Time-averaged and time-dependent energy-related quantities of waves propagating in inhomogeneous viscoelastic anisotropic media

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Summary

The energy-flux vector plays an important role in various wave propagation problems. In acoustics and seismology, the main attention has been devoted to the time-averaged energy flux of time-harmonic wave fields propagating in non-dissipative, isotropic and anisotropic media. In this paper, we investigate the energy-flux vector and other energy-related quantities of wave fields propagating in inhomogeneous anisotropic viscoelastic media. These quantities satisfy energy-balance equations, which have, as we show, formally different forms for real-valued wave fields with arbitrary time dependence and for time-harmonic wave fields. In case of time-harmonic wave fields, we do not study only time-averaged, but also time-dependent constituents of the energy-related quantities. We show that the energy-balance equations for time-harmonic wave fields can be obtained in two different ways. First, using real-valued wave fields in the real-valued equation motion and stress-strain relation. Second, using complex-valued wave fields in the complex-valued equation motion and stress-strain relation. Both approaches, when used for the Kelvin-Voigt viscoelastic model, yield the same expressions for the time-averaged and time-dependent constituents of all energy-related quantities and the same energy-balance equations. The latter approach is more powerful, as it can be applied to media of unrestricted anisotropy and viscoelasticity.

Key words

Viscoelastic anisotropic media, energy-flux vector, time-averaged energy-related quantities, time-dependent energy-related quantities.

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1 Introduction

The energy-flux vector is one of the basic characteristics of seismic wave fields propagating in the Earth's interior, and plays an important role in many applications in seismology and seismic exploration; both theoretical and practical. In this paper, the properties of energy-flux vector and of other energy-related quantities of waves propagating in anisotropic dissipative media, are studied. Only dissipative mechanisms, which can be described in the framework of linear viscoelasticity, are considered. All derivations are based on the equation of motion and on appropriate stress-strain relation.

A broad literature has been devoted to the energy flux of wave fields propagating in perfectly elastic, isotropic and anisotropic, media, and in viscoelastic isotropic media. For anisotropic viscoelastic media, however, the references are not so many. For an up-to-date review related to this problem see Červený and Pšenčík (2006). In spite of fast progress in this field, certain problems still remain open. In this paper, we try to discuss several of them.

The relations between the energy-flux vector and other energy-related quantities (kinetic, strain and dissipated energy and energy supplied by internal sources) are known in the literature under many names: energy-balance equation, energy-conservation equation, law on conservation of energy, energy-conservation theorem, Poynting theorem, etc. Throughout this paper, we shall consistently use the term *energy-balance equations*.

When studying energy-balance equations, two forms of wave fields are considered: a) real-valued wave fields of arbitrary time dependence, and b) time-harmonic wave fields. The energy-balance equations for these two cases are formally different. For example, the energy-balance equation for wave fields with arbitrary time dependence contains time-derivative of the sum of kinetic and strain (potential) energy, whereas the energy-balance equation for time-harmonic wave fields contains the *difference* between the kinetic and potential energy (the Lagrangian). An attempt to explain these differences is made in this paper.

For real-valued wave fields of arbitrary time dependence, the stress-strain relation is expressed in convolutory form. To avoid possible complications connected with this general convolutory form, we consider only a special case of a Kelvin-Voigt viscoelastic model, in which the stress-strain relation simplifies to standard multiplication. The Kelvin-Voigt stress-strain relation is quite sufficient for the purpose of our investigation. The energy-balance equation is derived, which gives the relation between the divergence of the energy flux vector and the time derivative of the sum of kinetic and strain energy, the dissipated energy and the energy supplied by internal sources. For non-dissipative media, the energy-balance equation reduces to the energy-conservation law, well-known from seismological textbooks, see, e.g., Červený (2001a). All the energy-related quantities, including the energy flux, are real valued and time dependent in this law.

We focus our attention on time-harmonic, real-valued or complex-valued wave fields. There are two approaches how to derive the energy-balance equations for time-harmonic waves.

(a) We specify the real-valued time-dependent energy-balance equation for real-valued, but time-harmonic wave fields. This is, of course, possible, as the energy-balance equation is valid for wave fields with arbitrary time dependence. Actually, two such real-valued energy-balance equations can be derived. The two energy-balance equations can be combined into one complete complex-valued energy-balance equation. All energy quantities in this complex-valued energy-balance equation are expressed as a sum of their time-averaged and time-dependent constituents.

(b) We start directly from the complex-valued equation of motion and complex-valued stress-strain relation, valid for time-harmonic waves. We obtain two energy-balance equations, one for time-averaged energy quantities, and the other for time-dependent energy quantities, and combine them again into one complete complex-valued energy-balance equation. This equation is not limited to the Kelvin-Voigt viscoelastic model, but it is valid for unrestricted viscoelasticity and anisotropy.

The complete complex-valued energy-balance equations, derived by the above two independent approaches, fully coincide if both are specified for the Kelvin-Voigt viscoelasticity model. The expressions for all time-averaged and time-dependent energy-related quantities are the same too in both approaches.

In seismological literature, the main attention has been devoted to time-averaged energy-flux vector. The studies of time-dependent energy flux are more or less exceptional. They are related mostly to non-dissipative media (Hayes, 1980; Mann, Tichy and Romano, 1987; Boulanger and Hayes, 2000). For non-dissipative media, it has been common to speak on instantaneous energy flux. As the stress-strain relations for viscoelastic media are not, in general, instantaneous, we prefer to speak on time-dependent energy flux in dissipative media. A most thorough treatment of time-dependent energy flux in non-dissipative media can be found in Mann, Tichy and Romano (1987), who also investigated the trains of waves. They showed that the time-dependent energy flux of train of waves may exhibit vortex behaviour of energy-velocity vector fields in certain cases. Some other authors discussed the time-dependent energy flux in relation to the reflection/transmission problem at a plane interface between two viscoelastic media. See Deschamps (1990), Ainslie and Burns (1995), Červený (2001b). Ainslie and Burns (1995) call the time-dependent energy flux vector the Deschamp's instantaneous pseudoenergy vector, Červený (2001b) calls it the instantaneous energy-flux vector or pseudoenergy flux vector. The solution of the reflection/transmission problem at a plane interface between two viscoelastic media is complicated by the appearence of interaction energy contributions, generated by interference of individual waves, involved in the R/T process. Červený (2001b) showed that the interaction contributions completely vanish if only time-dependent constituents of individual waves (incident, reflected, transmitted; P, S1, S2) are considered.

In Section 2, basic equations describing wave propagation in viscoelastic anisotropic inhomogeneous media are introduced and the stress-strain relations are briefly discussed. Simple Kelvin-Voigt stress-strain relation is used. In Section 3, the energy-balance equation is derived for real-valued arbitrarily time-dependent wave fields propagating in the Kelvin-Voigt viscoelasticity model. In Section 4, two real-valued energy-balance equations for real-valued time-harmonic wave fields are derived. Their time-averaged and time-dependent constituents are considered. Both real-valued energy-balance equations are combined into one complete complex-valued energy-balance equation. Finally, in Section

5, the complex-valued energy-balance equations for time-averaged and time-dependent energy quantities, are derived directly from complex-valued equation of motion for time-harmonic waves, and from the relevant complex-valued stress-strain relation.

