Calculation of a slowness vector from a ray vector in anisotropic media

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Abstract

The wave quantities needed in constructing the wave fields propagating in anisotropic elastic media are usually calculated as a function of the slowness vector or its direction called the wave normal. In some applications, however, it is desirable to calculate the wave quantities as a function of a ray direction. In this paper, a method of calculating the slowness vector for a specified ray direction is proposed. The method is applicable to general anisotropy of an arbitrary strength having arbitrary complex triplications. The slowness vector is determined by numerical solving a system of multivariate polynomial equations of 6^{th} order. Solving the equations, we obtain a complete set of slowness vectors corresponding to all wave types and to all branches of the triplicate wave surface. The system of equations is further specified for rays shot in a symmetry plane of an orthorhombic medium and for a transversely isotropic medium. The system is decoupled into two polynomial equations of the 4^{th} order for the P-SV waves, and separately into the equations for the SH waves, which yield an explicit closed-form solution. The presented approach is particularly advantageous in constructing ray fields, ray-theoretical Green functions and wave fields in strong anisotropy.

1. INTRODUCTION

The phase and group velocities of elastic waves are generally different in anisotropic media. The phase velocity (or its inverse called the slowness) describes the propagation of plane waves and it is directed along a wave normal. The group velocity determines the signal

propagation and energy transport, and it is directed along a ray (Červený 2001). Wave quantities necessary in constructing the wave fields are usually calculated as a function of the wave normal. This is way is simple and satisfactory in many cases. Yet still, in some applications, calculating the wave quantities for a specified ray vector is more appropriate and desirable. Such approach, however, has not yet been fully developed, because it conceals complications. Whereas the slowness surface (defined by a directional variation of the slowness) is of the 6^{th} order, the wave surface (defined by a directional variation of the group velocity) can be of up to 150^{th} order (Fedorov 1968; Musgrave 1970; Helbig 1994). The slowness surface is formed by three sheets corresponding to three types of waves (P, S1 and S2), and each wave quantity is unique for a specified slowness vector and a wave type. However, the wave surface can display complicated triplications with cusps points (Wolfe 1998; Červený 2002, Vavryčuk 2003a,b) when several different values of a wave quantity (including the slowness vector) can correspond to one specified ray and a wave type (see Fig. 1).

The problem of calculating the phase and group velocities for a specified ray has been treated by several authors. First, approximate formulae valid under various symmetries of weak anisotropy have been suggested by Vavryčuk (1997), Song & Every (2000), Pšenčík & Vavryčuk (2002) and Farra (2003). The weak anisotropy condition is employed by the authors, in order to linearize the problem, and make thus it more treatable. Second, formulae valid under the two-dimensional anisotropy and derived using the Stroh formalism were suggested by Wang (1995). Third, the group velocity along a general direction in three-dimensional strong triclinic anisotropy was calculated by Sharma (2002), who applied the Newton's method for solving non-linear simultaneous equations. His approach, however, has difficulties when the wave surface is triplicate.

In this paper, the problem is solved with use of a system of three multivariate polynomial equations of the 6^{th} order. The system allows calculating all slowness vectors corresponding to a specified ray direction. Subsequently, all other wave quantities can easily be evaluated. The approach can find applications in tracing rays, in constructing the wavefronts, and in calculating the ray-theoretical Green functions and wave fields in anisotropic media of arbitrary strength and symmetry (Every & Kim 1994; Wolfe 1998; Červený 2001).

