Coupling ray series

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Summary

We propose the coupling ray series, which yields the coupling ray theory in a similar way as the standard ray series yields the standard ray–theory solution of the elastodynamic equation. We insert the coupling ray series into the elastodynamic equation and obtain the coupling ray theory as the zero–order high–frequency asymptotic approximation.

The standard ray series yields a good approximation of the additional amplitude components but fails in the description of S–wave coupling. The coupling ray series well describes S–wave coupling but its approximation of the additional amplitude components is not satisfactory. We thus also propose more general ray series, which represent a generalization of both the standard ray series and coupling ray series.

Keywords

Elastic waves, anisotropy, heterogeneous media, ray series, coupling ray theory, travel time, amplitude, polarization.

1. Introduction

The elastodynamic equation for displacement u_i in the frequency domain reads

$$\tfrac{1}{\varrho}(\varrho\, a_{ijkl}u_{k,l})_{,j} + \omega^2 u_i = 0 \quad , \tag{1}$$

where

$$a_{ijkl} = \tfrac{1}{\varrho}\, c_{ijkl} \tag{2}$$

are the density–normalized elastic moduli.

1.1. Zero–order ray theory

For each elementary wave (a), where $(a)=(1)$ or $(a)=(2)$ for S waves and $(a)=(3)$ for a P wave, we express the displacement in terms of vectorial amplitude U_i and travel time $\tau^{(a)}$,

$$u_i = U_i \exp(\mathrm{i}\omega\tau^{(a)}) \quad . \tag{3}$$

Inserting (3) into (1), we obtain the elastodynamic equation in form

$$(\mathrm{i}\omega)^2\, N_i(U_k, \tau^{(a)}_{,l}) + \mathrm{i}\omega\, M_i(U_k, \tau^{(a)}_{,l}) + L_i(U_k) = 0 \quad , \tag{4}$$

where N_i, M_i, L_i are defined by relations

$$N_i(U_m, \tau^{(a)}_{,n}) = a_{ijkl}\tau^{(a)}_{,j}\tau^{(a)}_{,l}U_k - U_i \quad , \tag{5}$$

$$M_i(U_m, \tau^{(a)}_{,n}) = \tfrac{1}{\varrho}\,(\varrho\, a_{ijkl}\tau^{(a)}_{,l}U_k)_{,j} + a_{ijkl}\tau^{(a)}_{,j}U_{k,l} \quad , \tag{6}$$

$$L_i(U_m) = \tfrac{1}{\varrho}\,(\varrho\, a_{ijkl}U_{k,l})_{,j} \tag{7}$$

(Červený, 2001, eq. 2.4.41 but density normalized).

In the zero–order ray theory, we decompose 3 equations (4) according to the order in ω into 9 equations, and start with solving the highest–order equations.

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1.1.1. Anisotropic ray theory

Three highest–order equations

$$N_i(U_k, \tau^{(a)}_{,l}) = 0 \tag{8}$$

constitute the matrix Christoffel equation

$$\Gamma_{ik}\, U_k - U_i = 0 \quad , \tag{9}$$

where

$$\Gamma_{ik} = a_{ijkl}\, \tau^{(a)}_{,j} \tau^{(a)}_{,l} \tag{10}$$

is the Christoffel matrix. We select unit eigenvector $g^{(a)}_i$ and the corresponding eigenvalue $G^{(a)}$ of Γ_{ik}. Travel time $\tau^{(a)}$ is the solution of eikonal equation

$$G^{(a)}(x_m, \tau^{(a)}_{,n}) = 1 \quad , \tag{11}$$

where $G^{(a)}(x_m, \tau^{(a)}_{,n})$ is the eigenvalue of $\Gamma_{ik}(x_m, \tau^{(a)}_{,n})$ corresponding to eigenvector $g^{(a)}_i(x_m, \tau^{(a)}_{,n})$. In general, we denote by $g^{(ba)}_i$ the three eigenvectors and by $G^{(ba)}$ the corresponding eigenvalues of Christoffel matrix $\Gamma_{ik}(x_m, \tau^{(a)}_{,n})$. In this notation, $g^{(a)}_i = g^{(aa)}_i$ and $G^{(a)} = G^{(aa)}$.

