Quality factor Q in dissipative anisotropic media

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Summary

In an isotropic dissipative medium, the attenuation properties of rocks are usually specified by the quality factor Q, which is a dimensionless, direction-independent, scalar quantity. We propose a similar, but direction-dependent, dimensionless quality factor Q for anisotropic dissipative media.

We present an algorithm for the computation of quality factor Q of a homogeneous or an inhomogeneous time-harmonic plane wave propagating in a homogeneous viscoelastic anisotropic medium. The quality factor is defined as the ratio of the time-averaged complete stored energy and the dissipated energy, per unit volume (Buchen, 1971). The algorithm for the computation of Q is exact, and may be applied to homogeneous or inhomogeneous plane waves propagating in media of unrestricted anisotropy (including isotropy) and unrestricted viscoelasticity. Generally, the algorithm requires a solution of an algebraic equation of the sixth degree with complex-valued coefficients.

For weakly inhomogeneous plane waves, propagating in arbitrarily anisotropic, weakly dissipative media, the expression for the quality factor simplifies considerably. The solution of the algebraic equation of the sixth degree is no more required; simple explicit analytical expression for the quality factor Q is obtained. The approximate formula for the quality factor Q does not depend on the inhomogeneity of the considered plane wave; the concept of homogeneous plane waves is not needed and it is not used in the derivation of Q. The quality factor Q is inversely proportional to the intrinsic attenuation factor $\mathcal{A}^{in}$, $Q^{-1} = \mathcal{A}^{in}$, which is a measure of dissipative properties of rocks. The intrinsic attenuation factor $\mathcal{A}^{in}$ is proportional to the scalar product of the attenuation vector with the energy-flux vector. The proposed algorithm for calculation of Q is generalized to waves generated by point sources, and to high-frequency body waves, propagating in heterogeneous anisotropic viscoelastic media. The expression for Q simplifies further if weak anisotropy concept is used or if Q is specified for higher-symmetry anisotropy.

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Numerical examples show that the quality factors (or intrinsic attenuation factors) of plane waves propagating in anisotropic viscoelastic media may be strongly directionally dependent, even for weakly anisotropic media.

Keywords: anisotropy, weak attenuation, inhomogeneous plane waves, quality factor, viscoelasticity.

Introduction

The dissipative properties of rocks are commonly specified by the quality factor Q, or by its inverse Q^{-1}, called the loss (or dissipation) factor. The quality factor Q is a dimensionless quantity. In isotropic dissipative media, Q is direction independent, but in anisotropic media, Q depends on the direction of propagation. The directional dependency of the quality factor Q can be rather strong, even for weak anisotropy. For a review of definitions, terminology and observations in seismology and seismic exploration, devoted to isotropic dissipative media, see Aki and Richards (1981), Johnston and Toksöz (1981), Cormier (1989) or Carcione (2001, 2007). In these publications, many other references can be found.

In this paper, we are interested in the quality factor Q for anisotropic dissipative media. Only such dissipation mechanisms, which can be described in the framework of the linear viscoelasticity, are considered. For time-harmonic plane waves, propagating in homogeneous viscoelastic, isotropic or anisotropic media, the quality factor Q has been classically defined by the relation $Q^{-1} = W_d/E$, where W_d is the density of time-averaged dissipated energy and E is the density of time-averaged complete stored energy, $E = K + U$. Here K and U are densities of the time-averaged kinetic and strain energies, respectively, see Buchen (1971). Note that equality $U = K$ holds for perfectly elastic media, but generally $U \neq K$ for viscoelastic media. Some modified definitions of Q have been sometimes used in seismological literature. For example, Carcione (2001, pp. 58-59) uses the definition $Q^{-1} = W_d/2U$. This definition was adopted by Červený and Pšenčík (2006), and by Zhu and Tsvankin (2006, 2007). Some authors (Moczo et al., 1987, Krebes and Le, 1994; Carcione, 2001) define Q using the concept of complex velocity v_c, see more details in the Appendix A. In this paper, we consider systematically the classical definition $Q^{-1} = W_d/E$ of Buchen (1971). The definition is applicable to homogeneous and inhomogeneous plane waves propagating in isotropic as well as anisotropic viscoelastic media.

In the propagation of waves in viscoelastic, isotropic or anisotropic, media, an important role is played by inhomogeneous waves. The slowness vector $\mathbf{p}$ of waves propagating in viscoelastic media is always complex-valued, $\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A}$. Here $\mathbf{P}$ is the propagation vector, and $\mathbf{A}$ the attenuation vector. For $\mathbf{A}$ parallel to $\mathbf{P}$ or for $\mathbf{A} = \mathbf{0}$, the wave is called homogeneous; otherwise it is inhomogeneous. For plane waves propagating in homogeneous viscoelastic media, the inhomogeneity of the wave may be chosen arbitrarily. This is not the case for waves generated by a point source in a viscoelastic medium. The inhomogeneity of these waves cannot be chosen arbitrarily. It results from the solution of the problem under consideration. Only for unbounded homogeneous isotropic viscoelastic media, the waves generated by a point source are homogeneous. In all other cases, including homogeneous anisotropic viscoelastic media, the waves propagating from a point source to an receiver, are inhomogeneous. Although the inhomogeneity of the wave may be weak, the inhomogeneous waves must be always considered if the medium is viscoelastic and anisotropic. We show that only for certain privileged directions (the direction of energy flux), the attenuation of the wave does not depend on the strength of its inhomogeneity. The inhomogeneous waves play an importan role in many branches of physics. An excellent review of ultrasonic inhomogeneous waves, their generation and applications in nondestructive testing is given by Declercq (2005). For applications in seismology and acoustics, see also Borcherdt and Wennerberg (1985), Hosten et al. (1987).

In viscoelastic media, the stifnesses c_{ijkl} relating the stress and strain tensors are not real valued as in perfectly elastic media, but complex valued. We call them viscoelastic moduli. The imaginary parts of viscoelastic moduli are responsible for the dissipative properties of waves, propagating in viscoelastic media. The sign convention for the imaginary parts of the viscoelastic moduli must correspond to the sign convention used to describe the time-harmonic wave under consideration, particularly to the sign in the exponential time factor $\exp[\pm i\omega t]$. We use here the exponential time factor $\exp[-i\omega t]$, for which $c_{ijkl} = c^R_{ijkl} - ic^I_{ijkl}$. To simplify certain relations, throughout the paper we use the density-normalized viscoelastic moduli $a_{ijkl} = c_{ijkl}/\rho$:

$$a_{ijkl} = a^R_{ijkl} - ia^I_{ijkl} \ . \tag{1}$$

We assume that a_{ijkl} satisfy the symmetry relations,

$$a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij} \ , \tag{2}$$

although the relation $a^I_{ijkl} = a^I_{klij}$ has not yet been proved to hold generally, see Scott (1997), Carcione (2001). Consequently, the number of density-normalized viscoelastic moduli reduces from 81 to 21, and they can be described by 6×6 Voigt symmetric matrix $A_{\alpha\beta}$ $(\alpha, \beta = 1, 2, ...6)$, with $A_{\alpha\beta} = A^R_{\alpha\beta} - iA^I_{\alpha\beta}$. We assume that the matrix $A^R_{\alpha\beta}$ is positive definite, see Fedorov (1968), Auld (1973), Helbig (1994), Carcione (2001) and many others. Similarly, we assume that the matrix $A^I_{\alpha\beta}$ is positive definite or zero, see Červený and Pšenčík (2006, p.1302). These asumptions remain valid for both sign options of the exponential time factor. In seismological literature, both signs have been used, so that the reader must be careful in comparing the results of different authors. The density-normalized viscoelastic moduli a_{ijkl} are frequency dependent in viscoelastic media, but we do not write $a_{ijkl}(\omega)$, as we consider an arbitrary, but fixed $\omega > 0$ in the whole paper.