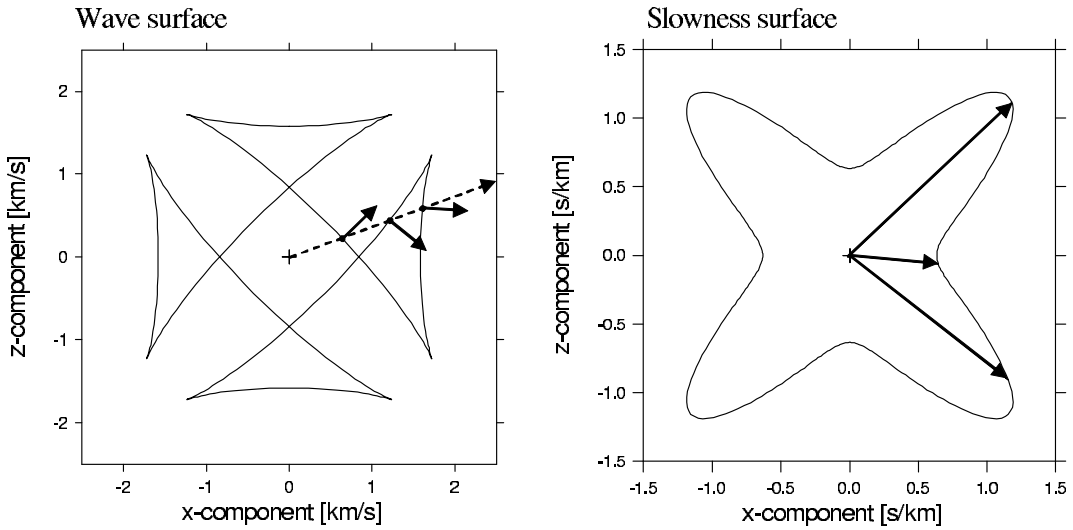

Figure 1. Wave and slowness surfaces for the SV wave in a transversely isotropic medium defined by the density-normalized elastic parameters: $a_{11} = a_{22} = a_{33} = 6.25$, $a_{44} = a_{55} = a_{66} = 2.50$, $a_{12} = a_{11} - 2a_{66} = 1.25$, and $a_{13} = a_{23} = 5.50$. Since the wave surface is triplicate, the ray (dashed line) crosses the wave surface at three points. The normals to the wave surface at these points define the directions of three slowness surfaces.

2. SLOWNESS SURFACE

The propagation of waves in elastic anisotropic media is described by the Christoffel equation which reads (Červený 2001, Eqs 2.2.35-37)

$$\det\left(\Gamma_{jk} - G\delta_{jk}\right) = 0. \qquad (1)$$

Tensor Γ_{jk} is the Christoffel tensor, δ_{jk} is the Kronecker delta, and G is the eigenvalue of Γ_{jk}. The Christoffel tensor Γ_{jk} can be defined either in terms of the slowness direction **n,**

$$\Gamma_{jk}(\mathbf{n}) = a_{ijkl} n_i n_l \ , \tag{2}$$

or in terms of the slowness vector $\mathbf{p} = \mathbf{n}/c$,

$$\Gamma_{jk}(\mathbf{p}) = a_{ijkl} p_i p_l \ , \tag{3}$$

where $a_{ijkl} = c_{ijkl}/\rho$ are the density-normalized elastic parameters, c_{ijkl} are the elastic parameters, ρ is the density of the medium, and c is the phase velocity of the wave. The tensor Γ_{jk} is positive-definite. It has three eigenvalues G, which are real and positive, and three unit eigenvectors **g**, called polarization vectors. Eigenvalues $G(\mathbf{n})$ and $G(\mathbf{p})$ read

$$G(\mathbf{n}) = a_{ijkl} n_i n_l g_j g_k = c^2 , \tag{4}$$

$$G(\mathbf{p}) = a_{ijkl} p_i p_l g_j g_k = 1 \ . \tag{5}$$

The slowness surface is defined by a set of all slowness vectors p(n) and can be determined by Eq. (5). Since the Christoffel tensor has three eigenvalues, the slowness surface consists of three sheets corresponding to three different waves (P, S1 and S2 waves). These sheets can be separated, but they can also touch or intersect each other. The degree of the slowness surface is 6, thus any straight line intersects the surface in 6 points at the most. The inner sheet (corresponding to the P wave) must be wholly convex (see Musgrave 1970, p. 92), but the other sheets (corresponding to the S1 and S2 waves) can also be locally concave or saddle shaped.

3. WAVE SURFACE

The group velocity vector $\mathbf{v} = \mathbf{v}(\mathbf{p})$ satisfies the equation (Červený 2001, Eq. 2.2.65)

$$v_i = \frac{1}{2}\frac{\partial G(\mathbf{p})}{\partial p_i}. \tag{6}$$

The direction $\mathbf{N}$ of the group velocity, $\mathbf{N} = \mathbf{v}/\sqrt{v_i v_i}$, is called the ray direction. If we differentiate Eq. (7) using the theorem on implicit functions, we obtain (Červený 1972, Eq. 15; Červený 2001, Eq. 3.6.15)

$$v_i = a_{ijkl}\, p_l \frac{D_{jk}}{D}, \tag{7}$$

where D_{jk} is the matrix of cofactors of $\hat{D}_{jk} = \Gamma_{jk} - G\delta_{jk}$,

$$\begin{aligned}
&D_{11} = (\Gamma_{22} - G)(\Gamma_{33} - G) - \Gamma_{23}^2, \quad D_{12} = D_{21} = \Gamma_{13}\Gamma_{23} - \Gamma_{12}(\Gamma_{33} - G),\\
&D_{22} = (\Gamma_{11} - G)(\Gamma_{33} - G) - \Gamma_{13}^2, \quad D_{13} = D_{31} = \Gamma_{12}\Gamma_{23} - \Gamma_{13}(\Gamma_{22} - G),\\
&D_{33} = (\Gamma_{11} - G)(\Gamma_{22} - G) - \Gamma_{12}^2, \quad D_{23} = D_{32} = \Gamma_{12}\Gamma_{13} - \Gamma_{23}(\Gamma_{11} - G),
\end{aligned} \tag{8}$$