The dot above a letter denotes the time derivative ($\dot{u}_i = \partial u_i / \partial t$), the index following the comma in the subscript denotes the partial derivative with respect to the relevant Cartesian coordinate ($u_{i,j} = \partial u_i / \partial x_j$), and the asterisk in the superscript denotes the complex conjugacy. The lower-case indices i, j, ..., m, n attain values 1, 2, 3; the lower-case indices α, β values 1, 2, .., 6. The Einstein summation convention over repeated indices is used throughout the paper.

2 Basic formulae

We use Cartesian coordinates x_i, time t, and denote the Cartesian components of the displacement vector $u_i(x_k, t)$, of the particle-velocity vector $v_i(x_k, t) = \dot{u}_i(x_k, t)$, of the stress tensor $\tau_{ij}(x_k, t)$, and of the strain tensor $e_{ij}(x_k, t)$. The strain tensor e_{ij} can be expressed in terms of the displacement vector u_i as follows:

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \ . \tag{1}$$

As we shall mostly work with particle-velocity vector v_i instead of the displacement vector u_i, we shall also use time derivative of (1) :

$$\dot{e}_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}) \ . \tag{2}$$

The equation of motion is given by the relation

$$\tau_{ij,j} + f_i = \rho \dot{v}_i \ , \tag{3}$$

where $\rho(x_i)$ is the density and $f_i(x_k, t)$ is the density of the body force vector representing internal sources.

Equations (2) and (3) must be supplemented by a stress-strain relation, connecting τ_{ij} and e_{ij}. Various anelastic material processes influence this relation (grain defects, grain-boundary processes, thermoelastic effects, etc.). An extensive literature is devoted to these anelastic processes. Many of them can be discussed within the framework of linear viscoelasticity, within which stress depends linearly on the strain and its time variation. The stress-strain relation is then expressed in a convolutory form, with the relaxation and creep function as the main ingredients. In this section, we consider one of the simplest stress-strain relations of viscoelastic anisotropic media, in which the convolutory form is reduced to standard multiplication and summation. More specifically, we consider the following relation between τ_{ij} and e_{ij}, see, for example, Auld (1973), Scott (1997), Shuvalov and Scott (1999), Carcione (2001):

$$\tau_{ij} = c^R_{ijkl}\, e_{kl} + \eta^R_{ijkl} \dot{e}_{kl} \ . \tag{4}$$

Here c^R_{ijkl} and η^R_{ijkl} are components of four-rank real-valued elasticity and viscosity tensors, respectively. We will also call individual components c^R_{ijkl} and η^R_{ijkl} the elasticity and viscosity moduli, respectively. They are real-valued and frequency-independent constants. The stress-strain relation (4) will allow us to discuss certain important properties of time-averaged and time-dependent energy-related quantities, in which we are interested, without unnecessary mathematical complications. Eq.(4) represents an extension of the standard Kelvin-Voigt model, which can be described in terms of me- chanical spring and dashpot acting in parallel, to anisotropy.

We assume that the elasticity and viscosity moduli satisfy the symmetry relations,

$$c^R_{ijkl} = c^R_{jikl} = c^R_{ijlk} = c^R_{klij} \ , \quad \eta^R_{ijkl} = \eta^R_{jikl} = \eta^R_{ijlk} = \eta^R_{lkij} \ . \tag{5}$$

Note that the relation $\eta^R_{ijkl} = \eta^R_{klij}$ has not yet been proved to hold generally, see, e.g., Carcione (2001). Relations (5) reduce the number of independent components of elasticity and viscosity tensors from 81 to 21. Due to these symmetries, the elasticity and viscosity tensors can be also expressed as symmetric 6×6 matrices (Voigt notation).

Individual quantities introduced in this section are expressed in the following units: displacement components u_i in meters (m), the particle-velocity components v_i in $m\,s^{-1}$, density ρ in $kg\,m^{-3}$, the components of body forces f_i in newton per cubic meter (N/m^3; $kg\,m^{-2}s^{-2}$), the strain components e_{ij} are dimensionles, the stress components τ_{ij} and elasticity moduli c^R_{ijkl} are in pascals ($Pa; kg\,m^{-1}s^{-2}$), and the viscosity moduli η^R_{ijkl} in pascal $\times$ second ($Pa\,s; kg\,m^{-1}s^{-1}$).

3 Energy-balance equation for wave fields with arbitrary time dependence

In this section, we consider the vectors v_i and f_i, and tensors τ_{ij}, e_{ij} to be real valued and time dependent. Multiplying (3) by $-v_i$ and (2) by $-\tau_{ij}$, and adding both, we obtain

$$(-v_i\tau_{ij})_{,j} = -\rho v_i \dot{v}_i - \tau_{ij}\dot{e}_{ij} + v_i f_i \ . \tag{6}$$

Equation (6) represents a very general form of the energy-balance equation. Physical meaning of all terms in (6), except $-\tau_{ij}\dot{e}_{ij}$ is the same as in perfectly elastic media and will be explained later. The term $-\tau_{ij}\dot{e}_{ij}$ depends considerably on the stress-strain relation under consideration. It can be separated into two terms; one corresponding to the strain energy and the other to the dissipated energy. This separation is not necessarily unique.

The situation simplifies if we consider Kelvin-Voigt stress-strain relation (4). Then the separation of the term $-\tau_{ij}\dot{e}_{ij}$ becomes unique and trivial. Inserting (4) into (6) yields

$$(-v_i\tau_{ij})_{,j} = -\rho v_i \dot{v}_i - c^R_{ijkl}\dot{e}_{ij}e_{kl} - \eta^R_{ijkl}\dot{e}_{ij}\dot{e}_{kl} + v_i f_i \ . \tag{7}$$

This can be expressed in a condensed form:

$$\mathrm{div}\mathbf{F} = -\tfrac{\partial}{\partial t}(K+U) - P_d + P_s \ , \tag{8}$$

where

$$\begin{aligned} F_j &= -v_i\tau_{ij} \ , \qquad K = \tfrac{1}{2}\rho v_i v_i \ , \quad U = \tfrac{1}{2}c^R_{ijkl}e_{ij}e_{kl} \ , \\ P_d &= \eta^R_{ijkl}\dot{e}_{ij}\dot{e}_{kl} \ , \quad P_s = v_i f_i \ . \end{aligned} \tag{9}$$

The vector $\mathbf{F}$ and the scalars K, U, P_d and P_s are real valued and time dependent. The meaning of individual quantities in (9) is as follows: $\mathbf{F}$ is energy flux vector (also called energy-flow vector, power-flow vector, Poynting vector, Poynting-Umov vector). It represents the amount of energy passing through a unit area perpendicular to it per unit time. K is the density of kinetic energy, U is the density of strain energy (also called potential energy), P_d is the density of dissipated energy per second and P_s is the density of energy supplied by internal sources per second.

