Comment to "qS-waves in a vicinity of the axis of symmetry of homogeneous transversely isotropic media", by M. Popov, G.F. Passos, and M.A. Botelho [Wave Motion 42 (2005) 191-201]

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Abstract

A new asymptotic formula for S waves propagating near the symmetry axis in transversely isotropic elastic media derived by Popov et al. [Wave Motion 42 (2005) 191-201] is discussed and commented. It is shown that the formula is a modification of the previously published formula by Vavryčuk [Geophys. J. Int. 138 (1999) 581-589]. The formula of Popov et al. (2005) is less accurate and valid under more restrictive conditions than that of Vavryčuk (1999).

Introduction

A high-frequency modelling of wavefields in transversely isotropic (TI) media is more complicated in directions near the symmetry axis than in other directions, because we observe a so-called S-wave kiss singularity along the axis. The S-wave singularity causes anomalies in the field of polarization vectors and possibly also in the shape of the slowness and wave surfaces. For these reasons, the standard zeroth-order ray theory is inapplicable and more sophisticated methods must be used to model the wavefield correctly. For example, Vavryčuk [1] proposed to apply higher-order ray theory, and Gridin [2] developed a high-frequency asymptotics using a uniform stationary phase method. Other approaches valid for kiss singularities in any type of anisotropy are presented in [3] and [4].

Recently, this topic was also studied by Popov et al. [5], where the authors claim that the previous results presented in [1] and [2] are either questionable or incorrect, and they propose another high-frequency asymptotics valid for directions near the singularity. In my comment, I will show that the assertion of Popov et al. [5] regarding the previously published results is not justified and that the results in [5] are not in contradiction with the previously published results. Moreover, I will show that the formulas in [5] are valid under stronger restrictions than those derived in [1] and [2].

Exact solution

The exact elastodynamic Green tensor $G_{ij}(\mathbf{x},t)$ for unbounded, homogeneous, anisotropic, elastic media can be expressed as follows (Burridge [6], Eq. 4.6; Wang and Achenbach [7], Eq. 13):

$$G_{kl}(\mathbf{x},t) = -\frac{H(t)}{8\pi^2\rho}\sum_{m=1}^{3}\int_{S(\mathbf{n})}\frac{g_k^m g_l^m}{\left(c^m\right)^3}\,\dot{\delta}\left(t-\frac{\mathbf{n}\cdot\mathbf{x}}{c^m}\right)dS(\mathbf{n})\ . \tag{1}$$

Superscript m = 1,2,3 denotes the type of wave (P, $S1$ and $S2$), $\mathbf{g} = \mathbf{g}(\mathbf{n})$ denotes the unit polarization vector, $c = c(\mathbf{n})$ is the phase velocity, ρ is the density of the medium, t is time, $H(t)$ is the Heaviside step function, $\dot{\delta}(t)$ is the time derivative of the Dirac delta function, $\mathbf{x}$ is the position vector of the observation point, and $\mathbf{n}$ is the direction of the slowness vector. The integration is over unit sphere $S(\mathbf{n})$. Formula (1) represents the exact solution for homogeneous, weakly as well as strongly anisotropic media, containing far-field as well as near-field waves, and is valid at all distances and directions including the shear-wave singularities. For an isotropic medium, integral (1) can be evaluated analytically to yield the well-known Stokes solution (see Mura [8], pp. 61-63). For a *transversely isotropic medium*, integral (1) has also been evaluated analytically, but only for the symmetry axis direction [9, 10]. For other directions, the integral should be evaluated either numerically, or it should be expanded asymptotically.