where $\Gamma_{jk} = \Gamma_{jk}(\mathbf{p})$, $G = G(\mathbf{p}) = 1$ and $D = D(\mathbf{p})$ is defined as

$$D = D_{kk} = D_{11} + D_{22} + D_{33}. \tag{9}$$

Obviously, Eq. (7) holds only for the so-called regular directions which are defined by $D \neq 0$. The equation is not applicable to so-called singular directions (singularities, acoustic axes, see Vavryčuk 2001, 2003c), which are characterized by $D = 0$. For singularities, two eigenvalues of the Christoffel tensor coincide

$$G_1(\mathbf{n}) = G_2(\mathbf{n}) \neq G_3(\mathbf{n}), \tag{10}$$

and the tensor is degenerate. Exceptionally, all three eigenvalues can coincide, but we do not consider this type of singularity here. In the singularity, the group velocity is calculated as (Červený 2001, Eq. 3.6.10)

$$v_i = a_{ijkl} p_l g_j g_k, \tag{11}$$

where $\mathbf{g} = \mathbf{g}(\mathbf{p})$ is the polarization vector of the degenerate wave. The vector is not unique, but it has an arbitrary direction in the plane perpendicular to the polarization vector of the non-degenerate wave. The polarization vectors near the singularity have exceptional properties depending on the type of the singularity (see Shuvalov 1998).

The wave surface (also called the group velocity surface) is defined by a set of all group-velocity vectors **v**. They can be parameterized by slowness vector **p**, $\mathbf{v} = \mathbf{v}(\mathbf{p})$, by slowness direction **n**, $\mathbf{v} = \mathbf{v}(\mathbf{n})$, or by ray direction **N**, $\mathbf{v} = \mathbf{v}(\mathbf{N})$. Compared with the slowness surface, the geometry of the wave surface is much more complicated. The degree of the wave surface may be significantly higher than that of the slowness surface (but must be less than 150; see Musgrave 1970, p. 92). If the slowness sheet of the wave in question contains saddle-shaped or concave areas, the function $\mathbf{v} = \mathbf{v}(\mathbf{N})$ is multivalued. This means that many group-velocity vectors can correspond to one specified ray direction. This effect is known as the "triplication" of the wave surface. The existence of triplications is conditioned by the existence of parabolic lines which separate convex, concave and saddle-shaped areas on the wave surface (Every & Kim 1994; Shuvalov & Every 1996, 1997). The parabolic lines on the slowness surface are mapped on to caustics (also called cusps, cuspidal lines or cuspidal edges). The triplications and caustics develop on the wave surface, particularly, near the singularities (see Vavryčuk 2003b).

4. THE SLOWNESS VECTOR FROM A RAY VECTOR

From equations (7) and (11) we can calculate the ray vector **N** as a function of the slowness vector **p**, $\mathbf{N} = \mathbf{N}(\mathbf{p})$. In this section we attempt to find the inverse relation, $\mathbf{p} = \mathbf{p}(\mathbf{N})$. We will derive the equations valid for general triclinic anisotropy and discuss their applicability.

4.1. System of equations

Equation (7) can be rewritten in the following form:

$$DN_i - \frac{1}{v} a_{ijkl} D_{jk} p_l = 0. \tag{12}$$

Taking into account that

$$v_i p_i = 1, \tag{13}$$

and consequently,

$$\frac{1}{v} = p_i N_i \ , \tag{14}$$

we get

$$DN_i - a_{ijkl} D_{jk} p_l p_m N_m = 0. \tag{15}$$

This equation can be viewed as the system of three linear equations in unknowns N_1, N_2 and N_3 provided that elastic parameters a_{ijkl} and components of the slowness vector p_1, p_2 and p_3 are known. However, the equation can also be viewed as the system of three coupled polynomial equations of the sixth order in three unknowns p_1, p_2 and p_3, provided that elastic

parameters a_{ijkl} and components of the ray direction N_1, N_2 and N_3 are known. Since the equation is invariant to inserting **N** or –**N**, it should yield solutions for a ray direction with both orientations ±**N**.

4.2. Spurious solutions

Equation (15) yields not only true slowness vectors for a specified ray direction, but also some spurious solutions, which must be rejected. First, we reject complex-valued solutions. Second, we have to reject some of the solutions with $D = 0$. These solutions were incorporated during the derivation of (15), when Eq. (7) was multiplied by D. Even though that all true slowness vectors lie in regular directions ($D \neq 0$), the complete set of solutions of (15) will always comprise also some spurious solutions characterized by $D = 0$. In order to discriminate whether the solution with $D = 0$ is a true singular solution or a spurious solution, we have to calculate the group velocity vector by Eq. (11) and to check, whether this vector is parallel or not to the specified ray direction. If the group velocity is parallel to the specified ray, then the slowness vector represents a true solution, which is pointed at the singularity. In the other case, the solution is spurious. Obviously, we have to reject also all slowness vectors, which yield the orientation opposite to the specified ray direction.