Equation (8) is the energy-balance equation for viscoelastic anisotropic media for arbitrary time dependent real-valued wave field. It immediately yields the well-known energy-balance equation for perfectly elastic media, in which $P_d = 0$. See Červený (2001, eq. 2.1.38).

Since the density of strain energy U should be positive, and the density of dissipated energy per second, P_d, non-negative, the 6×6 Voigt elasticity matrix must be positive-definite, and the 6×6 Voigt viscosity matrix positive definite or zero.

Individual energy quantities in (9) have the following dimensions: U and K are expressed in pascals ($Pa; kg\, m^{-1} s^{-2}$), P_d and P_s in pascals per second ($Pa\, s^{-1}; kg\, m^{-1} s^{-3}$), and $\mathbf{F}$ in watts per square meter ($W m^{-2}; kg\, s^{-3}$). Thus, all terms in the energy-balance equation (8) are expressed in pascals per second.

4 Energy-balance equations for real-valued time-harmonic wave fields

Let us now use the energy-balance equation (8) for time-harmonic waves with a frequency $\omega > 0$. In Secs. 4.1 and 4.2, we use two specifications of time-harmonic real-valued expressions for $v_i, f_i, \tau_{ij}, e_{ij}$, describing the time-harmonic waves.

4.1 Energy-balance equation for $\mathrm{Re}\mathbf{F}^+$

We express the real-valued time-dependent quantities $v_i, f_i, \tau_{ij}, e_{ij}$ as follows:

$$\begin{array}{ll} v_i = \frac{1}{2}(V_i e^{\mathrm{i}\omega t} + V_i^* e^{-\mathrm{i}\omega t}) \ , & e_{ij} = \frac{1}{2}(E_{ij} e^{\mathrm{i}\omega t} + E_{ij}^* e^{-\mathrm{i}\omega t}) \ , \\ f_i = \frac{1}{2}(\mathcal{F}_i e^{\mathrm{i}\omega t} + \mathcal{F}_i^* e^{-\mathrm{i}\omega t}) \ , & \tau_{ij} = \frac{1}{2}(T_{ij} e^{\mathrm{i}\omega t} + T_{ij}^* e^{-\mathrm{i}\omega t}) \ . \end{array} \tag{10}$$

Here V_i, F_i, E_{ij} and T_{ij} are time-independent, complex-valued, frequency-dependent quantities. We now insert (10) into (7). For example, for $-v_i \tau_{ij}$, we obtain:

$$\begin{aligned} -v_i \tau_{ij} &= -\tfrac{1}{4}(V_i e^{\mathrm{i}\omega t} + V_i^* e^{-\mathrm{i}\omega t})(T_{ij} e^{\mathrm{i}\omega t} + T_{ij}^* e^{-\mathrm{i}\omega t}) \\ &= -\tfrac{1}{4}(V_i T_{ij}^* + V_i^* T_{ij}) - \tfrac{1}{4}[V_i T_{ij} e^{2\mathrm{i}\omega t} + V_i^* T_{ij}^* e^{-2\mathrm{i}\omega t}] \\ &= -\tfrac{1}{2}\mathrm{Re}(V_i^* T_{ij}) - \tfrac{1}{2}\mathrm{Re}(V_i T_{ij} e^{2\mathrm{i}\omega t}) \ . \end{aligned} \tag{11}$$

Since v_i and τ_{ij} are real valued, the expression for $-v_i \tau_{ij}$ is also real-valued. The first term on the right-hand side of (11) is time-independent, and the second term is time harmonic, with frequency 2ω. We can immediately see that the first term represents the time average over one period. For such time-averaging, the second term is zero.

We introduce complex-valued frequency-dependent vectors $\mathbf{F}^+$, $\bar{\mathbf{F}}$ and $\tilde{\mathbf{F}}$,

$$\mathbf{F}^+ = \bar{\mathbf{F}} + \tilde{\mathbf{F}} \ , \tag{12}$$

where $\bar{\mathbf{F}}$ and $\tilde{\mathbf{F}}$ are given by relations

$$\bar{F}_j = -\tfrac{1}{2} V_i^* T_{ij} \ , \qquad \tilde{F}_j = -\tfrac{1}{2} V_i T_{ij} e^{2\mathrm{i}\omega t} \ . \tag{13}$$

Here $\bar{\mathbf{F}}$ is the complex-valued time-averaged energy-flux vector, $\tilde{\mathbf{F}}$ is the complex-valued time-dependent energy-flux vector, and $\mathbf{F}^+$ is the complete complex-valued energy-flux vector. Then (11) can be expressed as follows:

$$-v_i \tau_{ij} = \mathrm{Re} F_j^+ \ . \tag{14}$$

Quite analogous expressions can be derived also for other energy quantities in (7):

$$\begin{array}{ll} -\rho v_i \dot{v}_i = 2\omega \mathrm{Im} K^+ \ , & -c_{ijkl}^R \dot{e}_{ij} e_{kl} = -2\omega \mathrm{Im} U^+ \ , \\ v_i f_i = \mathrm{Re} P_s^+ \ , & -\eta_{ijkl}^R \dot{e}_{ij} \dot{e}_{kl} = -\mathrm{Re} P_d^+ \ , \end{array} \tag{15}$$

where

$$\begin{array}{ll} K^+ = \bar{K} + \tilde{K} \ , & U^+ = \bar{U} + \tilde{U} \ , \\ P_s^+ = \bar{P}_s + \tilde{P}_s \ , & P_d^+ = \bar{P}_d + \tilde{P}_d \ . \end{array} \tag{16}$$

The time-averaged quantities $\bar{K}, \bar{U}, \bar{P}_s$ and $\bar{P}_d$ are given by relations:

$$\begin{array}{ll} \bar{K} = \frac{1}{4} \rho V_i V_i^* \ , & \bar{U} = \frac{1}{4} c_{ijkl}^R E_{ij} E_{kl}^* \ , \\ \bar{P}_s = \frac{1}{2} V_i^* \mathcal{F}_i \ , & \bar{P}_d = \frac{1}{2} \omega^2 \eta_{ijkl}^R E_{ij} E_{kl}^* \ . \end{array} \tag{17}$$

The time-dependent quantities $\tilde{K}, \tilde{U}, \tilde{P}_s$ and $\tilde{P}_d$ are given by relations:

$$\begin{array}{ll} \tilde{K} = \frac{1}{4} \rho V_i V_i e^{2\mathrm{i}\omega t} \ , & \tilde{U} = -\frac{1}{4} c_{ijkl}^R E_{ij} E_{kl} e^{2\mathrm{i}\omega t} \ , \\ \tilde{P}_s = \frac{1}{2} V_i \mathcal{F}_i e^{2\mathrm{i}\omega t} \ , & \tilde{P}_d = -\frac{1}{2} \omega^2 \eta_{ijkl}^R E_{ij} E_{kl} e^{2\mathrm{i}\omega t} \ . \end{array} \tag{18}$$

We can immediately see that $\bar{K}$, $\bar{U}$ and $\bar{P}_d$ are real-valued , the latter two quantities thanks to symmetries of c_{ijkl}^R and η_{ijkl}^R, see (5). Consequently,

$$\begin{array}{lll} \mathrm{Re}\bar{K} = \bar{K} \ , & \mathrm{Re}\bar{U} = \bar{U} \ , & \mathrm{Re}\bar{P}_d = \bar{P}_d \ , \\ \mathrm{Im}\bar{K} = 0 \ , & \mathrm{Im}\bar{U} = 0 \ , & \mathrm{Im}\bar{P}_d = 0 \ . \end{array} \tag{19}$$

Inserting (14) and (15) into (7), and taking into account (19), we obtain the real-valued energy-balance equation for time-harmonic waves in the form:

$$\mathrm{div}(\mathrm{Re}\mathbf{F}^+) = 2\omega \mathrm{Im}(\tilde{K} - \tilde{U}) - \mathrm{Re} P_d^+ + \mathrm{Re} P_s^+ \ . \tag{20}$$

It is interesting to note that the real-valued energy-balance equation for $\mathrm{Re}\mathbf{F}^+$ does not depend on the time-averaged kinetic and strain energies.