The references, whose authors studied the quality factor Q of plane waves propagating in homogeneous viscoelastic anisotropic media are rare. For plane SH waves propagating in a plane of symmetry of an anisotropic medium, see Krebes and Le (1994), and Carcione (2001, 2007). For plane P waves propagating in TI and orthorhombic media, see recent papers by Zhu and Tsvankin (2006, 2007). The latter authors based their concept of the Q factor exclusively on *homogeneous* plane waves, which is not used in this paper. Exact expression for the quality factor Q for homogeneous plane SH waves propagating in homogeneous media of unrestricted viscoelasticity and anisotropy can be found in Carcione (2001, 2007). Krebes and Le (1994) used the exact definition of Q, given by Buchen (1971) for isotropic viscoelastic media, for inhomogeneous plane SH waves propagating in transversely isotropic viscoelastic media. Expression for Q valid for inhomogeneous plane waves propagating in media of unrestricted anisotropy and viscoelasticity can be found in Červený and Pšenčík (2006). Using Carcione's (2001) definition of Q and the so-called mixed specification of the slowness vector, Červený and Pšenčík (2006) derived a simple formula, in which Q was expressed in terms of viscoelastic moduli and complex-valued slowness and polarization vectors of homogeneous as well as inhomogeneous waves. To obtain exact results, it is necessary to solve an algebraic equation of the sixth degree with complex-valued coefficients.

Červený and Pšenčík (2007) show that the solution of the algebraic equation of the sixth degree can be avoided if weakly inhomogeneous plane waves propagating in weakly dissipative anisotropic media are considered. The relevant expressions for the quality factor Q are derived and discussed in this paper. Probably the first expression for the direction-dependent quality factor Q for generally anisotropic, weakly dissipative media was given by Gajewski and Pšenčík (1992). Gajewski and Pšenčík (1992) used a quite different approach to derive Q, based on the modification of the asymptotic ray-series method, in which it was assumed that a^I_{ijkl} are small quantities of the order ω^{-1} for $\omega \rightarrow \infty$. The method is very powerful and is valid for high-frequency elementary waves (with curved wavefronts) propagating in smoothly varying, weakly dissipative, generally anisotropic media. Moreover, the expression of Gajewski and Pšenčík (1992) is given in a simple compact analytical form. The paper, however, did not provide answers to two important questions: i) how is the derived expression for Q related to the definition $Q^{-1} = W_d/E$? ii) is the result also valid for inhomogeneous waves when inhomogeneous waves were not considered in the derivation? The answers to these two questions are given in this paper. It is shown, that the expression for Q, given by Gajewski and Pšenčík (1992) is consistent with the definition $Q^{-1} = W_d/E$, and it is valid even for weakly inhomogeneous waves.

The structure of the paper is following. We start with a brief review of theory of time-harmonic, homogeneous and inhomogeneous plane waves, propagating in general viscoelastic anisotropic media. We use the mixed specification of the slowness vector $\mathbf{p}$, see Červený and Pšenčík (2005), which reduces the problem of the determination of $\mathbf{p}$ into a solution of an algebraic equation of the sixth degree with complex-valued coefficients. Once the equation is solved, exact explicit analytical expressions for the quantities of our interest, including the exact expression for the quality factor Q, can be found for all P, S1 and S2 homogeneous and inhomogeneous plane waves. We show that the definition of the quality factor, $Q^{-1} = W_d/E$, is explicitly related to the direction of energy flux vector. For weakly inhomogeneous plane waves propagating in weakly dissipative anisotropic media, we substitute the numerical solution of the algebraic equation of the sixth degree by approximate, but extremely simple, first-order perturbation formula. All expressions then simplify considerably, including the expression for the quality factor Q. We show that the quality factor Q is independent of the inhomogeneity of the considered plane waves. Thus, Q defined in this way, represents a parameter characterizing intrinsic attenuation of the medium. It is not affected by the inhomogeneity of the plane wave. The concept of homogeneity of plane waves is redundant, and it is thus not used. It is further shown that the derived expression for Q can be used even for high-frequency elementary waves (with curved wavefronts), calculated along rays in heterogeneous, weakly dissipative, generally anisotropic media, and for waves generated by point sources. The presented numerical examples illustrate directional behaviour of Q, or more precisely $\mathcal{A}^{in} = Q^{-1}$ in a model used in our previous studies.

In the whole paper, we use Cartesian coordinates x_i and time t. The lower-case italic indices $i, j, ...$ take the values of 1,2,3, the Greek indices $\alpha, \beta, ...$ the values of 1,2,...,6. The Einstein summation convention over repeated indices is used. For time-harmonic waves, we consider the exponential time factor $\exp[-\mathrm{i}\omega t]$, where ω is a fixed, real-valued, positive circular frequency.

Time-harmonic plane waves in viscoelastic anisotropic media

The time-harmonic plane waves propagating in an unbounded homogeneous viscoelastic anisotropic medium can be described by the expression

$$u_i(x_j, t) = aU_i \exp[-\mathrm{i}\omega(t - p_n x_n)] \ . \tag{3}$$

Here u_i, U_i and p_i are Cartesian components of the complex-valued displacement vector $\mathbf{u}$, normalized polarization vector $\mathbf{U}$, $(\mathbf{U} \cdot \mathbf{U} = 1)$, and the slowness vector $\mathbf{p}$, respectively; and a is a complex-valued scalar amplitude. The normalized polarization vector $\mathbf{U}$, the slowness vector $\mathbf{p}$ and the scalar amplitude a are independent of $\mathbf{x}$ and t. The displacement vector $\mathbf{u}$ satisfies the equation of motion $\tau_{ij,j} = \rho \ddot{u}_i$, where τ_{ij} is the stress tensor, related to the infinitesimal strain tensor $e_{ij} = (u_{i,j} + u_{j,i})/2$ by the generalized Hooke's law $\tau_{ij} = c_{ijkl} e_{kl}$. The quantities c_{ijkl} are complex-valued viscoelastic moduli.

It follows from equation of motion that (3) represents a plane wave if and only if U_1, U_2 and U_3 satisfy a system of three linear equations

$$(\Gamma_{ik} - \delta_{ik}) U_k = 0, \qquad i = 1, 2, 3. \tag{4}$$

Here Γ_{ik} is the generalized complex-valued Christoffel matrix, given by the relation

$$\Gamma_{ik} = a_{ijkl} p_j p_l \ . \tag{5}$$

The system (4) can be solved only if

$$\det(\Gamma_{ik} - \delta_{ik}) = 0 \ . \tag{6}$$

This equation represents a constraint relation for the slowness vector $\mathbf{p}$.

We express the slowness vector $\mathbf{p}$ using the so-called mixed specification, proposed by Červený (2004), see also Červený & Pšenčík (2005):

$$\mathbf{p} = \sigma \mathbf{n} + \mathrm{i} D \mathbf{m} \ , \quad \mathbf{n} \cdot \mathbf{m} = 0 \ . \tag{7}$$

The two real-valued, mutually orthogonal unit vectors $\mathbf{n}$ and $\mathbf{m}$ and the real-valued parameter D uniquelly specify the plane wave under consideration, and can be chosen arbitrarily. The vector $\mathbf{n}$ is perpendicular to the wavefront and the vector $\mathbf{m}$ is tangent to the wavefront. The vectors $\mathbf{n}$ and $\mathbf{m}$ specify the propagation-attenuation plane. The real-valued parameter D is called the inhomogeneity parameter since it controls inhomogeneity of the plane wave. For $D = 0$, the plane wave is homogeneous, for $D \neq 0$ it is inhomogeneous. The complex-valued quantity σ in (7) can be determined from the constraint relation (6), which, after inserting (7) into (6), reads

$$\det[a_{ijkl}(\sigma n_j + \mathrm{i} D m_j)(\sigma n_l + \mathrm{i} D m_l) - \delta_{ik}] = 0 \ , \tag{8}$$

see Červený and Pšenčík (2005). Eq. (8) is an algebraic equation of the sixth degree with complex-valued coefficients for σ. It has six complex-valued roots, corresponding to P, S1 and S2 plane waves, propagating in the directions of $\mathbf{n}$ and $-\mathbf{n}$, respectively. The quantity σ is a function of a_{ijkl}, $\mathbf{n}$, $\mathbf{m}$ and D.

For certain simple cases, the algebraic equation of the sixth degree can be factorized to two equations; one of the fourth degree, the other of the second degree. For example, the quantity σ of a plane wave propagating in an isotropic viscoelastic medium or an SH wave propagating in a plane of symmetry of a monoclinic (orthorhombic, hexagonal) medium is a solution of the quadratic equation resulting from factorization of eq. (8). The formula (7) for the slowness vector can be then expressed in a closed analytical form.