5. ANISOTROPY OF HIGHER SYMMETRY

5. 1. Orthorhombic anisotropy

Let us study the previously derived equations in a symmetry plane of orthorhombic anisotropy (ORT). The elastic parameters are specified in the two-index notation as follows: a_{11}, a_{22}, a_{33}, a_{44}, a_{55}, a_{66}, a_{12}, a_{13}, a_{23}, the other parameters being zero. The symmetry plane is the x_1-x_3 plane. Since $p_2 = 0$, the system (13) of three equations in three unknowns p_1, p_2 and p_3 reduces to two equations in two unknowns p_1 and p_3. The equations read

$$A(N_1p_1+N_3p_3)p_1p_3^4+BN_1p_3^4+C(2N_1p_1+N_3p_3)p_1p_3^2+2D(N_1p_1+N_3p_3)p_1^3p_3^2+2EN_1p_3^2$$
$$+3F(N_1p_1+N_3p_3)p_1^5+G(3N_1p_1+2N_3p_3)p_1^3+H(3N_1p_1+N_3p_3)p_1+3N_1=0 \tag{16}$$

$$2A(N_1p_1+N_3p_3)p_1^2p_3^3+B(2N_1p_1+3N_3p_3)p_3^3+C(N_1p_1+2N_3p_3)p_1^2p_3+D(N_1p_1+N_3p_3)p_1^4p_3$$
$$+E(N_1p_1+3N_3p_3)p_3+GN_3p_1^4+2HN_3p_1^2+3I(N_1p_1+N_3p_3)p_3^5+3N_3=0$$

where

$$A=-a_{11}a_{33}a_{44}+a_{13}^2a_{44}+2a_{13}a_{44}a_{55}-a_{33}a_{55}a_{66},$$
$$B=a_{33}a_{44}+a_{33}a_{55}+a_{44}a_{55},$$
$$C=a_{11}a_{33}+a_{11}a_{44}-a_{13}^2-2a_{13}a_{55}+a_{33}a_{66}+a_{44}a_{55}+a_{55}a_{66},$$
$$D=-a_{11}a_{33}a_{66}-a_{11}a_{44}a_{55}+a_{13}^2a_{66}+2a_{13}a_{55}a_{66}, \tag{17}$$
$$E=-a_{33}-a_{44}-a_{55},$$
$$F=-a_{11}a_{55}a_{66},$$
$$G=a_{11}a_{55}+a_{11}a_{66}+a_{55}a_{66},$$
$$H=-a_{11}-a_{55}-a_{66},$$
$$I=-a_{33}a_{44}a_{55}.$$

The system of equations can be even more simplified. Since two waves are always polarized in the symmetry plane (the P and SV waves), and the third wave is always transverse (the SH wave), the system can be decoupled into two systems of equations: for the P-SV waves, and separately for the SH waves. The equations for the SH waves can be solved explicitly. If we apply the equations for the group velocities derived by Musgrave (1970, Eqs 9.3.6, 9.3.7, 9.3.8), we obtain for the P-SV waves,

$$\left(-a_{11}a_{33}+a_{13}^2+2a_{13}a_{55}\right)(N_1p_1+N_3p_3)p_1p_3^2+(a_{33}+a_{55})N_1p_3^2+(a_{11}+a_{55})(2N_1p_1+N_3p_3)p_1$$
$$-2a_{11}a_{55}(N_1p_1+N_3p_3)p_1^3-2N_1=0 \tag{18}$$

$$\left(-a_{11}a_{33}+a_{13}^2+2a_{13}a_{55}\right)(N_1p_1+N_3p_3)p_1^2p_3+(a_{11}+a_{55})N_3p_1^2+(a_{33}+a_{55})(N_1p_1+2N_3p_3)p_3$$
$$-2a_{33}a_{55}(N_1p_1+N_3p_3)p_3^3-2N_3=0$$

and for the SH waves,

$$p_1=\sqrt{\frac{a_{44}}{a_{66}}}\frac{N_1}{\sqrt{a_{44}N_1^2+a_{66}N_3^2}},\ p_3=\sqrt{\frac{a_{66}}{a_{44}}}\frac{N_3}{\sqrt{a_{44}N_1^2+a_{66}N_3^2}}\ . \tag{19}$$