The energy-balance equation (20) is often expressed in its time-averaged form. It reads, see (19) and (20),

$$\mathrm{div}(\mathrm{Re}\bar{\mathbf{F}}) = -\bar{P}_d + \mathrm{Re}\bar{P}_s \ . \tag{21}$$

The real-valued vector $\mathrm{Re}\bar{\mathbf{F}}$ is sometimes called the active part of the energy flux vector $\bar{\mathbf{F}}$. The interpretation of the energy-balance equation (21) is simple. The outwardly-directed active part of the energy flux is balanced by energy supplied by internal sources decreased by the dissipated energy. This energy-balance equation is not influenced at all by kinetic and strain energies.

4.2 Energy-balance equation for $\mathrm{Im}\mathbf{F}^+$

We now express the real-valued, time-dependent quantities v_i, f_i, e_{ij} and τ_{ij} in a way alternative to (10):

$$\begin{aligned} \hat{v}_i &= \tfrac{1}{2\mathrm{i}}(V_i e^{\mathrm{i}\omega t} - V_i^* e^{-\mathrm{i}\omega t}) \ , \quad & \hat{e}_{ij} &= \tfrac{1}{2\mathrm{i}}(E_{ij} e^{\mathrm{i}\omega t} - E_{ij}^* e^{-\mathrm{i}\omega t}) \ , \\ \hat{f}_i &= \tfrac{1}{2\mathrm{i}}(\mathcal{F}_i e^{\mathrm{i}\omega t} - \mathcal{F}_i^* e^{-\mathrm{i}\omega t}) \ , \quad & \hat{\tau}_{ij} &= \tfrac{1}{2\mathrm{i}}(T_{ij} e^{\mathrm{i}\omega t} - T_{ij}^* e^{-\mathrm{i}\omega t}) \ . \end{aligned} \tag{22}$$

We use here the circumflex above letters to distinguish the representation (22) from (10).

We use the representation (22) in (3), and representation (10) in (2). We obtain

$$\hat{\tau}_{ij,j} + \hat{f}_i = \rho \dot{\hat{v}}_i \ , \qquad \dot{e}_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}) \ . \tag{23}$$

Both equations in (23) are expressed in consistent quantities, so that they can be treated together. Multiplying the first equation in (23) by -v_i and the second by -$\hat{\tau}_{ij}$, adding both, and using (4), we obtain:

$$(-v_i\hat{\tau}_{ij})_{,j} = -\rho v_i \dot{\hat{v}}_i - c^R_{ijkl}\dot{e}_{ij}\hat{e}_{kl} - \eta^R_{ijkl}\dot{e}_{ij}\dot{\hat{e}}_{kl} + v_i\hat{f}_i \ . \tag{24}$$

Equation (24) represents an alternative form of the energy-balance equation (7) specified for time-harmonic waves. The frequency under consideration is again $\omega > 0$. All the terms in (24) are real valued and time dependent. They are given by expressions:

$$\begin{aligned} & -v_i\hat{\tau}_{ij} = \mathrm{Im}F_j^+ \ , \qquad & & -\rho v_i \dot{\hat{v}}_i = -2\omega \mathrm{Re}K^+ \ , \\ & -c^R_{ijkl}\dot{e}_{ij}\hat{e}_{kl} = 2\omega\mathrm{Re}U^+ & & -\eta^R_{ijkl}\dot{e}_{ij}\dot{\hat{e}}_{kl} = -\mathrm{Im}P_d^+ \ , \\ & v_i\hat{f}_i = \mathrm{Im}P_s^+ \ . \end{aligned} \tag{25}$$

Here the complete energy flux F_j^+ is given by (12) with (13), and the complete energy quantities K^+, U^+, P_d^+ and P_s^+ by (16), with (17) and (18). Inserting (25) into (24), and taking into account that $\mathrm{Im}\bar{P}_d = 0$, see (19), we obtain the real-valued energy-balance equation for $\mathrm{Im}F_j^+$:

$$\mathrm{div}(\mathrm{Im}\mathbf{F}^+) = -2\omega\mathrm{Re}(K^+ - U^+) - \mathrm{Im}\tilde{P}_d + \mathrm{Im}P_s^+ \ . \tag{26}$$

The energy-balance equation (26) does not depend on the time-averaged dissipated energy. Analogously to (21), we obtain from (26) the energy-balance equation for $\mathrm{Im}\bar{\mathbf{F}}$ in the time-averaged form,

$$\mathrm{div}(\mathrm{Im}\bar{\mathbf{F}}) = -2\omega(\bar{K} - \bar{U}) + \mathrm{Im}\bar{P}_s \ . \tag{27}$$

The imaginary part of $\bar{\mathbf{F}}$ is sometimes called the reactive part of the energy flux vector $\bar{\mathbf{F}}$. Eq.(27) allows the following interpretation. The reactive part of the energy-flux vector is generated by the difference of time-averaged strain and kinetic energies and by imaginary part of the time-averaged energy supplied by internal sources per second. In perfectly elastic media, where $\bar{K} = \bar{U}$, in the absence of internal sources, the reactive part of $\bar{\mathbf{F}}$ is zero.

4.3 Complete complex-valued energy-balance equation

Let us combine both real-valued energy-balance equations for $\mathrm{Re}\mathbf{F}^+$ and $\mathrm{Im}\mathbf{F}^+$, (20) and (26). Multiplying (26) by the imaginary unit and adding it to (20), we obtain

$$\mathrm{div}\mathbf{F}^+ = -2\mathrm{i}\omega(K^+ - U^+) - P_d^+ + P_s^+ \ . \tag{28}$$

We call (28) the complete complex-valued energy-balance equation for time-harmonic waves. All energy-related quantities in (28) are complex valued, and are given by (12) and (16). They include both time-averaged and time-dependent energy constituents.

The complete complex-valued energy-balance equation (28) can be also expressed in its time-averaged form. The time-averaged form reads

$$\mathrm{div}\bar{\mathbf{F}} = -2\mathrm{i}\omega(\bar{K} - \bar{U}) - \bar{P}_d + \bar{P}_s \ . \tag{29}$$

Remember that $\mathrm{Im}\bar{K} = \mathrm{Im}\bar{U} = \mathrm{Im}\bar{P}_d = 0$ in (29), see(19).