Then the vectorial amplitude reads

$$U_i = A^{(a)}\, g^{(a)}_i \quad . \tag{12}$$

The only remaining free parameter is the zero–order ray–theory amplitude $A^{(a)}$, but we still have 3 middle–order and lower–order equations. Amplitude $A^{(a)}$ is determined by transport equation

$$g^{(a)}_i\, M_i(A^{(a)} g^{(a)}_k, \tau^{(a)}_{,l}) = 0 \quad . \tag{13}$$

Inserting (6), the transport equation reads

$$(\varrho\, g_i a_{ijkl} \tau_{,l} g_k A^2)_{,j} = 0 \quad . \tag{13'}$$

Other 2 middle–order equations

$$g^{(ba)}_i\, M_i(U_k, \tau^{(a)}_{,l}) = 0 \quad , \qquad (b) \neq (a) \quad , \tag{14}$$

require that the polarization vector is approximately constant in all directions and that the amplitude is approximately constant perpendicularly to the ray. These 2 equations are not satisfied by the zero–order vectorial amplitude. Moreover, the S–wave polarization vectors $g^{(1)}_i$ and $g^{(2)}_i$ may rapidly rotate about the ray.

These 2 equations may approximately be satisfied by the first–order additional components, or by coupling between waves. The additional components are useful between P and S waves, but should be avoided between S waves with similar propagation velocities. Coupling works fine between S waves, but we shall see that coupling does not represent a good approximation to additional components between P and S waves.

1.1.2. Isotropic ray theory

The anisotropic ray theory is fully applicable to P waves in isotropic media. There is thus no difference for P waves between the anisotropic and isotropic ray theories.

The eigenvalues of the Christoffel matrix corresponding to S waves are equal. The travel times are thus also equal,

$$\tau^{(1)} = \tau^{(2)} \quad , \tag{15}$$

and the vectorial amplitude of S waves is expressed in form

$$U_i = A^{(1)}\, g_i^{(1)} + A^{(2)}\, g_i^{(2)} \quad . \tag{16}$$

Then also one of 2 equations (14), for $(b) \neq (3)$, can be satisfied: the S–wave polarization vector does not rotate about the ray.

An isotropic medium may also serve as an approximation to a weakly anisotropic medium. In a weakly anisotropic medium, we may thus apply two different standard high–frequency asymptotic ray theories for S waves: the isotropic ray theory based on the assumption of equal velocities of both S waves, and the anisotropic ray theory assuming both S waves strictly decoupled. In the isotropic ray theory, the S–wave polarization vectors do not rotate about the ray, whereas in the anisotropic ray theory they coincide with the eigenvectors of the Christoffel matrix which may rotate rapidly about the ray.

1.2. Coupling ray theory

In "weakly anisotropic" models, at moderate frequencies, the S–wave polarization tends to remain unrotated round the ray, but is partly attracted by the rotation of the eigenvectors of the Christoffel matrix. The intensity of the attraction increases with frequency. This behaviour of the S–wave polarization is described by the coupling ray theory proposed by Coates & Chapman (1990). The frequency–dependent coupling ray theory is the generalization of both the zero–order isotropic and anisotropic ray theories and provides continuous transition between them. The coupling ray theory is applicable to S waves at all degrees of anisotropy, from isotropic to considerably anisotropic velocity models.

We aim to propose the coupling ray series, which yields the coupling ray theory in a similar way as the standard ray series yields the standard ray–theory solution of the elastodynamic equation.

2. Standard ray series

Standard ray series (anisotropic ray series and isotropic ray series) for elementary wave (a) read

$$u_i = \sum_{n=0}^{+\infty} (\mathrm{i}\omega)^{-n}\, U_i^{[n]} \exp(\mathrm{i}\omega\tau^{(a)}) \tag{17}$$

(Červený, 2001, eq. 5.7.1 but the opposite sign of the odd–order terms). Inserting series (17) into elastodynamic equation (1), we obtain the system of equations

$$N_i(U_k^{[n]}, \tau_{,l}^{(a)}) + M_i(U_k^{[n-1]}, \tau_{,l}^{(a)}) + L_i(U_k^{[n-2]}) = 0 \tag{18}$$

(Červený, 2001, eq. 5.7.3), where $U_k^{[-1]} = 0$ and $U_k^{[-2]} = 0$, i.e., M_i is missing in (18) for $n = 0$ and L_i is missing in (18) for $n = 0, 1$.