It is often useful to express the slowness vector $\mathbf{p}$ in the form:

$$\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A} \ , \tag{9}$$

where $\mathbf{P}$ and $\mathbf{A}$ are real-valued vectors, called the propagation and attenuation vector, respectively. Both $\mathbf{P}$ and $\mathbf{A}$ are situated in the propagation-attenuation plane. Once σ is determined, $\mathbf{P}$ and $\mathbf{A}$ are given by the relations:

$$\mathbf{P} = \mathbf{n}\mathrm{Re}\sigma \ , \qquad \mathbf{A} = \mathbf{n}\mathrm{Im}\sigma + D\mathbf{m} \ . \tag{10}$$

The real-valued phase-velocity $\mathcal{C}$ in the direction of $\mathbf{n}$ and the attenuation angle γ (the angle between $\mathbf{P}$ and $\mathbf{A}$) are then expressed as follows:

$$\mathcal{C} = 1/|\mathbf{P}| = 1/|\mathrm{Re}\sigma| \ , \quad \cos\gamma = \epsilon\, \mathrm{Im}\sigma/[(\mathrm{Im}\sigma)^2 + D^2]^{1/2} \ , \tag{11}$$

where $\epsilon = \mathrm{Re}\sigma/|\mathrm{Re}\sigma| = \pm 1$.

Let us mention that there are other possible specifications of the slowness vector. Since they may lead to certain complications, we use here exclusively the mixed specification (7) of the slowness vector. For other possible specifications and problems related to their use, see Červený and Pšenčík (2005) or the Appendix B.

Once σ is found, we can also simply determine the complex-valued polarization vectors $\mathbf{U}$ from (4) and from the normalization condition $\mathbf{U} \cdot \mathbf{U} = 1$.

Exact expressions for the quality factor Q

For σ and $\mathbf{U}$ known, we can also compute densities of time-averaged energy related quantities, e.g. the kinetic K, strain U, complete $E = K + U$ and dissipated W_d energies, the complex-valued Poynting vector $\mathbf{F}$ (also called the complex-valued energy flux), the real-valued Poynting vector $\mathbf{S} = \mathrm{Re}\mathbf{F}$ (also called the energy flux), and the real-valued energy-velocity vector $\boldsymbol{\mathcal{U}} = \mathbf{S}/E$. For details see Červený and Pšenčík (2006). Here we are interested in the quality factor Q, which is given by the relation:

$$Q = E/W_d \ . \tag{12}$$

In (12), we can use the relations

$$E = \mathbf{P} \cdot \mathbf{S} \ , \qquad W_d = 2\mathbf{A} \cdot \mathbf{S}, \tag{13}$$

derived by Carcione & Cavallini (1993), see also Červený & Pšenčík (2006). In (13), $\mathbf{P}$ and $\mathbf{A}$ are given by (10). Inserting (13) into (12), we obtain alternative expressions for Q

$$Q = \frac{E}{2\mathbf{A}\cdot\mathbf{S}} = \frac{\mathbf{P}\cdot\mathbf{S}}{2\mathbf{A}\cdot\mathbf{S}} = (2\mathbf{A}\cdot\boldsymbol{\mathcal{U}})^{-1} \ , \tag{14}$$

as $\boldsymbol{\mathcal{U}} = \mathbf{S}/E = \mathbf{S}/\mathbf{P}\cdot\mathbf{S}$. For the time-averaged energy-velocity vector $\boldsymbol{\mathcal{U}}$, we can use the formula derived by Červený & Pšenčík (2006):

$$\mathcal{U}_i = \mathrm{Re}(a_{ijkl}U_kU_j^*p_l)/(P_n\mathrm{Re}(a_{njkl}U_kU_j^*p_l)) \ . \tag{15}$$

Equations (14) for Q, (10) for $\mathbf{P}$ and $\mathbf{A}$ and (15) for $\boldsymbol{\mathcal{U}}$ play a basic role in this paper. They are exact, valid for homogeneous and inhomogeneous plane waves propagating in isotropic or anisotropic, perfectly elastic or viscoelastic media. No approximation has been used in their derivations, and no restriction has been imposed on the anisotropy or viscoelasticity of the medium and on the inhomogeneity parameter D of the plane wave.

Quality factor Q for weakly inhomogeneous plane waves in anisotropic, weakly dissipative media

In this section, we derive approximate expressions for the quality factor Q of a weakly inhomogeneous plane wave propagating in an anisotropic, weakly dissipative medium. In practical problems of seismology and seismic exploration, the dissipation is usually small, and inhomogeneity of the considered waves is also weak. In such a case, the numerical solution of the algebraic equation (8) of the sixth degree can be substituted by its approximate explicit solution, and the equation for Q can be simplified considerably.

The problem of weakly inhomogeneous plane waves propagating in weakly dissipative media was treated by Červený & Pšenčík (2007) who used the first-order perturbation method for its solution. As a reference (nonperturbed) case, a homogeneous plane wave with a real-valued slowness vector ($D = 0$) propagating in a perfectly elastic medium ($a^I_{ijkl} = 0$) was considered. The symbols corresponding to this reference case are denoted here by the upper indices 0. The solutions for the reference case are well-known. For σ^0, $\mathbf{P}^0$ and $\mathbf{A}^0$, we have

$$\sigma^0 = 1/\mathcal{C}^0 \ , \quad \mathbf{P}^0 = \mathbf{n}/\mathcal{C}^0 \ , \quad \mathbf{A}^0 = \mathbf{0} \ . \tag{16}$$

The symbol $\mathcal{C}^0$ denotes the phase velocity in the direction of $\mathbf{n}$ in the reference medium. The polarization vector $\mathbf{U}^0$ of the reference homogeneous wave can be determined from the equation (4) specified for the reference, perfectly elastic medium:

$$(\Gamma^0_{ik} - \delta_{ik})U^0_k = 0, \qquad i = 1,2,3. \tag{17}$$

Here, as in eq. (4), we consider $\mathbf{U}^0$ to be normalized, $U^0_iU^0_i = 1$. The energy-velocity vector $\boldsymbol{\mathcal{U}}^0$ of a homogeneous plane wave propagating in the reference perfectly elastic medium is given by the relation:

$$\mathcal{U}^0_i = a^0_{ijkl}p^0_lU^0_jU^0_k \ , \tag{18}$$

see Červený (2001, eq. 2.2.65). Note an obvious identity, which follows from the relation $a^0_{ijkl}p^0_jp^0_lU^0_iU^0_k = 1$, see (17) and (18):

$$\mathcal{U}^0_in_i = \mathcal{C}^0 \ . \tag{19}$$

The inhomogeneity of the studied plane wave (specified by D) and the imaginary parts of viscoelastic moduli (a^I_{ijkl}) are considered as small perturbations. The unit vectors $\mathbf{n}$ and $\mathbf{m}$ remain fixed and are not perturbed. Under these assumptions, Červený & Pšenčík (2007) derived the following approximate expression for the quantity σ:

$$\sigma = 1/\mathcal{C}^0[1 + \mathrm{i}\mathcal{A}^{in}/2 - \mathrm{i}D(\boldsymbol{\mathcal{U}}^0 \cdot \mathbf{m})] \ . \tag{20}$$

Červený & Pšenčík (2007) call the quantity $\mathcal{A}^{in}$ the *intrinsic attenuation factor* since it depends only on the intrinsic dissipation, not on D. It is given by the relation

$$\mathcal{A}^{in} = a^I_{ijkl} p^0_j p^0_l U^0_i U^0_k \ . \tag{21}$$

For the perturbed propagation vector $\mathbf{P}$ and the attenuation vector $\mathbf{A}$, we obtain, see (10):

$$\mathbf{P} = \mathbf{n}/\mathcal{C}^0 \ , \quad \mathbf{A} = \mathbf{n}\mathcal{A}^{in}/2\mathcal{C}^0 + D[\mathbf{m} - \mathbf{n}(\boldsymbol{\mathcal{U}}^0 \cdot \mathbf{m})/\mathcal{C}^0] \ . \tag{22}$$

Thus, the perturbed propagation vector $\mathbf{P}$ is the same as the unperturbed one, $\mathbf{P} = \mathbf{P}^0$. The perturbed attenuation vector $\mathbf{A}$, however, differs from the unperturbed one significantly. The unperturbed attenuation vector $\mathbf{A}^0$ is zero, see (16), but the perturbed one depends on the perturbations a^I_{ijkl} as well as on the inhomogeneity parameter D.