4.4 Time-dependent energy-related quantities

The time-dependent energy-related quantities $\tilde{K}$, $\tilde{U}$, $\tilde{P}_s$ and $\tilde{P}_d$ are given by relations (18). They can be expressed in many alternative forms. For example, we can introduce time-independent arguments (phase shifts) χ_K, χ_U, χ_s and χ_d of complex-valued quantities V_iV_i, $V_i\mathcal{F}_i$, $c^R_{ijkl}E_{ij}E_{kl}$ and $\eta^R_{ijkl}E_{ij}E_{kl}$ as follows:

$$\begin{aligned} V_iV_i &= |V_iV_i|e^{\mathrm{i}\chi_K}, \quad c^R_{ijkl}E_{ij}E_{kl} = |c^R_{ijkl}E_{ij}E_{kl}|e^{\mathrm{i}\chi_U} \ , \\ V_i\mathcal{F}_i &= |V_i\mathcal{F}_i|e^{\mathrm{i}\chi_s}, \quad \eta^R_{ijkl}E_{ij}E_{kl} = |\eta^R_{ijkl}E_{ij}E_{kl}|e^{\mathrm{i}\chi_d} \ . \end{aligned} \tag{30}$$

The time-dependent energy-related quantities $\tilde{K}$, $\tilde{U}$, $\tilde{P}_s$ and $\tilde{P}_d$ then read

$$\begin{aligned} \tilde{K} &= \tfrac{1}{4}\rho|V_iV_i|e^{2\mathrm{i}\omega t+\mathrm{i}\chi_K}, \quad \tilde{U} = -\tfrac{1}{4}|c^R_{ijkl}E_{ij}E_{kl}|e^{2\mathrm{i}\omega t+\mathrm{i}\chi_U} \ , \\ \tilde{P}_s &= \tfrac{1}{2}|V_i\mathcal{F}_i|e^{2\mathrm{i}\omega t+\mathrm{i}\chi_s}, \quad \tilde{P}_d = -\tfrac{1}{2}\omega^2|\eta^R_{ijkl}E_{ij}E_{kl}|e^{2\mathrm{i}\omega t+\mathrm{i}\chi_d} \ . \end{aligned} \tag{31}$$

As only the exponential functions in (31) are complex valued, we obtain real and imaginary parts of the energy-related quantities from (31) by only replacing the exponential functions by cosines and sines, respectively. For example,

$$\mathrm{Re}\tilde{K} = \tfrac{1}{4}\rho|V_iV_i|\cos(2\omega t + \chi_K), \quad \mathrm{Im}\tilde{K} = \tfrac{1}{4}\rho|V_iV_i|\sin(2\omega t + \chi_K). \tag{32}$$

Let us now discuss the time-dependent constituents of the *complex-valued energy-flux vector* $\tilde{\mathbf{F}}$. The expression for its j-th Cartesian component reads:

$$\tilde{F}_j = -\tfrac{1}{2}|V_i T_{ij}| e^{\mathrm{i}(2\omega t + \chi_j)}, \tag{33}$$

where χ_j is given by the relation

$$V_i T_{ij} = |V_i T_{ij}| e^{\mathrm{i}\chi_j}. \tag{34}$$

The vector $\tilde{\mathbf{F}} = \mathrm{Re}\tilde{\mathbf{F}} + \mathrm{iIm}\tilde{\mathbf{F}}$ can be expressed as follows

$$\mathrm{Re}\tilde{\mathbf{F}} = \mathbf{F}_1 \cos\, 2\omega t - \mathbf{F}_2 \sin 2\omega t \ , \quad \mathrm{Im}\tilde{\mathbf{F}} = \mathbf{F}_1 \sin\, 2\omega t + \mathbf{F}_2 \cos 2\omega t \ , \tag{35}$$

where $\mathbf{F}_1$ and $\mathbf{F}_2$ are real-valued and time-independent vectors with components

$$F_{1j} = -\tfrac{1}{2}\mathrm{Re}(V_i T_{ij}) \ , \quad F_{2j} = -\tfrac{1}{2}\mathrm{Im}(V_i T_{ij}) \ . \tag{36}$$

4.5 Complete complex-valued energy-related quantities

Using (16), (17) and (31), we easily obtain expressions for complete complex-valued energy-related quantities K^+, U^+, P_d^+ and P_s^+.

For complete complex-valued energy-flux vector $\mathbf{F}^+ = \mathrm{Re}\mathbf{F}^+ + \mathrm{iIm}\mathbf{F}^+$, we obtain

$$\mathrm{Re}\mathbf{F}^+ = \mathrm{Re}\bar{\mathbf{F}} + \mathbf{F}_1 \cos\, 2\omega t - \mathbf{F}_2 \sin 2\omega t \ , \quad \mathrm{Im}\mathbf{F}^+ = \mathrm{Im}\bar{\mathbf{F}} + \mathbf{F}_1 \sin\, 2\omega t + \mathbf{F}_2 \cos 2\omega t \ , \tag{37}$$

Thus, $\mathrm{Re}\mathbf{F}^+$ of time-harmonic waves varies periodically with time and frequency 2ω around its time-averaged part $\mathrm{Re}\bar{\mathbf{F}}$. Its behaviour is completely specified by three time-independent vectors, namely $\mathrm{Re}\bar{\mathbf{F}}$, $\mathbf{F}_1$ and $\mathbf{F}_2$. Similarly, oscillations of $\mathrm{Im}\mathbf{F}^+$ are fully specified by three independent vectors $\mathrm{Im}\bar{\mathbf{F}}$, $\mathbf{F}_1$ and $\mathbf{F}_2$. If $\mathbf{F}_1$ and $\mathbf{F}_2$ are parallel, the oscillations are linear. If $\mathbf{F}_1$ and $\mathbf{F}_2$ make a non-zero angle, the oscillations are elliptical.

In various applications, it is useful to know the complete complex-valued energy flux through a plane specified by a normal $\mathbf{n}$. We obtain

$$\begin{aligned} \mathrm{Re}\mathbf{F}^+ \cdot \mathbf{n} &= -\tfrac{1}{2}\mathrm{Re}(V_i^* T_{ij} n_j) - \tfrac{1}{2}|V_i T_{ij} n_j| \cos(2\omega t + \chi) \ , \\ \mathrm{Im}\mathbf{F}^+ \cdot \mathbf{n} &= -\tfrac{1}{2}\mathrm{Im}(V_i^* T_{ij} n_j) - \tfrac{1}{2}|V_i T_{ij} n_j| \sin(2\omega t + \chi) \ , \end{aligned} \tag{38}$$

where

$$\cos\chi = \frac{\mathrm{Re}(V_i T_{ij} n_j)}{|V_i T_{ij} n_j|} \ , \quad \sin\chi = \frac{\mathrm{Im}(V_i T_{ij} n_j)}{|V_i T_{ij} n_j|} \ . \tag{39}$$

Relations (38) and (39) may play an important role in the solution of reflection/transmission problem on a plane interface. They can be used to decide whether the energy flux of the wave under consideration is pointing away from the interface, or opposite. Actually, according to (38) it may be pointing away from the interface for some times, and to the interface for other times within a period.