2.1. Anisotropic ray theory

Equation (18) for $n = 0$ reads

$$\Gamma_{ik}\, U_k^{[0]} - U_i^{[0]} = 0 \quad , \tag{19}$$

where Christoffel matrix Γ_{ik} is defined by (10). Then travel time $\tau^{(a)}$ is the solution of eikonal equation (11), and the zero–order vectorial amplitude is

$$U_i^{[0]} = A^{(a)}\, g_i^{(a)} \quad , \tag{20}$$

where the zero–order ray–theory amplitude $A^{(a)}$ is determined by transport equation (13).

For $n > 0$, we define amplitude components $U^{(b)[n]}$ with respect to eigenvectors of the Christoffel matrix,

$$U_i^{[n]} = \sum_{(b)} U^{(b)[n]}\, g_i^{(ba)} \tag{21}$$

(Červený, 2001, eq. 5.7.9). We decompose each higher–order vectorial amplitude $U_i^{[n]}$, $n > 0$, into principal component $U^{(a)[n]}$ and two additional components $U^{(b)[n]}$, $(b) \neq (a)$:

$$U_i^{[n]} = U^{(a)[n]}\, g_i^{(a)} + \sum_{(b)\neq(a)} U^{(b)[n]}\, g_i^{(ba)} \quad . \tag{22}$$

One of the three equations (18) for $n = 1$ is transport equation (13) for amplitude $A^{(a)}$. Other two equations (18) for $n = 1$,

$$g_i^{(ba)} \left[N_i(U_k^{[1]}, \tau_{,l}^{(a)}) + M_i(U_k^{[0]}, \tau_{,l}^{(a)}) \right] = 0 \quad , \tag{23}$$

can be satisfied by additional components $U^{(b)[1]}$, $(b) \neq (a)$. Since

$$N_i(g_m^{(ba)}, \tau_{,n}^{(a)}) = (G^{(ba)} - 1)\, g_i^{(ba)} \quad , \tag{24}$$

equation (23) yields

$$U^{(b)[1]} = -\frac{g_i^{(ba)} M_i(A^{(a)} g_m^{(a)}, \tau_{,n}^{(a)})}{G^{(ba)} - 1} \tag{25}$$

(Červený, 2001, eq. 5.7.14).

There are additional components in the first–order solution, but no coupling between S waves in the zero–order solution. Coupling between S waves can be calculated as series composed of approximate higher–order additional components, but these series often do not converge or converge slowly. Vavryčuk (1999) demonstrated the convergence on a numerical example of two S waves propagating vertically in a weakly transversely isotropic medium with the axis of symmetry rotating in the horizontal plane.

2.2. Isotropic ray theory

In an isotropic medium, which may also serve as an approximation to a weakly anisotropic medium, anisotropic ray theory applies to P waves, but the eigenvalues of the Christoffel matrix corresponding to S waves are equal,

$$G^{(1)} = G^{(2)} \quad . \tag{26}$$

Then

$$\tau^{(1)} = \tau^{(2)} \quad , \tag{27}$$

$$A^{(1)} = A^{(2)} \quad , \tag{28}$$

$$U_i^{[0]} = (r^{(1)[0]}\, g_i^{(1)} + r^{(2)[0]}\, g_i^{(2)})\, A^{(1)} \quad , \tag{29}$$

and we obtain 2 equations

$$g_i^{(B)} \sum_{(A)} M_i(r^{(A)[0]} A^{(A)} g_k^{(A)}, \tau_{,l}^{(A)}) = 0 \quad , \tag{30}$$

which represent the transport equation for $A^{(1)} = A^{(2)}$ and the decoupling equation for 2–D unit vector $(r^{(1)[0]}, r^{(2)[0]})$. The solution of the decoupling equation guaranties that the S–wave polarization vector

$$e_i = r^{(1)[0]}\, g_i^{(1)} + r^{(2)[0]}\, g_i^{(2)} \tag{31}$$

does not rotate about the ray.

In the isotropic ray theory, for each $n > 0$, we have two principal amplitude components $U^{(1)[n]}$ and $U^{(2)[n]}$ and one additional component $U^{(3)[n]}$.

