian coordinates is not influenced by variations of $\mathbf{e}_I$ along the ray at all.

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