4.6 Peak values of energy-related quantities

Under peak values we understand the maximum values of energy-related quantities as a function of time t. These values, of course, differ from the time-averaged values , but as them, they are time independent. They are composed of time-averaged value of the corresponding quantity and the peak value of its time-dependent constituent. In the following, we present only expressions for the latter values.

From (31), we obtain

$$\begin{aligned} \tilde{K}^{\text{peak}} &= \tfrac{1}{4}\rho|V_iV_i| \ , \quad \tilde{U}^{\text{peak}} = \tfrac{1}{4}\omega|c^R_{ijkl}E_{ij}E_{kl}| \ , \\ \tilde{P}_s^{\text{ peak}} &= \tfrac{1}{2}|V_i\mathcal{F}_i| \ , \quad \tilde{P}_d^{\text{ peak}} = \tfrac{1}{2}\omega^2|\eta^R_{ijkl}E_{ij}E_{kl}| \ . \end{aligned} \tag{40}$$

We can see that all the above peak values are real valued. The peak values are the same for the complex-valued energy-related quantities and their real and imaginary parts.

For the j-th component $\tilde{F}_j$ of the time-dependent constituent of the complex-valued energy flux, we obtain from (33),

$$\tilde{F}_j^{\text{peak}} = \tfrac{1}{2}|V_iT_{ij}| \ . \tag{41}$$

We can also specify the peak value of the time-dependent constituent of the energy flux through a plane specified by a normal $\mathbf{n}$,

$$(\tilde{\mathbf{F}}\cdot\mathbf{n})^{\text{peak}} = \tfrac{1}{2}|V_iT_{ij}n_j| \ . \tag{42}$$

5 Direct derivation of energy-balance equation for complex-valued time-harmonic wave fields

In previous sections, we derived energy-balance equations for time-harmonic waves considering two real-valued representations of a wave field. In the derivation, we considered the special case of the stress-strain relation, corresponding to Kelvin-Voigt viscoelasticity model (4).

In this section, we consider the complex-valued representation of the time-harmonic wave field, and the frequency-dependent stress-strain relation for an arbitrary viscoelastic model. In spite of this generalization, the derivation is simpler and more transparent, and directly combines both approaches, used independently in Section 4. We compare the results that we obtain here for general viscoelasticity and the complex-valued representation of the wave field, with those for Kelvin-Voigt viscoelasticity model and real-valued representation of the wave field, obtained in Sec.4. We shall see that both approaches give exactly the same final equations for the Kelvin-Voigt viscoelasticity model. This indicates that the general approach, presented in this section, based on complex-valued representation of the wave field, may be used as a generalization of the approach based on the real-valued representation of the wave field.

In the following, we consider vectors v_i, f_i and tensors τ_{ij}, e_{ij} to be complex-valued time-harmonic quantities:

$$\begin{aligned} v_i &= V_i e^{\mathrm{i}\omega t} \ , \quad & \tau_{ij} &= T_{ij} e^{\mathrm{i}\omega t} \ , \\ e_{ij} &= E_{ij} e^{\mathrm{i}\omega t} \ , \quad & f_i &= \mathcal{F}_i e^{\mathrm{i}\omega t} \ . \end{aligned} \tag{43}$$

We use the correspondence principle (Carcione, 2001). According to it the equation of motion, the time derivative of the strain tensor and the stress-strain relation in viscoelastic anisotropic media i.e.,

$$\tau_{ij,j} + f_i = \rho \dot{v}_i \ , \qquad \dot{e}_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}) \ , \tag{44}$$

and

$$\tau_{ij} = c_{ijkl} e_{kl} \tag{45}$$

have the same form as in perfectly elastic anisotropic media; only the moduli c_{ijkl} in (45) and the quantities V_i, $\mathcal{F}_i$, T_{ij} and E_{ij} in (43) are complex valued and frequency dependent.

5.1 Complex-valued frequency-dependent stress-strain relation

For time-harmonic waves propagating in a linear viscoelastic anisotropic medium, the complex-valued frequency-dependent stress-strain relation is given by (45), where

$$c_{ijkl} = \mathrm{Re}[c_{ijkl}(\omega)] + \mathrm{i}\mathrm{Im}[c_{ijkl}(\omega)] \ . \tag{46}$$

We will tacitly assume that c_{ijkl} are frequency-dependent, and omit the argument ω in the following. We assume that c_{ijkl} satisfy the symmetry relations

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij} \ , \tag{47}$$

which reduce the number of independent moduli from 81 to 21. Thus, we can construct the 6×6 symmetric Voigt viscoelasticity matrices $\mathrm{Re}(C_{\alpha\beta})$ and $\mathrm{Im}(C_{\alpha\beta})$. A proof that $\mathrm{Im}(C_{\alpha\beta}) = \mathrm{Im}(C_{\beta\alpha})$ has not yet been given in literature, but this problem is not discussed here. We further assume that the 6×6 Voigt matrices $\mathrm{Re}(C_{\alpha\beta})$ and $-\mathrm{Im}(C_{\alpha\beta})$ are positive definite and non-negative definite, respectively, for $\omega > 0$ under consideration.

As an example, consider the stress-strain relation (4) for Kelvin-Voigt viscoelasticity model. Taking into account (43), Eq.(4) yields

$$\tau_{ij} = (c^R_{ijkl} + \mathrm{i}\omega\eta^R_{ijkl}) e_{kl} \ . \tag{48}$$

Consequently, the complex-valued, frequency-dependent viscoelastic anisotropic moduli c_{ijkl} for Kelvin-Voigt model (4) are given by the relation

$$c_{ijkl} = c^R_{ijkl} + \mathrm{i}\omega\eta^R_{ijkl} \ . \tag{49}$$

The real part of c_{ijkl} is frequency independent, and the imaginary part of c_{ijkl} depends linearly on frequency.

5.2 Energy-balance equation for time-averaged quantities

Multiplying the first equation of (44) by $-v_i^*$ and the complex-conjugate of second equation of (44) by $-\tau_{ij}$, and adding both, we obtain

$$(-v_i^*\tau_{ij})_{,j} = -\rho v_i^* \dot{v}_i - \tau_{ij}\dot{e}_{ij}^* + v_i^* f_i \ . \tag{50}$$

We insert the stress-strain relation (45) into (50), perform the time derivatives, and multiply it by $\frac{1}{2}$, to get :

$$(-\tfrac{1}{2}v_i^*\tau_{ij})_{,j} = -\tfrac{1}{2}\mathrm{i}\omega\rho v_i v_i^* + \tfrac{1}{2}\mathrm{i}\omega c_{ijkl} e_{ij}^* e_{kl} + \tfrac{1}{2}v_i^* f_i \ . \tag{51}$$