3. Coupling ray series

We define the "coupling ray series"

$$u_i = \sum_{n=0}^{+\infty} \sum_{(a)} (\mathrm{i}\omega)^{-n}\, U_i^{(a)[n]} \exp(\mathrm{i}\omega\tau^{(a)}) \quad , \tag{32}$$

where

$$U_i^{(a)[n]} = r^{(a)[n]}\, A^{(a)}\, g_i^{(a)} \quad , \tag{33}$$

i.e.,

$$\boxed{u_i = \sum_{n=0}^{+\infty} \sum_{(a)} (\mathrm{i}\omega)^{-n}\, r^{(a)[n]}\, A^{(a)}\, g_i^{(a)} \exp(\mathrm{i}\omega\tau^{(a)}) \quad . \tag{34}}$$

Here travel time $\tau^{(a)}$ is the solution of eikonal equation (11) corresponding to unit eigenvector $g_i^{(a)}$, and the zero–order ray–theory amplitude $A^{(a)}$ is determined by transport equation (13).

The Christoffel equation is satisfied:

$$N_i\big(r^{(a)[n]} A^{(a)} g_k^{(a)}, \tau_{,l}^{(a)}\big) = 0 \quad . \tag{35}$$

Equations for amplitude coefficients $r^{(a)[n]}$ read

$$\sum_{(a)} \Big[M_i(r^{(a)[n]} A^{(a)} g_k^{(a)}, \tau_{,l}^{(a)}) + L_i(r^{(a)[n-1]} A^{(a)} g_k^{(a)}) \Big] \exp(\mathrm{i}\omega\tau^{(a)}) = 0 \quad , \tag{36}$$

where $r^{(a)[-1]} = 0$, i.e., L_i is missing in (36) for $n = 0$.

Coupling between S waves is nicely described by the zero–order solution $r^{(a)[0]}$ of equation (36), and often even by the zero approximation to $r^{(a)[0]}$ described in Section 3.2.

Unfortunately, there are no additional components analogous to the standard ray series. The additional components can be calculated as series composed of approximate solutions of the coupling equations, but these series converge slowly, see Section 3.3.

3.1. General coupling equations

Multiplying (36) by $g_i^{(b)}$, we obtain three coupling equations for $r^{(b)[n]}$, $b = 1, 2, 3$, $n \geq 0$,

$$g_i^{(b)} \sum_{(a)} \Big[M_i(r^{(a)[n]} A^{(a)} g_k^{(a)}, \tau_{,l}^{(a)}) + L_i(r^{(a)[n-1]} A^{(a)} g_k^{(a)}) \Big] \exp(\mathrm{i}\omega\tau^{(a)}) = 0 \quad . \tag{37}$$

We express

$$M_i(r^{(b)[n]} A^{(b)} g_m^{(b)}, \tau_{,n}^{(b)}) = r^{(b)[n]}\, M_i(A^{(b)} g_m^{(b)}, \tau_{,n}^{(b)}) + a_{ijkl}\, g_k^{(b)} (r_{,j}^{(b)[n]} \tau_{,l}^{(b)} + \tau_{,j}^{(b)} r_{,l}^{(b)[n]})\, A^{(b)} \, . \tag{38}$$

Multiplying this equation (38) by $g_i^{(b)}$ and considering the transport equation (13), we obtain

$$g_i^{(b)} M_i(r^{(b)[n]} A^{(b)} g_m^{(b)}, \tau_{,n}^{(b)}) = 2 V_j^{(b)} r_{,j}^{(b)[n]} A^{(b)} \quad , \tag{39}$$

where

$$V_j^{(a)} = \tfrac{1}{2} (a_{ijkl} + a_{ilkj})\, g_i^{(a)} g_k^{(a)} \tau_{,l}^{(a)} \tag{40}$$

is the ray–velocity vector. We insert (39) into (37) and obtain

$$2V_j^{(b)} r_{,j}^{(b)[n]} A^{(b)} \exp(\mathrm{i}\omega\tau^{(b)}) = -\sum_{(a)\neq(b)} g_i^{(b)} M_i(r^{(a)[n]} A^{(a)} g_m^{(a)}, \tau_{,n}^{(a)}) \exp(\mathrm{i}\omega\tau^{(a)})$$
$$- \sum_{(a)} g_i^{(b)} L_i(r^{(a)[n-1]} A^{(a)} g_m^{(a)}, \tau_{,n}^{(a)}) \exp(\mathrm{i}\omega\tau^{(a)}) \quad , \quad (41)$$