Since the perturbation of the polarization vector $\mathbf{U}^0$ is purely imaginary, see Červený and Pšenčík (2007), the perturbed energy-velocity vector $\boldsymbol{\mathcal{U}}$ is the same as the unperturbed one, see (15):

$$\mathcal{U}_i = \mathcal{U}^0_i = a^0_{ijkl} p^0_l U^0_i U^0_k \ . \tag{23}$$

If we now insert expressions for $\mathbf{A}$ from (22) and for $\boldsymbol{\mathcal{U}}$ from (23) into (14), we obtain a final expression for the quality factor

$$Q^{-1} = a^I_{ijkl} p^0_j p^0_l U^0_i U^0_k \ . \tag{24}$$

The term depending on D in the expression for $\mathbf{A}$ in (22) vanished when multiplied by $\boldsymbol{\mathcal{U}}(= \boldsymbol{\mathcal{U}}^0)$. Thus, the *quality factor in eq. (24) does not depend on* D although weakly inhomogeneous plane waves (with $D \neq 0$) can be considered. The quality factor Q, given by (24), is the same and independent of D for any homogeneous or weakly inhomogeneous plane wave.

We have thus shown that for weakly inhomogeneous plane waves, propagating in weakly dissipative anisotropic media, the exact quality factor $Q = E/W_d$ reduces to the approximate form given by (24). It does not depend on the inhomogeneity parameter D of the plane wave under consideration. It depends only on intrinsic dissipation parameters of the medium. Let us emphasize again that this conclusion holds only approximately. It is based on the first-order perturbation formulae for weakly inhomogeneous plane waves propagating in weakly dissipative anisotropic media.

Similarly as the phase velocity in a weakly anisotropic medium can be expressed in terms of weak-anisotropy parameters (Farra and Pšenčík, 2003), the quality factor Q for weakly dissipative media can be expressed in terms of weak-attenuation parameters, see the Appendix C.

Attenuation coefficient α of plane waves

We discuss here the attenuation coefficient α of an arbitrary time-harmonic homogeneous or inhomogeneous plane wave (3), specified by vectors $\mathbf{n}$, $\mathbf{m}$ and the inhomogeneity parameter D, along an arbitrarily oriented straight-line profile in a homogeneous viscoelastic anisotropic medium. Using (9) in the formula (3) for the plane wave, the formula can be rewritten in the following form:

$$u_i(x_j, t) = aU_i \exp[-\mathrm{i}\omega(t - P_n x_n)] \exp[-\omega A_n x_n] \ . \tag{25}$$

We now introduce the plane wave dissipative filter by the relation

$$\mathcal{D}(\omega) = \exp[-\omega A_n x_n] \ . \tag{26}$$

Equation (25) then reads

$$u_i(x_j, t) = aU_i \mathcal{D}(\omega) \exp[-\mathrm{i}\omega(t - P_n x_n)] \ . \tag{27}$$

The physical meaning of the plane-wave dissipation filter is obvious. It describes variation of amplitudes along an arbitrarily chosen profile. Let us specify the straight-line profile under consideration by a unit real-valued vector $\mathbf{l}$. Let us further introduce the distance $s - s_0$, measured along the profile $\mathbf{l}$, with an arbitrarily selected initial point s_0 on the profile so that $s - s_0$ is positive in the direction of $\mathbf{l}$. Then the plane-wave dissipation filter (26) along the profile $\mathbf{l}$ reads:

$$\mathcal{D}(\omega) = \exp[-\omega(s - s_0)\mathbf{A} \cdot \mathbf{l}] = \exp[-\alpha(\mathbf{l})(s - s_0)] \ . \tag{28}$$

Here we have introduced a quantity $\alpha(\mathbf{l})$

$$\alpha(\mathbf{l}) = \omega \mathbf{A} \cdot \mathbf{l} \ . \tag{29}$$

We call $\alpha(\mathbf{l})$ the *attenuation coefficient* (or absorption coefficient) of a studied homogeneous or inhomogeneous plane wave along the profile $\mathbf{l}$. The attenuation coefficient $\alpha(\mathbf{l})$ is real valued, and its dimension is $[distance]^{-1}$. For certain directions of $\mathbf{l}$, the attenuation coefficient may be negative, and may thus actually represent the exponential growth of amplitude (for example, if the profile $\mathbf{l}$ is oriented against the direction of propagation). In this case, we speak on negative attenuation coefficient.

Inserting $\mathbf{A}$ from (10) into (29), we obtain a formally simple expression for the attenuation coefficient $\alpha(\mathbf{l})$ along the profile $\mathbf{l}$:

$$\alpha(\mathbf{l}) = \omega \mathbf{A} \cdot \mathbf{l} = \omega[(\mathbf{n} \cdot \mathbf{l})\mathrm{Im}\sigma + (\mathbf{m} \cdot \mathbf{l})D] \ . \tag{30}$$

This expression for $\alpha(\mathbf{l})$ is exact, and may be used for arbitrary, homogeneous or inhomogeneous, plane wave (P, S1, S2) propagating in arbitrary, perfectly elastic or viscoelastic, isotropic or anisotropic, medium along any profile $\mathbf{l}$. The plane wave is specified by unit vectors $\mathbf{n}, \mathbf{m}$, and by the inhomogeneity parameter D. The straight line profile $\mathbf{l}$ can be chosen arbitrarily. For exact determination of $\alpha(\mathbf{l})$ using (30), we have to solve the algebraic equation (8) of the sixth degree for σ. For viscoelastic isotropic media, the algebraic equation of the sixth degree factorizes into quadratic algebraic equations for plane P and S waves.

For weakly inhomogeneous plane waves propagating in weakly dissipative media, equation (30) simplifies. We can use the first-order perturbation expression for the attenuation vector $\mathbf{A}$ given in (22) and insert it into (30). We obtain,

$$\alpha(\mathbf{l}) = \omega[\frac{\mathbf{n}\cdot\mathbf{l}}{2Q\mathcal{C}^0} + D(\mathbf{m}\cdot\mathbf{l} - \frac{(\mathbf{n}\cdot\mathbf{l})(\boldsymbol{\mathcal{U}}^0\cdot\mathbf{m})}{\mathcal{C}^0})] \ . \tag{31}$$

Here Q is the quality factor given by (24). Expression (31) is approximate, but otherwise it is quite general. The structural perturbations a^I_{ijkl} and the inhomogeneity parameter D are, of course, assumed to be small. Note that the effects of structural perturbations a^I_{ijkl} and of the inhomogeneity parameter D are coupled in eq. (31) even for weakly inhomogeneous waves propagating in weakly dissipative media.

In seismological structural studies, in seismic exploration or in non-destructive testing of materials by ultrasonic techniques, the researchers are mostly interested in the "intrinsic" dissipation, caused exclusively by a^I_{ijkl}. The attenuation coefficients, related fully or partly to the inhomogeneity of the wave D, are usually of no interest for them, as they are not related exclusively to the properties of tested rocks, but also to the properties of the wave used for testing. Therefore, a very important question is, whether there is a profile $\mathbf{l}$, along which the attenuation coefficient $\alpha(\mathbf{l})$ does not depend on D.

For exact answer, it would be necessary to study (30) numerically because $\mathrm{Im}\sigma$ in (30) also depends on D. However, for weakly inhomogeneous plane waves, propagating in weakly dissipative anisotropic media, we can use (31) and obtain the answer immediately: attenuation coefficient $\alpha(\mathbf{l})$, given by (31), does not depend on D if

$$\mathbf{m}\cdot\mathbf{l} - (\mathcal{C}^0)^{-1}(\mathbf{n}\cdot\mathbf{l})(\boldsymbol{\mathcal{U}}^0\cdot\mathbf{m}) = 0 \ . \tag{32}$$

This equation is satisfied only if the unit vector $\mathbf{l}$ is taken parallel to $\boldsymbol{\mathcal{U}}^0$,

$$\mathbf{l} \parallel \boldsymbol{\mathcal{U}}^0 \ . \tag{33}$$

In the following, we call the attenuation coefficient $\alpha(\mathbf{t})$ along the profile $\mathbf{t}$, where $\mathbf{t} = \boldsymbol{\mathcal{U}}^0/|\boldsymbol{\mathcal{U}}^0|$, simply the attenuation coefficient, and denote it by α. Consequently, for the profile $\mathbf{l}$ taken along $\boldsymbol{\mathcal{U}}^0$, the attenuation coefficient α is given by a simple relation

$$\alpha = \omega/2Q\mathcal{U}^0 \ . \tag{34}$$

Equation (34) shows that the attenuation coefficient measured along the direction of $\boldsymbol{\mathcal{U}}^0$ depends only on intrinsic attenuation parameters a^I_{ijkl}, and does not depend on the inhomogeneity parameter D of the studied plane wave. We remind the reader that this result was obtained with the use of the first-order perturbation method. Consequently, the result is approximate and valid only for weakly inhomogeneous plane waves propagating in weakly dissipative media. In this approximation, we can write $\boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{U}}^0$ in (31)-(34). A similar conclusion was first obtained by Deschamps and Assouline (2000) in the numerical study of plane waves in orthorhombic anisotropic media.