5. 2. Transverse isotropy

Let us assume that the medium is transversely isotropic (TI) with a vertical axis of symmetry. The elastic parameters are specified in the two-index notation as follows: $a_{11}=a_{22}$, a_{33}, $a_{44}=a_{55}$, a_{66}, $a_{13}=a_{23}$, $a_{12}=a_{11}-2a_{66}$, the other parameters being zero. Since the medium is rotationally symmetric around the vertical axis, it is sufficient to study all quantities and equations in the x_1-x_3 plane only. Similarly to orthorhombic anisotropy, the system (13) of three equations in three unknowns p_1, p_2 and p_3 reduces to two equations in two unknowns p_1 and p_3. The form of equations is identical to (16), but the coefficients (17) slightly differ:

$$A=a_{44}\left(-a_{11}a_{33}+a_{13}^2+2a_{13}a_{44}-a_{33}a_{66}\right),$$
$$B=a_{44}(2a_{33}+a_{44}),$$
$$C=a_{11}a_{33}+a_{11}a_{44}-a_{13}^2-2a_{13}a_{44}+a_{33}a_{66}+a_{44}^2+a_{44}a_{66},$$
$$D=-a_{11}a_{33}a_{66}-a_{11}a_{44}^2+a_{13}^2a_{66}+2a_{13}a_{44}a_{66}, \tag{20}$$
$$E=-a_{33}-2a_{44},$$

$$F = -a_{11}a_{44}a_{66},$$

$$G = a_{11}a_{44} + a_{11}a_{66} + a_{44}a_{66},$$

$$H = -a_{11} - a_{44} - a_{66},$$

$$I = -a_{33}a_{44}^2.$$

The system of equations can be even more simplified. Similarly as for orthorhombic media in a symmetry plane, the system can be decoupled into two systems of equations: for the P-SV waves, and separately for the SH waves. If we apply equations for the group velocities derived by Musgrave (1970, Eqs 8.2.2a, 8.2.8, 8.2.9), we obtain for the P-SV waves

$$\begin{aligned}&\left(-a_{11}a_{33} + a_{13}^2 + 2a_{13}a_{44}\right)(N_1p_1 + N_3p_3)p_1p_3^2 + (a_{33} + a_{44})N_1p_3^2 + (a_{11} + a_{44})(2N_1p_1 + N_3p_3)p_1 \\ &-2a_{11}a_{44}(N_1p_1 + N_3p_3)p_1^3 - 2N_1 = 0\end{aligned} \tag{21}$$

$$\begin{aligned}&\left(-a_{11}a_{33} + a_{13}^2 + 2a_{13}a_{44}\right)(N_1p_1 + N_3p_3)p_1^2p_3 + (a_{11} + a_{44})N_3p_1^2 + (a_{33} + a_{44})(N_1p_1 + 2N_3p_3)p_3 \\ &-2a_{33}a_{44}(N_1p_1 + N_3p_3)p_3^3 - 2N_3 = 0\end{aligned}$$

and for the SH waves,

$$p_1 = \sqrt{\frac{a_{44}}{a_{66}}}\frac{N_1}{\sqrt{a_{44}N_1^2 + a_{66}N_3^2}},\ p_3 = \sqrt{\frac{a_{66}}{a_{44}}}\frac{N_3}{\sqrt{a_{44}N_1^2 + a_{66}N_3^2}}\ . \tag{22}$$

Similar formulae can be derived also for tetragonal and cubic symmetries.

6. NUMERICAL PROCEDURE

Equations (15), (18) and (21) represent systems of multivariate polynomial equations, which can be solved numerically using, for example, the MATHEMATICA software package. We obtain a set of real- and complex-valued slowness vectors **p**. We reject all complex-valued solutions. Then we calculate group velocity vectors **v** corresponding to all real-valued slowness vectors **p** using Eq. (11). We select only those solutions, for which the following condition is satisfied

$$\frac{N_i v_i}{v} = 1 \ , \tag{23}$$

where $v = \sqrt{v_i v_i}$ is the group velocity. Obviously, the equality in (23) is approximate and depends on the precision of the procedure for calculating the roots of the multivariate polynomial equations. Using Eq. (23), we skip also the solutions corresponding to the opposite orientation of the ray. Finally, we have to skip all multiplicities in the solutions. Once the set of true slowness vectors is known, we can trace rays or calculate all other wave quantities necessary for constructing wave fields such as phase velocities, polarization vectors or the Gaussian curvatures of the slowness and wave surfaces.

Equation (15) has also some limitations. First, it cannot be applied for calculating slowness vectors in conical points of the wave surface, when an infinite set of slowness vectors corresponds to a single ray direction. Second, the equation is not suitable for calculating the slowness vectors under extremely weak anisotropy. In this case, the medium is closely degenerate ($D \to 0$), and the solution can be distorted by numerical errors. Instead, a modified approach should be applied such as iterations or the linearized formulae obtained by perturbation theory.