Expressing c_{ijkl} using (46), we obtain finally,

$$\mathrm{div}\bar{\mathbf{F}} = -2\mathrm{i}\omega(\bar{K} - \bar{U}) - \bar{P}_d + \bar{P}_s \ , \tag{52}$$

where

$$\begin{aligned}
&\bar{F}_j = -\tfrac{1}{2}v_i^*\tau_{ij} = -\tfrac{1}{2}V_i^* T_{ij}, && \bar{K} = \tfrac{1}{4}\rho v_i v_i^* = \tfrac{1}{4}\rho V_i^* V_i \ ,\\
&\bar{U} = \tfrac{1}{4}\mathrm{Re}(c_{ijkl}) e_{ij}^* e_{kl} = \tfrac{1}{4}\mathrm{Re}(c_{ijkl}) E_{ij}^* E_{kl} \ , && \bar{P}_d = \tfrac{1}{2}\omega\mathrm{Im}(c_{ijkl}) e_{ij}^* e_{kl} = \tfrac{1}{2}\omega\mathrm{Im}(c_{ijkl}) E_{ij}^* E_{kl},\\
&\bar{P}_s = \tfrac{1}{2}v_i^* f_i = \tfrac{1}{2}V_i^* \mathcal{F}_i \ .
\end{aligned} \tag{53}$$

Here the quantities $\bar{\boldsymbol{F}}$ and $\bar{P}_s$ are complex valued, $\bar{K}$, $\bar{U}$ and $\bar{P}_d$ real valued. Eq. (52) is exactly the same as energy-balance equation (29) for time-averaged energy quantities. For (49), which corresponds to the Kelvin-Voigt viscoelastic model, the quantities in (53) correspond exactly to the time-averaged energy-related quantities $\bar{F}_j$, $\bar{K}$, $\bar{U}$, $\bar{P}_d$, and $\bar{P}_s$ defined in Section 4, see (13) and (17).

5.3 Energy-balance equation for time-dependent quantities

Multiplying the first equation of (44) by $-v_i$ and the second equation of (44) by $-\tau_{ij}$, and adding both, we obtain

$$(-v_i\tau_{ij})_{,j} = -\rho v_i \dot{v}_i - \tau_{ij}\dot{e}_{ij} + v_i f_i \ . \tag{54}$$

This is formally the same as in (6), but individual quantities are not real valued, but complex valued and frequency dependent. Taking the time-derivatives in (54), inserting the stress-strain relation (45), and multiplying the result by $\frac{1}{2}$, we get:

$$\mathrm{div}\tilde{F} = -2\mathrm{i}\omega(\tilde{K} - \tilde{U}) - \tilde{P}_d + \tilde{P}_s \ . \tag{55}$$

Here

$$\begin{aligned}
\tilde{F}_j &= -\tfrac{1}{2}v_i\tau_{ij} = -\tfrac{1}{2}V_i T_{ij} e^{2\mathrm{i}\omega t} \ ,\\
\tilde{K} &= \tfrac{1}{4}\rho v_i v_i = \tfrac{1}{4}\rho V_i V_i e^{2\mathrm{i}\omega t} \ ,\\
\tilde{U} &= -\tfrac{1}{4}\mathrm{Re}(c_{ijkl}) e_{ij} e_{kl} = -\tfrac{1}{4}\mathrm{Re}(c_{ijkl}) E_{ij} E_{kl} e^{2\mathrm{i}\omega t} \ ,\\
\tilde{P}_d &= -\tfrac{1}{2}\omega\mathrm{Im}(c_{ijkl}) e_{ij} e_{kl} = -\tfrac{1}{2}\omega\mathrm{Im}(c_{ijkl}) E_{ij} E_{kl} e^{2\mathrm{i}\omega t} \ ,\\
\tilde{P}_s &= \tfrac{1}{2}v_i f_i = \tfrac{1}{2}V_i \mathcal{F}_i e^{2\mathrm{i}\omega t} \ .
\end{aligned} \tag{56}$$

Inserting (49) into (56), we can see that the resulting expressions are the same as the expressions (13) and (18) for time-dependent energy-related quantities.

5.4 Complete energy-balance equation

In Sections 5.2 and 5.3, we derived two energy-balance equations: one for time-averaged energy-related quantities, (52), and the other for time-dependent energy-related quantities (55). It is natural to combine both these equations in such a way to get the complete energy-balance equation, containing both the time-averaged and time-dependent energy-related constituents. Summing up (52) and (55), we obtain the complete energy-balance equation

$$\mathrm{div}\mathbf{F}^+ = -2\mathrm{i}\omega(K^+ - U^+) - P_d^+ + P_s^+ \ . \tag{57}$$

Here the energy-related quantities with "+" in the superscript denote again the sum of time-averaged and time-dependent energy-related quantities. Using (53) and (56), we obtain:

$$\begin{array}{rcl} F_j^+ &=& -\frac{1}{2}v_i^*\tau_{ij} - \frac{1}{2}v_i\tau_{ij} = -\mathrm{Re}(v_i)\tau_{ij} \ , \\ K^+ &=& \frac{1}{4}\rho v_i v_i^* + \frac{1}{4}\rho v_i v_i = \frac{1}{2}\rho\mathrm{Re}(v_i)v_i \ , \\ U^+ &=& \frac{1}{4}\mathrm{Re}(c_{ijkl})e_{ij}^*e_{kl} - \frac{1}{4}\mathrm{Re}(c_{ijkl})e_{ij}e_{kl} = -\frac{1}{2}\mathrm{i}\mathrm{Re}(c_{ijkl})\mathrm{Im}(e_{ij})e_{kl} \ , \\ P_d^+ &=& \frac{1}{2}\omega\mathrm{Im}(c_{ijkl})e_{ij}^*e_{kl} - \frac{1}{2}\omega\mathrm{Im}(c_{ijkl})e_{ij}e_{kl} = -\mathrm{i}\omega\mathrm{Im}(c_{ijkl})\mathrm{Im}(e_{ij})e_{kl} \ , \\ P_s^+ &=& \frac{1}{2}v_i^*f_i + \frac{1}{2}v_if_i = \mathrm{Re}(v_i)f_i \ . \end{array} \tag{58}$$

We call $\mathbf{F}^+$ the complete energy-flux vector, and K^+, U^+, P_d^+ and P_s^+ the complete energy-related quantities. It is interesting to note a simplicity of the expressions for the complete energy-related quantities, particularly of $\mathbf{F}^+, K^+$ and P_s^+.

It is not difficult to check that, for (49), Equations (57) with (58) give exactly the same results as obtained in Section 4, see (28), and (12), (13), (16) - (18).

Let us briefly discuss the complete energy flux $\mathbf{F}^+$ from the physical point of view. It is given by the relation, see (58),

$$F_j^+ = -\mathrm{Re}(v_i)\tau_{ij} \ . \tag{59}$$

This immediately yields

$$\mathrm{Re}F_j^+ = -\mathrm{Re}(v_i)\mathrm{Re}(\tau_{ij}) \ , \quad \mathrm{Im}F_j^+ = -\mathrm{Re}(v_i)\mathrm{Im}(\tau_{ij}) \ . \tag{60}$$

Both expressions in (60) are real valued, and contain the time-averaged and time-dependent constituents, see (58). The explanation of (60) is simple: In order to obtain physically meaningful quantities in energy considerations, the complex-valued quantities must be replaced by their real-valued parts according to (60). When seeking $\mathrm{Re}F_j^+$, one should replace the complex-valued term $-v_i\tau_{ij}$ by $-\mathrm{Re}(v_i)\mathrm{Re}(\tau_{ij})$, when seeking $\mathrm{Im}F_j^+$, one should replace $-v_i\tau_{ij}$ by $-\mathrm{Re}(v_i)\mathrm{Im}(\tau_{ij})$.