which reads

$$\boxed{\begin{aligned} V_j^{(b)} r_{,j}^{(b)[n]} = &-\sum_{(a)\neq(b)} \tfrac{1}{2}\, g_i^{(b)} M_i(r^{(a)[n]} A^{(a)} g_m^{(a)}, \tau_{,n}^{(a)})\, (A^{(b)})^{-1} \exp[\mathrm{i}\omega(\tau^{(a)}-\tau^{(b)})] \\ &- \sum_{(a)} \tfrac{1}{2}\, g_i^{(b)} L_i(r^{(a)[n-1]} A^{(a)} g_m^{(a)}, \tau_{,n}^{(a)})\, (A^{(b)})^{-1} \exp[\mathrm{i}\omega(\tau^{(a)}-\tau^{(b)})] \ . \quad (42) \end{aligned}}$$

We express equation (6) in form

$$M_i(U_m, \tau_{,n}) = \tfrac{1}{\varrho}\,(\varrho\, a_{ijkl}\tau_{,l})_{,j} U_k + (a_{ijkl} + a_{ilkj})\tau_{,j} U_{k,l} \quad , \qquad (43)$$

and insert the arguments used in (42),

$$M_i(r^{(a)[n-1]} A^{(a)} g_m^{(a)}, \tau_{,n}^{(a)}) = \tfrac{1}{\varrho}\,(\varrho\, a_{ijkl}\tau_{,l}^{(a)})_{,j} g_k^{(a)} r^{(a)[n-1]} A^{(a)}$$
$$+ (a_{ijkl} + a_{ilkj})\tau_{,j}^{(a)} \left[g_k^{(a)} (r^{(a)[n-1]} A^{(a)})_{,l} + g_{k,l}^{(a)} r^{(a)[n-1]} A^{(a)}\right] \quad . \qquad (44)$$

We decompose $g_{i,j}^{(a)}$ into the directions of $g_i^{(c)}$,

$$g_{i,j}^{(a)} = -\sum_{(c)} g_i^{(c)} \varphi_j^{(ca)} \quad . \qquad (45)$$

Then

$$M_i(r^{(a)[n-1]} A^{(a)} g_m^{(a)}, \tau_{,n}^{(a)}) = \tfrac{1}{\varrho}\,(\varrho\, a_{ijkl}\tau_{,l}^{(a)})_{,j} g_k^{(a)} r^{(a)[n-1]} A^{(a)}$$
$$+ (a_{ijkl} + a_{ilkj})\tau_{,j}^{(a)} \left[g_k^{(a)} (r^{(a)[n-1]} A^{(a)})_{,l} - \sum_{(c)} g_k^{(c)} \varphi_l^{(ca)} r^{(a)[n-1]} A^{(a)}\right] \quad . \qquad (46)$$

We define quantities

$$W_j^{(abc)} = \tfrac{1}{2}(a_{ijkl} + a_{ilkj})\, g_i^{(a)} g_k^{(b)} \tau_{,l}^{(c)} \quad , \qquad (47)$$

and quantities

$$\widetilde{W}_j^{(abc)} = \tfrac{1}{2}(a_{ijkl} + a_{ilkj})\, g_i^{(a)} g_k^{(bc)} \tau_{,l}^{(c)} \quad . \qquad (48)$$

We insert (46) into (42) and consider definitions (40), (47) and (48),

$$\boxed{\begin{aligned} V_j^{(b)} r_{,j}^{(b)[n]} = &-\sum_{(a)\neq(b)} \Bigg[\widetilde{W}_j^{(baa)} (r^{(a)[n]} A^{(a)})_{,j} \\ &+ \left(\tfrac{1}{2}\tfrac{1}{\varrho} g_i^{(b)} (\varrho a_{ijkl}\tau_{,l}^{(a)})_{,j} g_k^{(a)} - \sum_{(c)} W_j^{(bca)} \varphi_j^{(ca)}\right) r^{(a)[n]} A^{(a)}\Bigg] \\ &\times (A^{(b)})^{-1} \exp[\mathrm{i}\omega(\tau^{(a)} - \tau^{(b)})] \\ &-\tfrac{1}{2}\tfrac{1}{\varrho} \sum_{(a)} g_i^{(b)} [\varrho\, a_{ijkl} (r^{(a)[n-1]} g_k^{(a)} A^{(a)})_{,l}]_{,j} (A^{(b)})^{-1} \exp[\mathrm{i}\omega(\tau^{(a)} - \tau^{(b)})] \quad . \\ &\qquad (49) \end{aligned}}$$

This more explicit equation for the coefficients of the coupling ray series is still exact.