If we use the profile $\mathbf{l}$ along the energy-velocity vector $\boldsymbol{\mathcal{U}}^0$, the final expression for the dissipation filter in weakly dissipative anisotropic media reads

$$\mathcal{D}(\omega) = \exp[-\omega(s - s_0)/2Q\mathcal{U}^0] \ . \tag{35}$$

Here Q is given by (24), and $\mathcal{U}^0 = |\boldsymbol{\mathcal{U}}^0|$ by (23). For weakly inhomogeneous plane waves, the attenuation coefficient α and the dissipation filter $\mathcal{D}(\omega)$ do not depend on the inhomogeneity parameter D of the wave and are the same as for a homogeneous wave. Equations (34) and (35) remain valid also for isotropic media, where $\mathcal{U}^0$ and Q are direction independent.

Eq.(31) can be used to investigate the attenuation coefficients $\alpha(\mathbf{l})$ in other directions $\mathbf{l}$ than the direction of the energy-velocity vector $\boldsymbol{\mathcal{U}}^0$. For other two interesting examples see the Appendix D.

Weakly inhomogeneous waves in heterogeneous anisotropic, weakly dissipative media

The results derived in previous sections for weakly inhomogeneous plane waves propagating in homogeneous, weakly dissipative, anisotropic media can be simply generalized for high-frequency elementary waves (with a curved wavefront) propagating in heterogeneous weakly dissipative anisotropic media. Only smoothly varying heterogeneous media are considered. In this case, it is natural to consider the ray method, particularly its zero-order approximation. The ray method is based on the concept of elementary waves propagating along rays, and on the local application of plane-wave theory along rays. The computation of rays in heterogeneous, perfectly elastic, anisotropic media is well understood, even for models with structural interfaces. For a detailed review with many references see Červený (2001, sec. 3.6). A decisive role in our treatment is played by the fact that the ray trajectory is a trajectory of the energy flux, i.e., it is the trajectory tangent to the energy-velocity vector at any of its points. Thus, for the infinitesimal length element $\mathrm{d}s$ of the ray, we can use the expression (35) for the dissipation filter, as it corresponds to the direction of the energy-velocity vector,

$$\mathcal{D}(\omega) = \exp[-\omega \mathrm{d}s/2Q\mathcal{U}^0] \; . \tag{36}$$

Here Q is again given by (24) and $\mathcal{U}^0 = |\boldsymbol{\mathcal{U}}^0|$ follows from (23). In heterogeneous media, Q and $\mathcal{U}^0$ vary along the ray. Multiplication of dissipation filters along the whole ray, from the initial point S to the end point R, gives

$$\mathcal{D}(\omega) = \exp\left[-\frac{\omega}{2}\int_S^R \frac{\mathrm{d}s}{Q\mathcal{U}^0}\right] \; , \tag{37}$$

where the integration is taken along the ray. If we take the integration variable corresponding to the real-valued travel time T, $\mathrm{d}T = \mathrm{d}s/\mathcal{U}^0$, we obtain

$$\mathcal{D}(\omega) = \exp\left[-\frac{\omega}{2}\int_S^R Q^{-1}\mathrm{d}T\right] \; . \tag{38}$$

Thus, the dissipation filter for weakly dissipative, heterogeneous, anisotropic media is given by a simple expression (38). Its computation requires merely the quadratures of Q^{-1}, where Q^{-1} is given by (24), along the ray.

In seismology, when considering weakly dissipative isotropic media, the quantity $t^*(R,S)$ has been often used. It is related to $\mathcal{D}(\omega)$ by the relation $\mathcal{D}(\omega) = \exp[-\mathrm{i}\frac{\omega}{2}t^*(R,S)]$. Consequently, we can simply generalize $t^*(R,S)$ for anisotropic media,

$$t^*(R,S) = \int_S^R Q^{-1}\mathrm{d}T \; , \tag{39}$$

where Q^{-1} is again given by (24), and may vary along the ray.

Equations (38) and (39) are valid for weakly inhomogeneous elementary waves propagating in heterogeneous, weakly dissipative, but generally anisotropic media. They remain valid even for media with structural interfaces, and for the point S representing a point source. In such a case, effects of attenuation on the R/T coefficients must be, however, also taken into account. Consequently, equation (38) or (39) can be used in the derivation of the ray-theory Green function in smoothly varying, weakly dissipative, anisotropic media.

Equations analogous to (38) and (39), with the quality factor Q given by (24), have been derived by Gajewski and Pšenčík (1992). Gajewski and Pšenčík (1992) used a method different from that presented here, which was fully based on the ray theory concepts. The most valuable result they obtained was the expression for Q, which fully coincides with (24). The inhomogeneous waves, however, were not considered in their derivation. Nevertheless, the result of Gajewski and Pšenčík (1992) for Q is valid even for weakly inhomogeneous elementary waves as we have shown here. Moreover, it is consistent with the definition of Q, $Q^{-1} = W_d/E$. Note that the same method, based on ray theory concepts, has also been used by Moczo et al. (1987) for smoothly varying isotropic weakly dissipative media.

Equations analogous to those derived in this section could be also obtained if we considered the well-known expressions for the perturbation of travel time along the ray due to the small perturbation of elastic moduli, see Červený (1982), Červený and Jech (1982), Hanyga (1982), Farra (1989), Jech and Pšenčík (1989) etc. It would be only necessary to substitute the real-valued structural perturbation by $-\mathrm{i}a^I_{ijkl}$. For a brief exposition, see Červený (2001, sec. 3.9).

Numerical examples

In this section we illustrate behaviour of the intrinsic attenuation factor $\mathcal{A}^{in}$, which represents inverse of the quality factor $Q^{-1} = \mathcal{A}^{in}$, in different sections of a weakly viscoelastic anisotropic medium and for varying size of the imaginary part of the viscoelastic modulus A_{33}. Computations of both quantities are based on eqs (21) and/or (24). As mentioned in the text, for weakly inhomogeneous waves, the mentioned formulae are independent of the choice of the inhomogeneity parameter D. It means that the plots presented in this section hold for homogeneous as well as weakly inhomogeneous waves.

In the following, we consider Cartesian coordinate system with x_1 and x_2 axes horizontal and x_3 axis vertical, positive downwards. The coordinate system is right handed. We study behaviour of the intrinsic attenuation factor $\mathcal{A}^{in}$ in vertical planes whose orientation is specified later. In each such plane, we construct polar diagrams, in which $\mathcal{A}^{in}$ is shown as a function of the direction specified by the unit vector $\mathbf{n}$, perpendicular to the wavefront. The vector $\mathbf{n}$ is parameterized by the propagation angle i; $\mathbf{n} \equiv (\sin i, 0, \cos i)$, with $i = 0^0$ upwards, and $i = 90^0$ to the right.