Asymptotic solution of Vavryčuk [1]

Since the polarization field is singular along the symmetry axis in TI, the standard asymptotics of (1), which is equivalent to the zeroth-order ray theory, fails along the axis and in its vicinity. Therefore, Vavryčuk [1,11] proposed to apply a more general approach and derived an asymptotic formula using higher-order ray theory. The formula consists of the zeroth- and the first-order terms of the ray series and it can be expressed as follows (Vavryčuk [1], Eqs 2 and 5):

$$G_{kl}^{S\,far}(\mathbf{x},t)=\frac{1}{4\pi\rho}\frac{1}{v^{SV}\sqrt{K^{SV}}}\frac{g_k^{SV}g_l^{SV}}{r}\delta\left(t-\tau^{SV}\right)+\frac{1}{4\pi\rho}\frac{1}{v^{SH}\sqrt{K^{SH}}}\frac{g_k^{SH}g_l^{SH}}{r}\delta\left(t-\tau^{SH}\right)$$
$$+\frac{1}{4\pi\rho}\frac{1}{\sqrt{a_{44}}}\frac{g_k^{SH}g_l^{SH}-g_k^{SH\perp}g_l^{SH\perp}}{r^2\sin^2\vartheta}\left[H\left(t-\tau^{SH}\right)-H\left(t-\tau^{SV}\right)\right] \tag{2}$$

where K is the Gaussian curvature of the slowness surface, v is the group velocity, τ is the traveltime, t is time, r is the distance from the source to the receiver, ρ is the density of the medium, and a_{ij} are the density normalized elastic parameters. The superscript denotes the type of wave. Vector $\mathbf{g}$ is the polarization vector calculated as the eigenvector of the Christoffel tensor,

$$\mathbf{g}^{SV}=\begin{bmatrix}\cos\theta\cos\varphi\\ \cos\theta\sin\varphi\\ -\sin\theta\end{bmatrix},\ \mathbf{g}^{SH}=\begin{bmatrix}\sin\varphi\\ -\cos\varphi\\ 0\end{bmatrix},\ \mathbf{g}^{SH\perp}=\begin{bmatrix}\cos\varphi\\ \sin\varphi\\ 0\end{bmatrix}, \tag{3}$$

where θ denotes the inclination of $\mathbf{g}^{SV}$ from the horizontal plane and φ denotes the angle defined in the x_1-x_2 plane. The ray vector defines the direction of the receiver from the source and it is expressed as

$$\mathbf{N}=\begin{bmatrix}\sin\vartheta\cos\varphi\\ \sin\vartheta\sin\varphi\\ \cos\vartheta\end{bmatrix}, \tag{4}$$

where ϑ denotes the inclination of the ray from the vertical. The Gaussian curvature, group velocity and the traveltime for the SH wave can be expressed explicitly in a closed form [12]

$$K^{SH} = a_{66}^2 a_{44} \left[\frac{\sin^2 \vartheta}{a_{66}} + \frac{\cos^2 \vartheta}{a_{44}} \right]^2, \; v^{SH} = \left[\frac{\sin^2 \vartheta}{a_{66}} + \frac{\cos^2 \vartheta}{a_{44}} \right]^{-\frac{1}{2}}, \; \tau^{SH} = r \sqrt{\frac{\sin^2 \vartheta}{a_{66}} + \frac{\cos^2 \vartheta}{a_{44}}} \,. \tag{5}$$

Determining the Gaussian curvature, group velocity and the traveltime for the SV wave is more involved and should be calculated numerically.

Formula (2) consists of three terms: first two terms are the zeroth-order terms of the ray series and describe the standard far-field asymptotics of the SV and SH waves; the third term is the first-order term of the ray series called the "coupling term". The coupling term describes the interaction between the SV an SH waves and it is significant near the symmetry axis [13]. Formula (2) is valid for all directions of rays, assuming that the wavefront has no triplications and the receivers are far from the source. The essentially same formula has been obtained by Gridin [2], who derived the asymptotic Green function not only for directions near the symmetry axis but also for other difficult situations such as the directions near cuspidal edges at the wavefront due to triplications. The validity of (2) for directions near the symmetry axis was furthermore confirmed directly by a numerical comparison of (1) and (2) assuming a smooth source-time function of a point source situated in two differently anisotropic homogeneous media (a cracked medium and sandstone, see [1])

Asymptotic solution of Popov et al. [5]