7. EXAMPLE

The efficiency of the procedure is exemplified on tracing rays generated by a point source situated in a vertically inhomogeneous transversely isotropic medium with a vertical axis of symmetry. The approach is applied in order to compare ray fields calculated by specifying two different initial conditions of rays at the source: the rays specified by initial ray directions $\mathbf{N}_0$ and by initial slowness vector directions $\mathbf{n}_0$.

As the medium we assume the Bazhenov shale (Vernik & Liu, 1997, Appendix A, 12.507 ft) with the following elastic parameters (in km^2/s^2): $a_{11} = a_{22} = 23.52$, $a_{33} = 10.89$, $a_{44} = a_{55} = 5.29$, $a_{66} = 9.42$, $a_{12} = 4.69$, $a_{13} = a_{23} = 9.46$. These values characterize the medium at the source, which is situated at the depth of 1 km. The elastic parameters at other depths are calculated as:

$$a_{kl} = a_{kl}(z_0)[1 + \varepsilon(z - z_0)]^2, \tag{24}$$

where $\varepsilon = 0.3\,\mathrm{km}^{-1}$. The medium is strongly anisotropic with anisotropy strength 38.0%, 26.1% and 28.6% of *P*, *SV* and *SH* waves, respectively. The anisotropy strength is calculated as

$$a = 200\% \frac{c^{\max} - c^{\min}}{c^{\max} + c^{\min}}, \tag{25}$$

where $c^{\max}$ and $c^{\min}$ are the maximum and minimum phase velocities of the wave.

We shall study the rays of the SV wave excited by the source. The SV-wave surface displays two triplications: one is near the symmetry axis, and the other is near the symmetry plane. The triplication near the symmetry axis is more noticeable (see Fig. 2). The rays are shot from the source in the *x*-*z* plane in the angular interval $\langle 0°, 360° \rangle$ in steps of 15°. Figure 3a shows rays shot in an equidistant step of initial directions of a ray, figure 3b shows rays with and equidistant step of initial directions of a slowness vector. In both cases the angular step is 15°. The figures show a very irregular distribution of rays when the equidistant

angular step in directions of the slowness vector is required (see Fig. 3b). The rays concentrate in directions close to the vertical and horizontal axes. The other directions are covered very sparsely. If we perform the ray tracing and require equidistant initial directions of rays, we obtain a more regular and dense ray field (see Fig. 3a). This is even more clearly seen in Figs 3c and 3d, which show rays in the vicinity of the source. Fig. 3c also shows that for some initial directions, we have to trace not one but three different rays. This happens for the rays deviated from the vertical axis by 0° or by ±15°. In these cases, the rays hit the triplicate wave surface, and correspond to its three different branches. The similar situation appears for rays shot in the horizontal plane, but the triplication in the horizontal plane is almost indistinguishable and the three rays coincide.

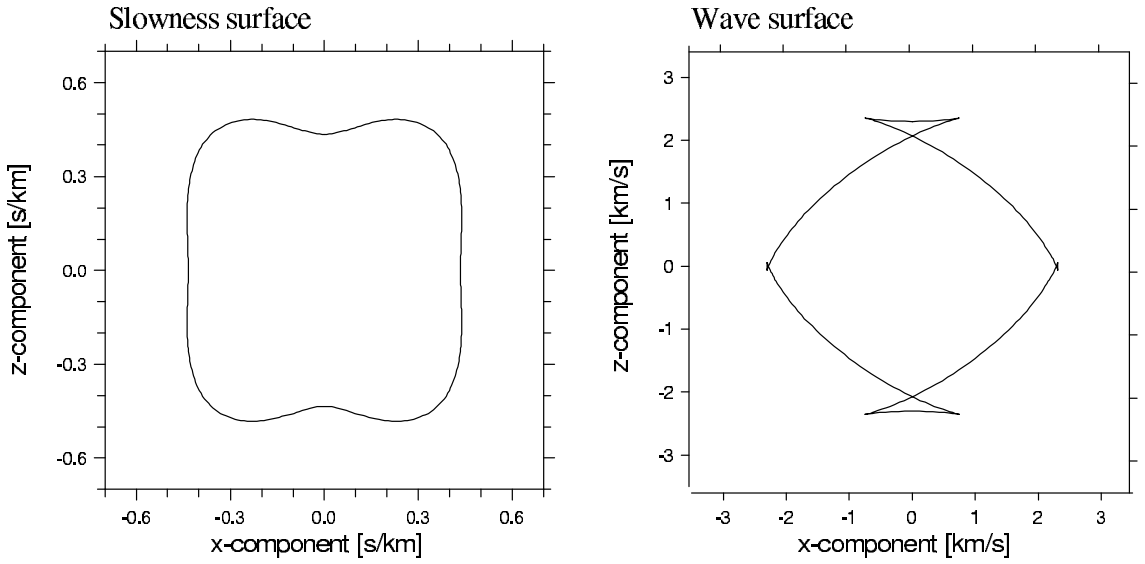

Figure 2. The SV wave and slowness surfaces for the Bazhenov shale. For elastic parameters, see the text.