We consider the model of an anisotropic viscoelastic medium proposed by Jakobsen et al. (2003) and used by Červený and Pšenčík (2005, 2006). As in the previous papers, we choose again the model corresponding to the frequency of approximately 35 Hz. For simplicity, we consider the density of 1000 kg/m^3. The 6×6 matrix of complex-valued, density-normalized viscoelastic moduli $\mathbf{A_C}$ in the Voigt notation, measured in (km/s)^2 reads:

$$\mathbf{A_C} = \mathbf{A_1} - \mathrm{i}\mathbf{A_2}, \tag{40}$$

where

$$\mathbf{A_1} = \begin{pmatrix} 46.631 & 5.983 & 4.278 & 0. & 0. & 0. \\ & 46.631 & 4.278 & 0. & 0. & 0. \\ & & 19.931 & 0. & 0. & 0. \\ & & & 13.444 & 0. & 0. \\ & & & & 13.444 & 0. \\ & & & & & 20.324 \end{pmatrix}$$

and

$$\mathbf{A_2} = \begin{pmatrix} 0.033 & 0.022 & 0.156 & 0. & 0. & 0. \\ & 0.033 & 0.156 & 0. & 0. & 0. \\ & & 1.312 & 0. & 0. & 0. \\ & & & 0.055 & 0. & 0. \\ & & & & 0.055 & 0. \\ & & & & & 0.005 \end{pmatrix}.$$

Both matrices $\mathbf{A_1}$ and $\mathbf{A_2}$ are positive definite. The matrix $\mathbf{A_1}$ corresponds to an anisotropic medium of hexagonal symmetry with vertical axis of symmetry.

Let us rotate the above matrices so that the axis of symmetry of the matrix $\mathbf{A_1}$, which is vertical in (40), becomes horizontal and parallel with the x_1 axis. The rotated matrix $\mathbf{A_1}$ thus describes a transversely isotropic medium with a horizontal axis of symmetry (HTI). In Figure 1, we study behaviour of the intrinsic attenuation factor $\mathcal{A}^{in}$ in four vertical planes, whose azimuths (angles between the plane and the x_1 axis) are 0^0, 30^0, 60^0 and 90^0. For 0^0, the vertical plane contains the x_1 axis, i.e. axis of symmetry, and thus the vertical plane represents the symmetry plane. For 90^0, the considered vertical plane is perpendicular to the x_1 axis, i.e., axis of symmetry and thus the plane represents the isotropy plane. In order to make the values of $\mathcal{A}^{in}$ comparable in the mentioned four vertical planes, we use the same scales in all four plots. Red colour corresponds to the wave with greatest phase velocity, i.e. P wave. Blue colour corresponds to the slower of the S waves, SV wave in our case. $\mathcal{A}^{in}$ of the SH wave is black. We can see that for the azimuth of 0^0, $\mathcal{A}^{in}$ of the P wave exceeds $\mathcal{A}^{in}$ of both S waves in the horizontal direction. Maximum values of $\mathcal{A}^{in}$ of the SV wave are for propagation angles close to 45^0. Behaviour of $\mathcal{A}^{in}$ of the SH wave is very similar to the behaviour of $\mathcal{A}^{in}$ of the P wave, but it is significantly smaller (SH wave is least attenuated). We can see that $\mathcal{A}^{in}$ decreases with increasing azimuth. We can also see that outside the symmetry plane, the strongest attenuation is connected with the SV wave. The directional character of the $\mathcal{A}^{in}$ of the SV wave outside the symmetry plane becomes similar to that of the P wave for 0^0, i.e. $\mathcal{A}^{in}$ of the SV wave has two lobes. For 90^0, it is in the isotropy plane, $\mathcal{A}^{in}$ of all waves have circular character. Attenuation of P and SH wave is practically zero, the attenuation of the SV has finite, but in comparison with other azimuths, small value.

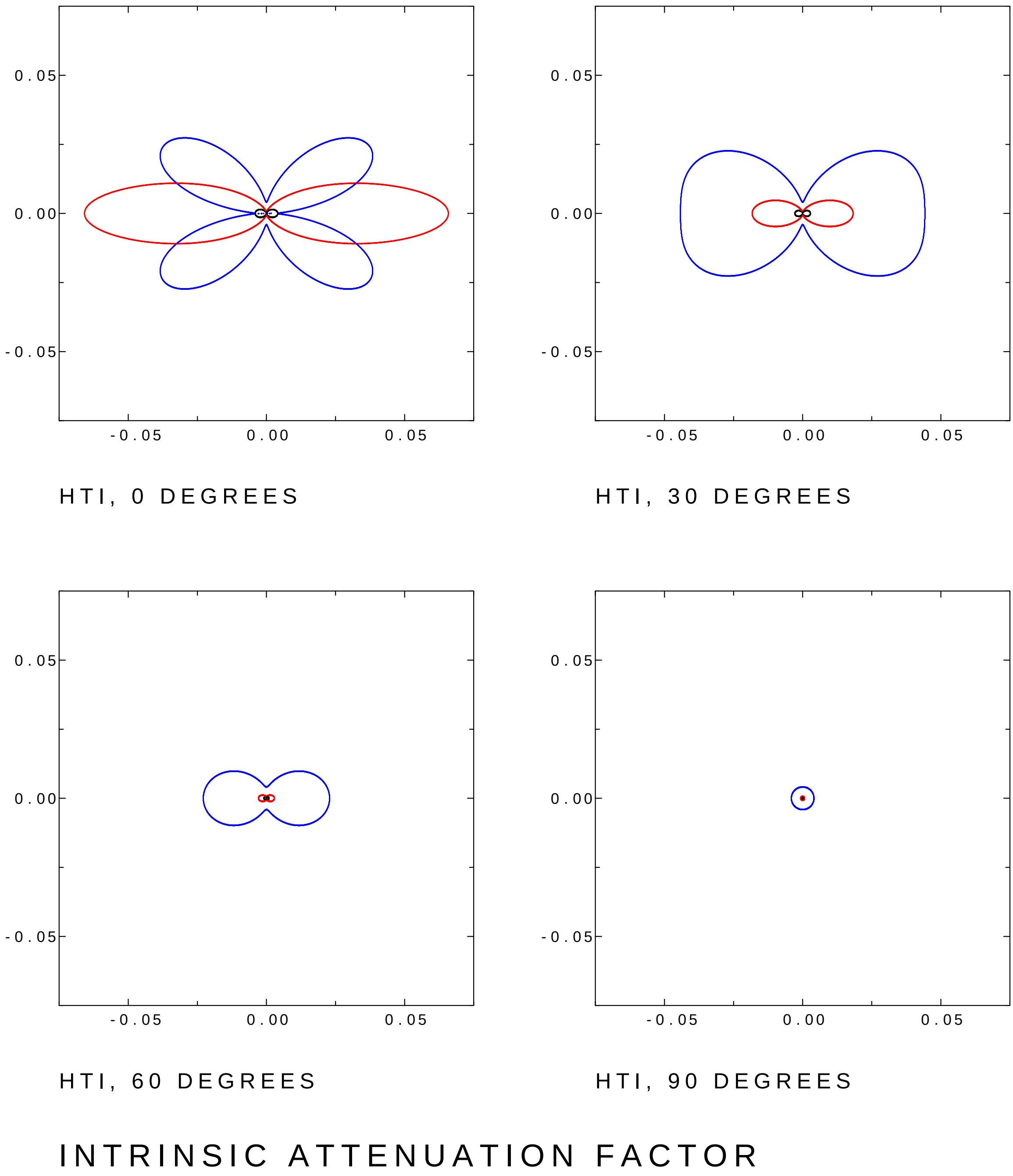

Fig.1 Polar diagrams of $\mathcal{A}^{in}$ for varying azimuth (angle from the horizontal axis of symmetry of the medium specified by $\mathbf{A_1}$). Red corresponds to the fastest plane wave (P wave), black to the faster of S waves (SH wave) and blue to the slowest wave (SV wave). The plot for 0^0 is made in the plane of symmetry of $\mathbf{A_1}$, the plot for 90^0 is made in the isotropy plane.