First, let us complete the formulas of Popov et al. [5] and transform them into the time domain to express the Green function in a form comparable with (2). Expressing equations (12) and (13) of [5] valid for the force along the x_1-axis also for forces along the other axes,

summing the contributions of the SV and SH waves, and transforming the equations into the time domain, we obtain the Green function in the following form:

$$G_{kl}^{S\,far}(\mathbf{x},t)=\frac{1}{4\pi\rho}\frac{1}{C^{SV}}\frac{\hat{g}_k^{SV}\hat{g}_l^{SV}}{r}\delta\left(t-\hat{\tau}^{SV}\right)+\frac{1}{4\pi\rho}\frac{1}{C^{SH}}\frac{g_k^{SH}g_l^{SH}}{r}\delta\left(t-\hat{\tau}^{SH}\right)$$
$$+\frac{1}{4\pi\rho}\frac{1}{\sqrt{a_{44}}}\frac{g_k^{SH}g_l^{SH}-g_k^{SH\perp}g_l^{SH\perp}}{r^2\sin^2\vartheta}\left[H\left(t-\hat{\tau}^{SH}\right)-H\left(t-\hat{\tau}^{SV}\right)\right], \tag{6}$$

where

$$\hat{\mathbf{g}}^{SV}=\begin{bmatrix}\cos\varphi\\ \sin\varphi\\ 0\end{bmatrix}, \tag{7}$$

$$\hat{\tau}^{SV}=\frac{r}{\sqrt{a_{44}}}\left[1-\frac{1}{2}\sin^2\vartheta\left(1+\frac{a_{44}(a_{33}-a_{44})}{(a_{13}+a_{44})^2-a_{11}(a_{33}-a_{44})}\right)\right], \tag{8}$$

$$\hat{\tau}^{SH}=\frac{r}{\sqrt{a_{44}}}\left[1-\frac{1}{2}\sin^2\vartheta\left(1-\frac{a_{44}}{a_{66}}\right)\right], \tag{9}$$

$$C^{SV}=-\frac{(a_{13}+a_{44})^2-a_{11}(a_{33}-a_{44})}{a_{33}-a_{44}}, \tag{10}$$

$$C^{SH}=a_{66}, \tag{11}$$

and the other quantities have the same meaning as in (2). Comparing formulas (2) and (6) we find that they are not very different. The only difference is that some quantities such as the polarization vector of the SV wave, and the traveltimes and the amplitude factors of the SV and SH waves have slightly different forms. However, it can be shown that these differences are minor and disappear near the symmetry axis. Hence, for $\vartheta\to 0$,

$$\hat{\tau}^{SV} \cong \tau^{SV},\ \hat{\tau}^{SH} \cong \tau^{SH},\ \hat{\mathbf{g}}^{SV} \cong \mathbf{g}^{SV},\ C^{SV} \cong v^{SV}\sqrt{K^{SV}} \text{ and } C^{SH} \cong v^{SH}\sqrt{K^{SH}}, \tag{12}$$

and subsequently equations (2) and (6) coincide. Moreover, a closer inspection figures out that quantities $\hat{\tau}^{SV}$, **Chyba! Objekty nemohou být vytvořeny úpravami kódů polí.**, **Chyba! Objekty nemohou být vytvořeny úpravami kódů polí.**, **Chyba! Objekty nemohou být vytvořeny úpravami kódů polí.** and C^{SH} derived in [5] are just Taylor expansions of τ^{SV}, τ^{SH}, $\mathbf{g}^{SV}$, $v^{SV}\sqrt{K^{SV}}$ and $v^{SH}\sqrt{K^{SH}}$ for $\vartheta \to 0$ derived in [1]. This implies that eq. (2) is valid under less restrictive conditions than eq. (6): while eq. (2) is valid for all rays, eq (6) is valid just for rays near the symmetry axis.