The left–hand side of equation (42) or (49) represents the derivative of $r_{,j}^{(b)[n]}$ along a ray corresponding to polarization (b). Unfortunately, the right–hand side of equation (42) or (49) cannot be calculated exactly along this single ray because it contains mostly the quantities corresponding to rays of other two polarizations. We thus have to approximate the right–hand side of equation (42) or (49) using perturbation expansion and neglecting small terms which are hard to calculate. In this way, we obtain the zero approximation to the solution of the coupling equation (42) or (49). In many situations, even this zero approximation may be acceptable.

If the zero approximation is inaccurate, we may solve equation (49) iteratively, starting with the zero approximation. Inserting the current approximation to the right–hand side of the coupling equations, we obtain the next approximation by the next numerical integration.

We shall now derive an example of the zero approximation to equation (49).

3.2. Approximate coupling equations

Quantities (47) and (48) may roughly be approximated by the generalized ray–velocity vectors

$$V_j^{(abc)} = \tfrac{1}{2}(a_{ijkl} + a_{ilkj})\, g_i^{(ac)} g_k^{(bc)} \tau_{,l}^{(c)} \quad , \tag{50}$$

$$W_j^{(abc)} \approx \widetilde{W}_j^{(abc)} \approx V_j^{(abc)} \quad . \tag{51}$$

The identity holds only for

$$W_j^{(abb)} = \widetilde{W}_j^{(abb)} \quad , \qquad \widetilde{W}_j^{(aba)} = V_j^{(aba)} \quad . \tag{52}$$

and for

$$W_j^{(aaa)} = \widetilde{W}_j^{(aaa)} = V_j^{(aaa)} = V_j^{(a)} \quad . \tag{53}$$

There are 18 generalized ray–velocity vectors, but usually $V_j^{(abc)} \approx V_j^{(bac)}$. Three vectors $V_j^{(1)} = V_j^{(111)}$, $V_j^{(2)} = V_j^{(222)}$ and $V_j^{(3)} = V_j^{(333)}$ represent the standard ray–velocity vectors and are roughly parallel (exactly parallel in isotropic media). Six other vectors $V_j^{(abc)}$ with $a = b$ have similar directions (equal directions in isotropic media) as $V_j^{(111)}$, $V_j^{(222)}$ or $V_j^{(333)}$. Six vectors $V_j^{(13c)}$ and $V_j^{(23c)}$ are roughly perpendicular (exactly perpendicular in isotropic media) to $V_j^{(111)}$, $V_j^{(222)}$ or $V_j^{(333)}$. Three vectors $V_j^{(12c)}$ are usually small (zero in isotropic media) compared to $V_j^{(111)}$, $V_j^{(222)}$ and $V_j^{(333)}$.

We now apply exact equation (49) to zero–order coefficient $r_{,j}^{(b)[0]}$ and we perform several rough approximations. We neglect the term containing $(\varrho a_{ijkl} \tau_{,l}^{(a)})_{,j}$, and neglect $W_j^{(abc)}$ and $\widetilde{W}_j^{(abc)}$ with $a \neq b$, which are roughly perpendicular to the ray given by $V_j^{(b)}$:

$$V_j^{(b)} r_{,j}^{(b)[0]} = \sum_{(a)\neq(b)} W_j^{(bba)} \varphi_j^{(ba)} r^{(a)[0]} A^{(a)} (A^{(b)})^{-1} \exp[\mathrm{i}\omega(\tau^{(a)} - \tau^{(b)})] \quad . \tag{54}$$

Along reference ray (c), we further approximate in (54)

$$V_j^{(b)} \approx \sqrt{\frac{G^{(c)}}{G^{(bc)}}}\, \frac{V^{(bbc)}}{V^{(c)}}\, V_j^{(c)} \quad , \tag{55}$$

$$W_j^{(bba)} \approx \sqrt{\frac{G^{(c)}}{G^{(ac)}}}\, \frac{V^{(bbc)}}{V^{(c)}}\, V_j^{(c)} \quad , \tag{56}$$

$$A^{(a)} \approx A^{(b)} \quad , \tag{57}$$

and arrive at approximation

$$V_j^{(c)} r_{,j}^{(b)[0]} = \sum_{(a)\neq(b)} \sqrt{\frac{G^{(bc)}}{G^{(ac)}}}\, V_j^{(c)} \varphi_j^{(ba)} r^{(a)[0]} \exp[\mathrm{i}\omega(\tau^{(a)} - \tau^{(b)})] \quad . \tag{58}$$

Here, the numerical integration is performed along rays corresponding to polarization $g_i^{(c)}$.