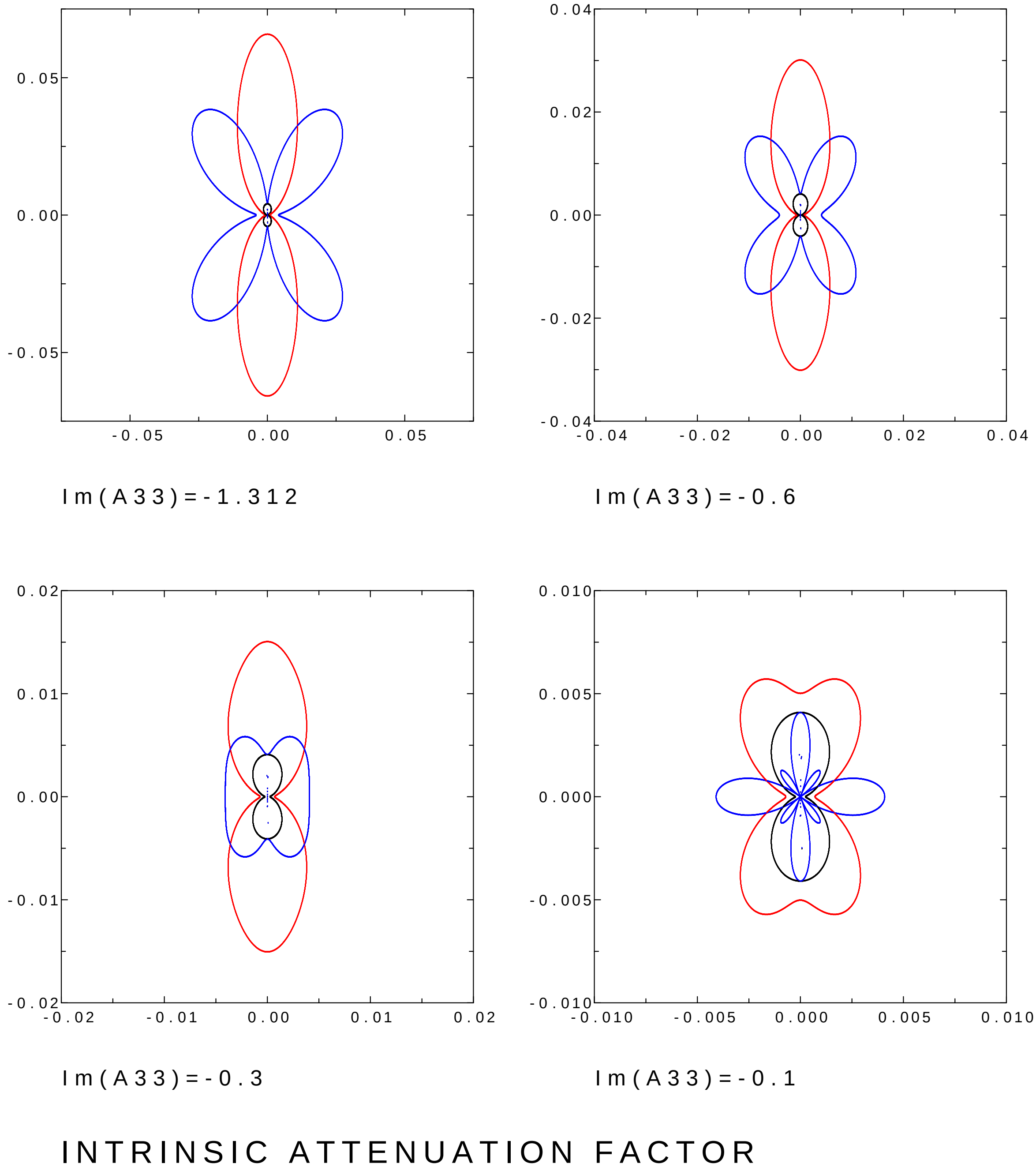

Fig.2 Polar diagrams of $\mathcal{A}^{in}$ for varying values of Im(A_{33} in the plane of symmetry of $\mathbf{A_1}$. Red corresponds to the fastest plane wave (P wave), black to the faster of S waves (SH wave) and blue to the slowest wave (SV wave).

The most important feature of the matrix $\mathbf{A_2}$ is the extremely great value of $(\mathbf{A_2})_{33}$ and small values of $(\mathbf{A_2})_{44}$, $(\mathbf{A_2})_{55}$ and $(\mathbf{A_2})_{66}$. They indicate that the attenuation of SH waves is considerably smaller than the attenuation of P and SV waves. In Figure 2, we show behaviour of the intrinsic attenuation factor $\mathcal{A}^{in}$ for varying values of ImA_{33}. The upper left plot shows $\mathcal{A}^{in}$ corresponding to the model (40). This plot coresponds to the upper left plot in Fig. 1, rotated by 90^0. In the following three plots, the value of ImA_{33} is successively reduced to -0.6, -0.3 and -0.1. One of the most interesting features is $\mathcal{A}^{in}$ of P wave, which exceeds in directions close to vertical the values of $\mathcal{A}^{in}$ of S waves. Another interesting feature is the simplification of the four-lobe character of $\mathcal{A}^{in}$ of the SV-wave (for Im$A_{33} = -0.3$) and then its considerable complication: eight lobes for Im$A_{33} = -0.1$. The two-lobe character of $\mathcal{A}^{in}$ of the SH wave remains unchanged. In the vertical direction values of $\mathcal{A}^{in}$ of the SH wave become comparable with the corresponding values of the SV wave. In contrast to Fig. 1, the scales of plots vary with varying ImA_{33}. We can see, expectably, that $\mathcal{A}^{in}$ decreases with decreasing absolute value of ImA_{33}.

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Appendix A

Complex velocity

It has been common to introduce complex velocity v_c when studying wave propagation in anelastic media. It is defined as $v_c^2 = (\mathbf{pp})^{-1}$, see e.g., Carcione (2001; eq. 4.28), Krebes and Le (1994), Moczo et al. (1987), etc. Carcione (2001; eq. 4.92) uses the complex velocity to define the quality factor Q of homogeneous plane waves in viscoelastic anisotropic media as

$$Q = \mathrm{Re}v_c^2/\mathrm{Im}v_c^2 \ . \qquad (A-1)$$

Krebes and Le (1994) tried to use this definition of Q also for inhomogeneous waves, but they came to the conclusion that (A-1) "gives a generally unreliable measure of the anelasticity". They used therefore (12) for definition of Q.

For weakly dissipative media ($\mathrm{Im}v_c^2 \ll 1$), eq. (A-1) can be rewritten in the following way:

$$Q = \mathrm{Re}v_c/2\mathrm{Im}v_c \ . \qquad (A-2)$$

Since v_c can be expressed through eqs (9) and (22), we can see that Q in (A-2) depends not only on intrinsic medium parameters through the intrinsic attenuation factor $\mathcal{A}^{in}$, but also on the inhomogeneity parameter D of the considered wave. This confirms the conclusion of Krebes and Le (1994) that Q defined in (A-1) is not a good measure of anelasticity for anisotropic media.

Let us now consider the relations (13):

$$\mathbf{P}\cdot\mathbf{S} = E \ , \quad \mathbf{A}\cdot\mathbf{S} = \frac{1}{2}W_d \ . \qquad (A-3)$$

Normalization of the above relations by the complete energy E leads to

$$\mathbf{P} \cdot \boldsymbol{\mathcal{U}} = 1 \ , \quad \mathbf{A} \cdot \boldsymbol{\mathcal{U}} = \frac{1}{2} Q^{-1} \ , \tag{$A-4$}$$

see eq. (12). If we express the energy-velocity vector $\boldsymbol{\mathcal{U}}$ as $\boldsymbol{\mathcal{U}} = \mathcal{U} \cdot \mathbf{t}$, where $\mathbf{t}$ is a unit vector in the direction of the energy flux $\mathbf{S}$, we can get from (A-4):

$$Q = (\mathrm{Re}\mathbf{p} \cdot \mathbf{t})/2(\mathrm{Im}\mathbf{p} \cdot \mathbf{t}) \ . \tag{$A-5$}$$

We can see that the generalization of the definition of Q for dissipative anisotropic media in terms of complex velocity should have the form of (A-5) rather than of (A-2). Let us emphasize that eq. (A-5) is exact, and fully corresponds to the definition (12). For weakly inhomogeneous plane waves propagating in anisotropic, weakly dissipative media, eqs (9) and (22) yield for $\mathbf{p} \cdot \mathbf{t}$ the following simple expression:

$$\mathbf{p} \cdot \mathbf{t} = \mathcal{U}^{-1}(1 + \frac{\mathrm{i}}{2} \mathcal{A}^{in}) \ . \tag{$A-6$}$$

We can see that (A-5) with (A-6) defines Q as a parameter, which depends only on material parameters, not on D, and that

$$Q = (\mathcal{A}^{in})^{-1} \ . \tag{$A-7$}$$

For an isotropic weakly dissipative medium, for which $\mathbf{p}$ and $\mathbf{t}$ are parallel and $\mathcal{U} = \mathcal{C} = \mathrm{Re} v_c$, eq.(A-6) yields

$$v_c^{-1} = \mathcal{C}^{-1}(1 + \frac{\mathrm{i}}{2} Q^{-1}) \ , \tag{$A-8$}$$

which is identical with eq. (10) of Moczo et al. (1987).