Numerical example

Here, I demonstrate a different range of validity of eqs (2) and (6) numerically. I use two transversely isotropic models: first, the Mesaverde immature sandstone [14] with the density-normalized elastic parameters (in $\mathrm{km^2/s^2}$): $a_{11} = a_{22} = 22.36$, $a_{33} = 18.91$, $a_{13} = 8.49$, $a_{44} = a_{55} = 6.61$, $a_{66} = 8.00$ and $\rho = 2.46\,\mathrm{gcm^{-3}}$, and second, a theoretical model of anisotropy caused by thin layers (Baptie et al. [15], model PTL3) with the density-normalized elastic parameters (in $\mathrm{km^2/s^2}$): $a_{11} = a_{22} = 28.58$, $a_{33} = 17.37$, $a_{13} = 6.64$, $a_{44} = a_{55} = 5.63$, $a_{66} = 9.21$ and $\rho = 2.60\,\mathrm{gcm^{-3}}$. The Mesaverde immature sandstone displays anisotropy of 8.3, 4.6 and 9.5% for the P, SV and SH waves; the model of the layered medium displays anisotropy of 24.8, 15.2 and 24.5% for the P, SV and SH waves. The wavefield is generated by a single point force $\mathbf{f} = (1, 1, 0)^T$. The source-time function is a one-sided pulse defined by eq. (10) of [1]. The wavefield is calculated by eqs (2) and (6) and is compared with the exact solution (1).

Figure 1 shows waveforms of the S waves propagating in both studied anisotropy models. The waves are recorded at three observation points that lie in the x_1-x_3 plane at distances from the source of 257.1 km for the sandstone and 237.3 km for the layered medium. This distance corresponds to 100 wavelengths of the S wave propagating in the direction of the symmetry axis. The large distance is used to satisfy well the far-field approximation. The rays from the source to observation points deviate from the symmetry

axis by angles 10º, 20º and 30º. The waveforms at the observation points are calculated using three different approaches: first, using a numerical integration of the exact solution (1); second, using eq. (2) proposed by Vavryčuk [1]; and third, using eq. (6) derived from equations published by Popov et al. [5]. A comparison of the waveforms reveals that the approximate waveforms calculated using eq. (2) coincide within the width of the line with the exact solution (2). The waveforms calculated using eq. (6) approximate the exact solution well for rays deviating by 10º and 20º. However, we detect visible differences in the waveforms for the ray deviating by 30º. First, the approximation erroneously predicts no signal at the x_3-component; second, both the SV and SH waves have slightly shifted traveltimes; and third, the amplitudes of the SV and SH waves are slightly distorted at the horizontal components. These differences increase with increasing strength of anisotropy and with increasing deviation between the ray and the symmetry axis. Obviously, the less accuracy of eq. (6) compared with eq. (2) is produced by using Taylor expansions of polarization vectors, traveltimes and amplitudes instead of using the exact quantities.

Conclusions

The assertion of Popov et al. [5] that the asymptotic formulas presented by Vavryčuk [1] and Gridin [2] are either questionable or erroneous is unjustified and misleading. The solution published in [5] is not in contradiction with the previously published results. The solution is just a modification of the formula presented in [1]. The modification consists in substituting some quantities standing in the formula of Vavryčuk [1] by their Taylor expansions. This substitution is justified for small deviations of a ray from the symmetry axis and is advantageous in cases, when we prefer to avoid calculating such quantities as the SV-wave traveltime or the Gaussian curvature of the SV slowness surface. These quantities stand in the previously published formulas and evaluating them is more involved than evaluating the other quantities needed. On the other hand, the formula of Popov et al. [5] is less accurate and valid under more restrictive conditions than that of Vavryčuk [1].