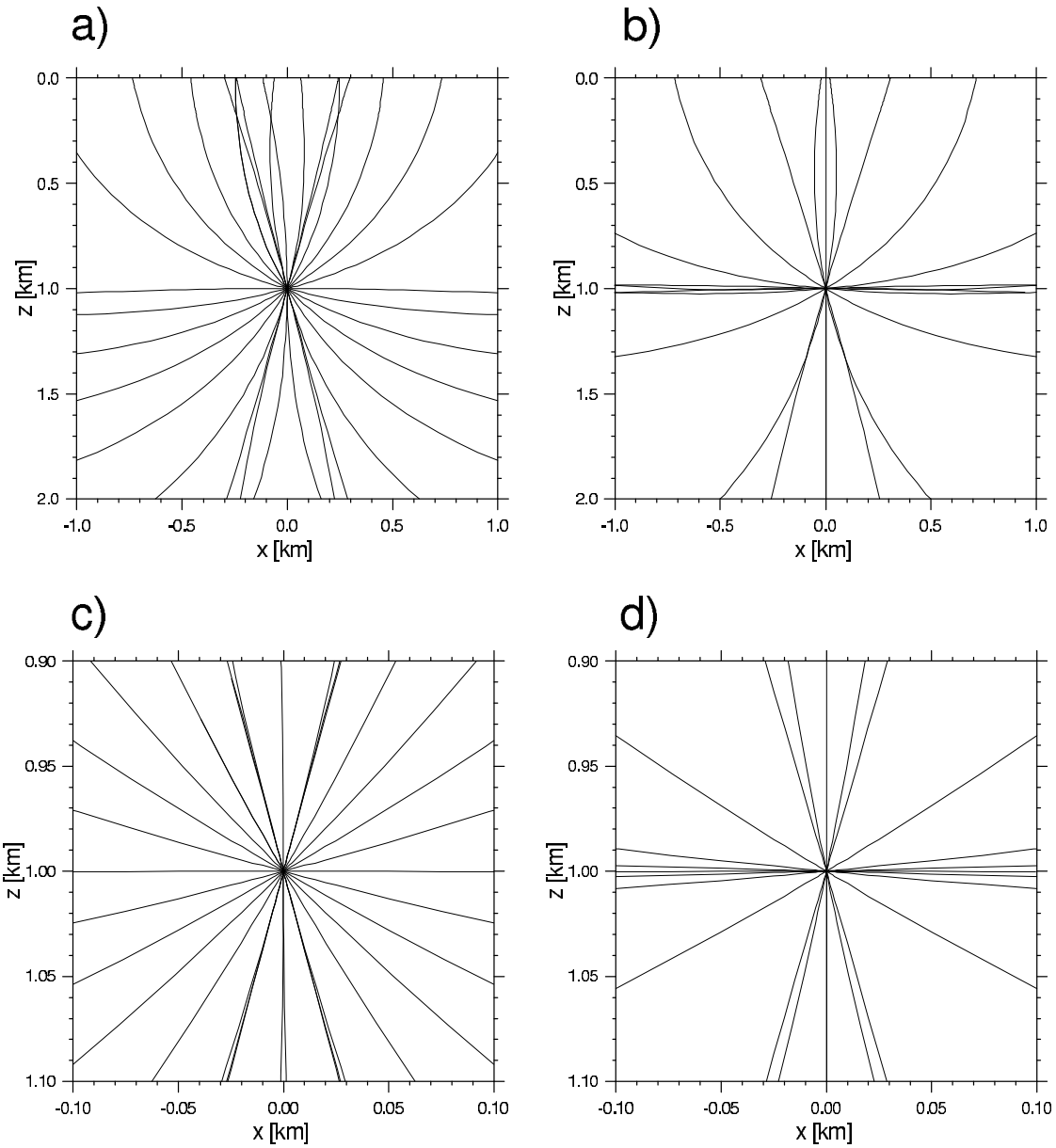

Figure 3. The ray field generated by a point source in the Bazhenov shale. (a) Rays are shot in an equidistant step of 15° in initial ray directions. (b) Rays are shot in an equidistant step of 15° in initial slowness vector directions. (c) A close view on the ray field from (a). (d) A close view on the ray field from (b).