In the absence of internal sources ($f_i = 0$), the first equation of (44) yields $\tau_{ik,k} = \rho\dot{v}_i$, so that $v_i = -\mathrm{i}\tau_{ik,k}/\rho\omega$. Then the complete energy flux can be expressed as a function of stress-field only,

$$F_j^+ = \mathrm{Im}(\tau_{ik,k})\tau_{ij}/\rho\omega \ . \tag{61}$$

For plane waves propagating in homogeneous viscoelastic anisotropic media, $\tau_{ik,k} = \mathrm{i}\omega\tau_{ik}p_k$. Then (61) reduces to

$$F_j^+ = \mathrm{Re}(\tau_{ik}p_k)\tau_{ij}/\rho \ . \tag{62}$$

5.5 Alternative complex-valued representation

Alternatively to the complex-valued representation (43), a representation with the exponential time factor $\exp[-\mathrm{i}\omega t]$ has been also often used:

$$\begin{aligned} v_i = V_i e^{-\mathrm{i}\omega t} \,, \quad & \tau_{ij} = T_{ij} e^{-\mathrm{i}\omega t} \,, \\ e_{ij} = E_{ij} e^{-\mathrm{i}\omega t} \,, \quad & f_i = \mathcal{F}_i e^{-\mathrm{i}\omega t} \,. \end{aligned} \tag{63}$$

This representation is particularly common in seismic ray theory, see, e.g., Červený (2001a).

Without a detailed treatment, we show the changes introduced by a new complex-valued representation (63) with respect to equations described in Section 5.1-5.4.

First, the stress-strain relation (45) remains valid, but $\mathrm{Im}(c_{ijkl})$ has an opposite sign. Consequently, we have to assume that the 6×6 Voigt matrix $-\mathrm{Im}(C_{\alpha\beta})$ is positive definite or zero.

Second, the complete energy-balance equation (57) transforms into:

$$\mathrm{div}\boldsymbol{F}^{+} = 2\mathrm{i}\omega(K^{+} - U^{+}) - P_d^{+} + P_s^{+} \,, \tag{64}$$

where $\boldsymbol{F}^{+}$, K^{+}, U^{+} and P_s^{+} are given by (58). The only formal difference is in P_d^{+}, which has an opposite sign than in (58). The negative sign at P_d^{+} in (64), however, remains valid, as $\mathrm{Im}(c_{ijkl})$ has an opposite sign. Note that Eq.(64) has been used by Červený and Pšenčík (2006).

Third, the complex-valued energy-related quantities in (64) must be taken complex-conjugate to those in (58).

6 Concluding remarks

Two approaches are used to investigate the energy-flux vector and other energy-related quantities of time-harmonic wave fields propagating in inhomogeneous anisotropic viscoelastic media. The first approach is limited to Kelvin-Voigt viscoelasticity model, and is based on the equation of motion and stress-strain relations for the real-valued time-dependent wave fields. The second approach is based on equation of motion and stress-strain relation for the complex-valued time-harmonic wave fields. The second approach is applicable for unrestricted viscoelasticity and anisotropy. Consequently. the second approach is considerably more general than the first.

In both approaches, the energy flux vector and other energy-related quantities are expressed as the sum of two contributions, namely the time-averaged and time-dependent constituents. Comparison of both approaches shows, that they yield the same results for Kelvin-Voigt viscoelasticity model. This indicates that the second approach is a better alternative for applications as it is considerably more general.

For high-frequency wave fields, only the time-averaged energy contributions are often considered. In certain applications, however, it is useful to consider also time-dependent contributions. The time-dependent energy-flux vector is known under different names (instantaneous Deschamp's pseudovelocity vector, or pseudoenergy flux), and was investigated as an independent special entity (Červený, 2001a). It is, however, shown in this paper that the time-dependent energy flux is merely one of the two constituents of the complete energy flux, and does not exist independently. Only the superposition of the time-averaged energy flux and time-dependent energy flux gives the complete energy flux.

The energy-balance equations for individual waves are derived and discussed, both for real-valued and complex-valued energy flux. They show the relation of the energy flux to other energy-related quantities, as the kinetic energy, strain energy, dissipated energy and energy supplied by internal sources. These energy-balance equations, however, are valid only for individual non-interfering waves. It would be useful to investigate the energy-related quantities also for trains of waves. The trains of waves can be generated by several sources (Mann et al., 1987), in a vicinity of a structural interface by waves generated by incidence of a wave at the interface, etc. Mann et al. (1987) investigated the energy flux of trains of waves in non-dissipative homogeneous fluids. They found that, even in this simple case, the field of energy velocity may display vortices in certain regions. For anisotropic viscoelastic media, this problem is still open for further research.

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References

Ainslie, M.A., and Burns, P.W. (1995). Energy conserving reflection and transmission coefficients for a solid-solid boundary. *J. Acoust. Soc. Am.*, **98**, 2836–2840.

Auld, B.A. (1973). *Acoustic fields and waves in solids*, Wiley, New York.

Boulanger, P., and Hayes, M. (2000). Special inhomogeneous waves in cubic elastic materials. *ZAMP*, **51**, 1031–1038.

Carcione, J.M. (2001). *Wave fields in real media: Wave propagation in anisotropics anelastic and porous media.* Pergamon, Amsterdam.

Červený, V. (2001a). *Seismic ray theory.* Cambridge Univ. Press, Cambridge.

Červený, V. (2001b). Energy flux of time-harmonic waves in anisotropic inhomogeneous media. In: *Seismic waves in complex 3-D structures, Report 11*, pp. 301–314. Charles University, Faculty of Mathematics and Physics, Dept. of Geophysics, Praha. (Available online at "http: sw3d.mff.cuni.cz".)

Červený, V., and Pšenčík, I. (2006). Energy flux in viscoelastic anisotropic media. *Geophys. J. Int.*, in press.

Deschamps, M. (1990). Réflexion-réfraction de l'onde plane hétérogène: Repartition de l'énergie. *J. Acoust.*, **3**, 251–261.

Hayes, M. (1980). Energy flux for trains of inhomogeneous plane waves. *Proc. R. Soc. London A*, **370**, 417–429.

Mann, J.A., Tichy, I., and Romano, A.J. (1987). Instantaneous and time-averaged energy transfer in acoustic fields. *J. Acoust. Soc. Am.*, **82**, 17–29.

Scott, N.H. (1997). Energy flux and dissipation of inhomogeneous plane wave forms in Kelvin-Voigt viscoelasticity. *Int. J. Engng. Sci.*, **35**, 561–572.

Shuvalov, A.L., and Scott, N.H. (1999). On the properties of homogeneous viscoelastic waves. *Q.J. Mech. Appl. Math.*, **52**, 405–417.