If we consider S waves only, we use approximation $G_j^{(BC)} \approx G_j^{(AC)}$ and obtain coupling equation

$$\boxed{V_j^{(C)} r_{,j}^{(B)[0]} = \sum_{(A)\neq(B)} V_j^{(C)} \varphi_j^{(BA)} \exp[\mathrm{i}\omega(\tau^{(A)} - \tau^{(B)})]\; r^{(A)[0]}} \tag{59}$$

identical to equations 30 and 31 by Coates & Chapman (1990). Here $A, B, C = 1, 2$.

3.3. Deriving the additional component from infinite series of coupling terms

Additional component to $U^{(a)[0]} = r^{(a)[0]} A^{(a)}$ is determined by equation (25) and reads

$$U^{(b)[1]} = -\frac{g_i^{(ba)} M_i(r^{(a)[0]} A^{(a)} g_m^{(a)}, \tau_{,n}^{(a)})}{G^{(ba)} - 1} \quad . \tag{60}$$

This additional component should be correct if the velocities of waves (a) and (b) sufficiently differ.

We now consider the coupling ray theory which has no additional component (60), and estimate what happens due to coupling between waves (a) and (b).

The small component coupled to $U^{(a)[0]} = r^{(a)[0]} A^{(a)}$ is described by coefficients $r^{(b)[n]}$. Their development along the ray is determined by equation (42). For the development of the zero–order coefficient we apply approximation $g_i^{(b)} \approx g_i^{(ba)}$ and obtain

$$V_j^{(b)} r_{,j}^{(b)[0]} = \tfrac{1}{2}(G^{(ba)} - 1)\, U^{(b)[1]}\, (A^{(b)})^{-1} \exp[\mathrm{i}\omega(\tau^{(a)} - \tau^{(b)})] \quad . \tag{61}$$

We approximate the development of the higher–order coefficients by

$$V_j^{(b)} r_{,j}^{(b)[n]} = -\tfrac{1}{2} g_i^{(b)} L_i(r^{(b)[n-1]} A^{(b)} g_m^{(b)}, \tau_{,n}^{(b)})\, (A^{(b)})^{-1} \quad . \tag{62}$$

The right–hand sides of both equations (61) and (62) behave approximately according to $\exp[\mathrm{i}\omega(\tau^{(a)} - \tau^{(b)})]$. Because we assume that the velocities of waves (a) and (b) sufficiently differ, other functions in (61) and (62) should change considerably more slowly along the ray.

Coefficient $r^{(b)[0]}$ should thus approximately behave as

$$r^{(b)[0]} A^{(b)} \approx \frac{1}{2}\, \frac{G^{(ba)} - 1}{\tau_{,i}^{(a)} V_i^{(b)} - \tau_{,i}^{(b)} V_i^{(b)}}\, \frac{U^{(b)[1]}}{\mathrm{i}\omega} \exp[\mathrm{i}\omega(\tau^{(a)} - \tau^{(b)})] \quad , \tag{63}$$

and coefficients $r^{(b)[n]}$, $n > 0$, which oscillate analogously to $r^{(b)[0]}$, should approximately behave as

$$r^{(b)[n]} A^{(b)} \approx -\frac{1}{2\mathrm{i}\omega}\, \frac{g_i^{(b)} L_i(r^{(b)[n-1]} A^{(b)} g_m^{(b)}, \tau_{,n}^{(b)})}{\tau_{,i}^{(a)} V_i^{(b)} - \tau_{,i}^{(b)} V_i^{(b)}} \quad . \tag{64}$$

The zero–order wavefield correction is

$$r^{(b)[0]}A^{(b)}\exp(\mathrm{i}\omega\tau^{(b)}) \approx \frac{1}{2}\,\frac{G^{(ba)}-1}{\tau^{(a)}_{,i}V^{(b)}_i-1}\,\frac{U^{(b)[1]}}{\mathrm{i}\omega}\exp(\mathrm{i}\omega\tau^{(a)}) \quad . \tag{65}$$