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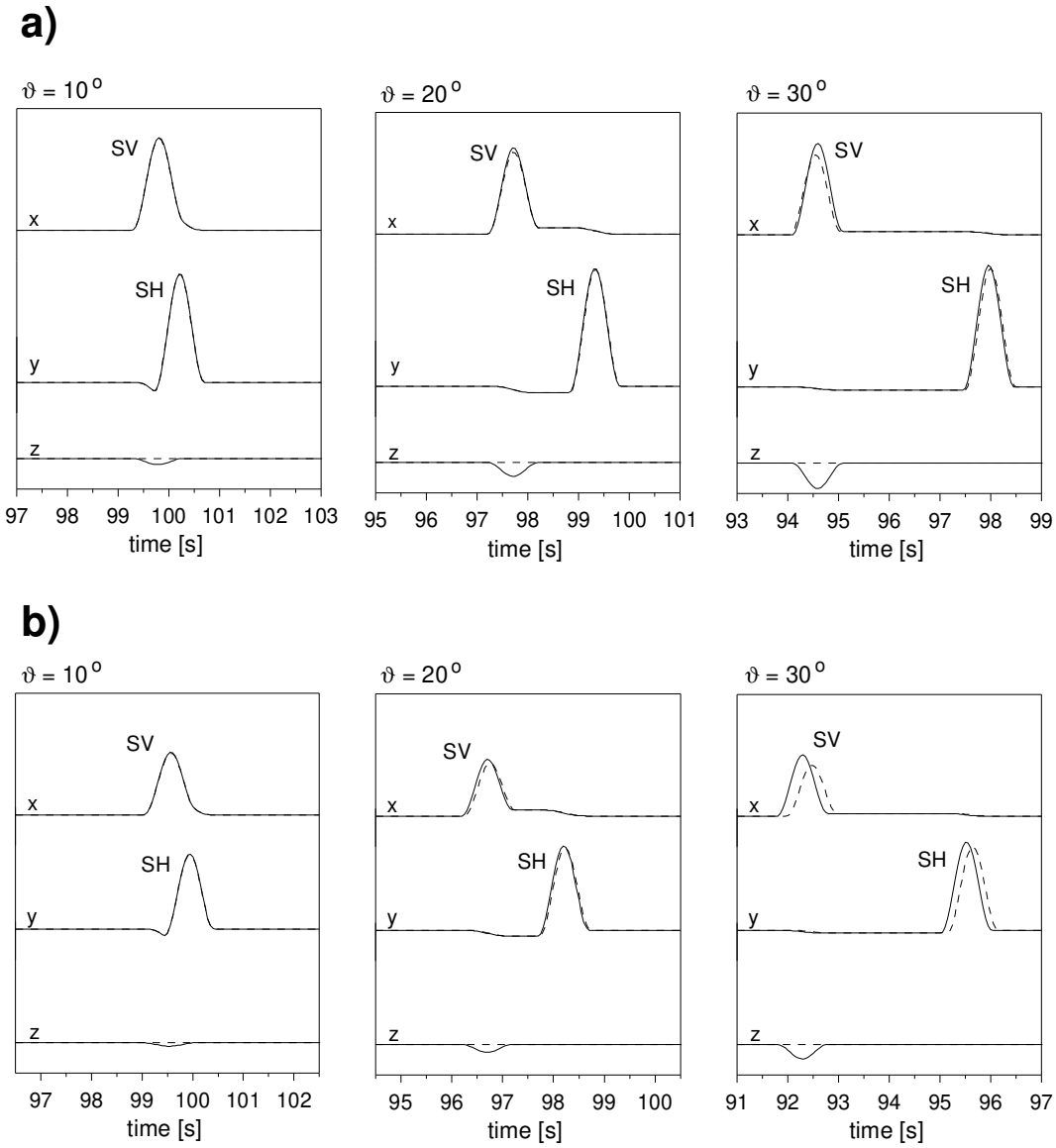

Figure 1. Waveforms of S waves propagating in the Mesaverde immature sandstone (a) and in the model of effective anisotropy caused by layering (b). Solid line – waveforms calculated using the exact formula (1) and using the approximate formula (2) proposed by Vavryčuk [1]. Dashed line – waveforms calculated using the approximate formula (6)

proposed by Popov et al. [5]. Angle ϑ defines the deviation of a ray from the symmetry axis. For parameters of the source and of distances of observation points from the source, see the text.

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