8. CONCLUSIONS

The calculation of a slowness vector from a specified ray direction in a general anisotropy with an arbitrary strength leads to solving a system of three coupled equations of the 6^{th} order in three unknowns. Such system can be solved numerically using, for example, the MATHEMATICA software package. The system of equations is simplified for a higher symmetry of anisotropy. In transverse isotropy and in a symmetry plane of orthorhombic, tetragonal or cubic anisotropy, the system can be further decoupled into equations for the P-SV waves and separately for the SH waves. The system of equations for the P-SV waves consists of two equations of the 4^{th} order in two unknowns. The problem for the SH waves can be solved algebraically.

The basic advantage of the proposed approach is that we obtain a full set of slowness vectors corresponding to a specified ray even in the case of complicated multiple triplications. From the slowness vectors we can calculate all wave quantities needed for modeling waves in anisotropic media. The approach works properly in regular directions as well as in all kinds of singularities (kiss, intersection, wedge or conical singularity). The approach is not applicable for conical points on the wave surface when an infinite set of slowness vectors correspond to a single ray direction. Also for extremely weak anisotropy, the approach can yield distorted results due to numerical errors. In this case, it is more appropriate to apply the linearized formulae obtained by perturbation theory.

References

Červený, V., 1972. Seismic rays and ray intensities in inhomogeneous anisotropic media, *Geophys. J. R. astr. Soc.*, **29**, 1-13.

Červený, V., 2001. *Seismic Ray Theory*, Cambridge University Press, Cambridge.

Červený, V., 2002. Fermat's variational principle for anisotropic inhomogeneous media, *Stud. Geophys. Geod.*, **46**, 567-588.

Every, A.G. & Kim, K.Y., 1994. Time-domain dynamic response functions of elastically anisotropic solids, *J. Acoust. Soc. Am.*, **95**, 2505-2516.

Farra, V., 2004. Improved first-order approximation of group velocities in weakly anisotropic media, *Stud. Geophys. Geod.*, **48**, 199-213.

Fedorov, F.I., 1968. *Theory of Elastic Waves*, Plenum Press. N.Y.

Helbig, K., 1994. *Foundations of Anisotropy for Exploration Seismics*, Pergamon, N.Y.

Musgrave, M.J.P., 1970. *Crystal Acoustics*, Holden-Day, London.

Pšenčík, I., & Vavryčuk, V., 2002. Approximate relation between the ray vector and the wave normal in weakly anisotropic media, *Stud. Geophys. Geod.*, **46**, 793-807.

Sharma, M.D., 2002. Group velocity along general direction in a general anisotropic medium, *Int. J. Solids Struct.*, **39**, 3277-3288.

Shuvalov, A.L., 1998. Topological features of the polarization fields of plane acoustic waves in anisotropic media, *Proc. R. Soc. Lond.*, **A 454**, 2911-2947.

Shuvalov, A.L. & Every, A.G., 1996. Curvature of acoustic slowness surface of anisotropic solids near symmetry axes, Phys. Rev., B 53, No. 22, 14.906-14.916.

Shuvalov, A.L. & Every, A.G., 1997. Shape of the acoustic slowness surface of anisotropic solids near points of conical degeneracy, J. acoust. Soc. Am., 101, 2381-2382.

Song, L.-P., & Every, A.G., 2000. Approximate formulae for acoustic wave group slownesses in weakly orthorhombic media, *J. Phys. D: Appl. Phys.*, **33**, L81-L85.

Vavryčuk, V., 1997. Elastodynamic and elastostatic Green tensors for homogeneous weak transversely isotropic media, *Geophys. J. Int.*, **130**, 786-800.

Vavryčuk, V., 2001. Ray tracing in anisotropic media with singularities, *Geophys. J. Int.*, **145**, 265-276.

Vavryčuk, V., 2003a. Generation of triplications in transversely isotropic media, *Phys. Rev. B*, **68**, art. no. 54107.

Vavryčuk, V., 2003b. Parabolic lines and caustics in homogeneous weakly anisotropic media. *Geophys. J. Int.*, **152**, 318-334.

Vavryčuk, V., 2003c. Behavior of rays near singularities in anisotropic media, *Phys. Rev. B*, **67**, art. no. 54105.

Vernik, L., & Liu, X., 1997. Velocity anisotropy in shales: A petrological study, *Geophysics*, **62**, 521-532.

Wang, L., 1995. Determination of the ray surface and recovery of elastic constants of anisotropic elastic media: a direct and inverse approach, *J. Phys.: Condens. Matter*, **7**, 3863-3880.

Wolfe, J.P. *Imaging Phonons. Acoustic Wave Propagation in Solids* (Cambridge University Press, Cambridge, 1998).