Approximating the spatial dependence of functions in (7) by oscillations caused by $\exp[\mathrm{i}\omega(\tau^{(a)}-\tau^{(b)})]$ only, equation (7) yields

$$g^{(b)}_i L_i(r^{(b)[n-1]}A^{(b)}g^{(b)}_m,\tau^{(b)}_{,n}) \approx (\mathrm{i}\omega)^2(G^{(ba)}-2\tau^{(a)}_{,i}V^{(b)}_i+1)\,r^{(b)[n-1]}A^{(b)} \quad . \tag{66}$$

The higher–order amplitude corrections then are

$$\frac{r^{(b)[n]}A^{(b)}}{\mathrm{i}\omega} \approx q\,r^{(b)[n-1]}A^{(b)} \quad , \tag{67}$$

where

$$q = -\frac{1}{2}\,\frac{G^{(ba)}-2\tau^{(a)}_{,i}V^{(b)}_i+1}{\tau^{(a)}_{,i}V^{(b)}_i-1} \quad . \tag{68}$$

Sum of all amplitude corrections is

$$\sum_{n=0}^{+\infty}\frac{r^{(b)[n]}A^{(b)}}{(\mathrm{i}\omega)^n} = \frac{1}{1-q}\,r^{(b)[0]}A^{(b)} \quad . \tag{69}$$

Equation (68) yields

$$\frac{1}{1-q} = \frac{2\,(\tau^{(a)}_{,i}V^{(b)}_i-1)}{G^{(ba)}-1} \quad . \tag{70}$$

Inserting (70) and (65) into (69), we obtain the sum of all wavefield corrections due to coupling:

$$\sum_{n=0}^{+\infty}\frac{r^{(b)[n]}A^{(b)}}{(\mathrm{i}\omega)^n}\exp(\mathrm{i}\omega\tau^{(b)}) \approx \frac{U^{(b)[1]}}{\mathrm{i}\omega}\exp(\mathrm{i}\omega\tau^{(a)}) \quad . \tag{71}$$

Sum of all these corrections is approximately equal to the first–order additional component of the anisotropic ray theory. Unfortunately, if we consider coupling ray series (34), a complete first–order additional component can only be obtained as an infinite series of the most pronounced coupling terms.

4. More general ray series

To be able to describe both additional components and coupling, we may construct more general ray series

$$u_i = \sum_{n=0}^{+\infty} \sum_{(a)} (\mathrm{i}\omega)^{-n}\, U_i^{(a)[n]} \exp(\mathrm{i}\omega\tau^{(a)}) \quad , \tag{72}$$

where

$$U_i^{(a)[n]} = \sum_{(b)} U^{(ba)[n]}\, g_i^{(ba)} \quad , \tag{73}$$

with

$$U^{(aa)[n]} = r^{(aa)[n]} A^{(a)} \quad . \tag{74}$$

For $(b) \neq (a)$ we take

$$U^{(ba)[0]} = 0 \quad . \tag{75}$$

Here travel time $\tau^{(a)}$ is the solution of eikonal equation (11) corresponding to unit eigenvector $g_i^{(a)}$, and the zero–order ray–theory amplitude $A^{(a)}$ is determined by transport equation (13).

Three coupling equations for $r^{(bb)[n]}$, $b = 1, 2, 3$, $n = 0, 1, ...$ read

$$\sum_{(a)} \Bigg\{ \sum_{(c)\neq(a)} g_i^{(b)} g_i^{(ca)} (G^{(ca)} - 1)\, U^{(ca)[n+1]} + \sum_{(c)} g_i^{(b)} \left[M_i(U^{(ca)[n]} g_k^{(a)}, \tau_{,l}^{(a)}) + L_i(U^{(ca)[n-1]} g_k^{(a)}) \right] \Bigg\} \exp(\mathrm{i}\omega\tau^{(a)}) = 0 \quad . \tag{76}$$

There are 6 free parameters $U^{(ba)[n+1]}$, $(b) \neq (a)$ for each $n = 0, 1, ...$ These free parameters should be optimized in order to obtain the best compromise between anisotropic ray theory and coupling ray theory.

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