Let us mention here one more interesting aspect suggesting preferential use of the mixed specification (7) of the slowness vector instead of the specification (9). Krebes and Le (1994) used the specification (9), in which they used the attenuation angle γ as a measure of inhomogeneity of a considered wave. They found that for both definitions of Q, which they used, the quality factor depended on the angle γ, and thus on the considered wave. If we use the mixed specification, in which the inhomogeneity of a wave is characterized by the inhomogeneity parameter D, we find that Q defined in (A-5) is independent of the inhomogeneity parameter of the wave; it depends only on the parameters of the medium.

Appendix B

Specifications of slowness vectors

The complex-valued slowness vectror $\mathbf{p}$ can be specified in several ways. In eq. (7), we presented very useful mixed specification. Below we present another one.

In seismological literature, the so-called directional specification of the slowness vector, $\mathbf{p} = \mathcal{C}^{-1}(\mathbf{N} + \mathrm{i}\delta\mathbf{M})$, following from (9), has been used to study the propagation of inhomogeneous plane waves, particularly in isotropic viscoelastic media. Here $\mathbf{N}$ and $\mathbf{M}$ are real-valued unit vectors in the direction of $\mathbf{P}$ and $\mathbf{A}$, and are assumed to be given.

The unknown real-valued quantities $\mathcal{C}$ and δ are to be determined by inserting the above expression for $\mathbf{p}$ into (6). The quantity $\mathcal{C}$ represents again the real-valued phase velocity in the direction of $\mathbf{N}$. For isotropic viscoelastic media, the directional specification of the slowness vector does not cause any difficulty; see Červený and Pšenčík (2005), where many other references can be found. For general viscoelastic anisotropic media, however, the situation is more complicated. Two real-valued coupled polynomial equations for $\mathcal{C}^2$ and δ are obtained, which are of the third degree in $\mathcal{C}^2$ and of the sixth degree in δ. The system of two coupled equations can be decoupled, but a more complicated non-polynomial form of equation for δ is obtained. The main problem in the directional specification is, that the equations yield non-physical solution for $\mathcal{C}^2$, namely $\mathcal{C}^2 < 0$, for certain combinations of $\mathbf{N}$ and $\mathbf{M}$ (called forbidden directions). See Krebes and Le (1994), Romeo (1994), Carcione and Cavallini (1995), Carcione (2001). The problem of forbidden directions is fully avoided by the use of the mixed specification (7). In it, the unit vector $\mathbf{M}$ is not given, but computed ($\mathbf{M} = \mathbf{A}/|\mathbf{A}|$). Similarly, the attenuation angle γ is not given, but computed from (11). The mixed specification (7) can be used quite universally, for homogeneous and inhomogeneous, P, S1 and S2 plane waves, propagating in isotropic or anisotropic, perfectly elastic or viscoelastic media. The viscoelasticity and anisotropy are not restricted, and the value of inhomogeneity parameter D can be arbitrary.

Appendix C

Weak-attenuation parameters

In eqs (21) and (24), we found

$$Q^{-1} = \mathcal{A}^{in} = a^I_{ijkl} p^0_j p^0_l U^0_i U^0_k \ , \qquad (C-1)$$

where $-a^I_{ijkl}$ are imaginary parts of the density-normalized viscoelastic moduli ($|a^I_{ijkl}| \ll 1$), see (1), p^0_i and U^0_i are components of the slowness and polarization vectors in the reference, perfectly elastic medium specified by a^R_{ijkl}, see (1). We can rewrite (C-1) in the following way

$$Q^{-1} = \mathcal{A}^{in} = (\mathcal{C}^0)^{-2} a^I_{ijkl} n_j n_l U^0_i U^0_k \ . \qquad (C-2)$$

The right-hand side of (C-2) has formally the same form as the expressions for the diagonal terms of the matrix $B_{mn} = a_{ijkl} n_j n_l e^m_i e^n_k$ introduced by Farra and Pšenčík (2003) for weakly anisotropic media. As shown by Farra and Pšenčík (2003), these diagonal terms can be expressed in terms of weak-anisotropy parameters, which can substitute, in weakly anisotropic media, standard density-normalized elastic moduli $A_{\alpha\beta}$. The right-hand side of eq. (C-2) can be expressed in a similar way in terms of weak-attenuation parameters.

For brevity, we concentrate here only on the P-wave expression for the quality factor Q_P. For a viscoelastic medium of weak, but unrestricted anisotropy, Q_P reads

$$Q_P^{-1} = \mathcal{A}_P^{in} = 2(\alpha/\mathcal{C}^0)^2 \Big(\epsilon^I_x n_1^4 + \epsilon^I_y n_2^4 + \epsilon^I_z n_3^4 + \delta^I_x n_1^2 n_3^2 + \delta^I_y n_2^2 n_3^2 + \delta^I_z n_1^2 n_2^2 + 2[(\epsilon^I_{16} n_2 + \epsilon^I_{15} n_3) n_1^3$$
$$+ (\epsilon^I_{24} n_3 + \epsilon^I_{26} n_1) n_2^3 + (\epsilon^I_{35} n_1 + \epsilon^I_{34} n_2) n_3^3 + (\chi^I_x n_1 + \chi^I_y n_2 + \chi^I_z n_3) n_1 n_2 n_3] \Big) \ . \qquad (C-3)$$

In (C-3), α is a formal parameter with dimension of velocity, used to define dimensionless weak-attenuation parameters, see below. The nonzero parameter α can be chosen arbitrarily. The Q factor does not depend on α. The weak-attenuation parameters in (C-3) are defined as follows:

$$\epsilon_x^I = \frac{A_{11}^I}{2\alpha^2}, \quad \epsilon_y^I = \frac{A_{22}^I}{2\alpha^2}, \quad \epsilon_z^I = \frac{A_{33}^I}{2\alpha^2},$$

$$\delta_x^I = \frac{A_{13}^I + 2A_{55}^I}{\alpha^2}, \quad \delta_y^I = \frac{A_{23}^I + 2A_{44}^I}{\alpha^2}, \quad \delta_z^I = \frac{A_{12}^I + 2A_{66}^I}{\alpha^2},$$

$$\chi_x^I = \frac{A_{14}^I + 2A_{56}^I}{\alpha^2}, \quad \chi_y^I = \frac{A_{25}^I + 2A_{46}^I}{\alpha^2}, \quad \chi_z^I = \frac{A_{36}^I + 2A_{45}^I}{\alpha^2},$$

$$\epsilon_{15}^I = \frac{A_{15}^I}{\alpha^2}, \quad \epsilon_{16}^I = \frac{A_{16}^I}{\alpha^2}, \quad \epsilon_{24}^I = \frac{A_{24}^I}{\alpha^2}, \quad \epsilon_{26}^I = \frac{A_{26}^I}{\alpha^2}, \quad \epsilon_{34}^I = \frac{A_{34}^I}{\alpha^2}, \quad \epsilon_{35}^I = \frac{A_{35}^I}{\alpha^2}, \qquad (C-4)$$

If we put the weak-attenuation parameters zero, Q_P^{-1} in (C-3) becomes zero too, i.e., the medium is perfectly elastic.

The expression for Q_P^{-1} in (C-3) depends, in addition to the weak-attenuation parameters (C-4), also on the square of the phase velocity $\mathcal{C}^0$ in the perfectly elastic reference medium. The phase velocity $\mathcal{C}^0$ is, of course, directionally dependent (although not as strongly as the Q factor). If the reference medium is weakly anisotropic, then the square of the phase velocity $\mathcal{C}^0$ can be written in the form

$$(\mathcal{C}^0)^2 = \alpha^2[1 + 2Y(n_i)] \ . \qquad (C-5)$$

Here $Y(n_i)$ stands for the linear combination of weak-anisotropy parameters, with coefficients containing powers of n_i. For exact expression for $Y(n_i)$ for P waves, see Pšenčík and Farra (2005; eq. 12). Since both weak-attenuation and weak-anisotropy parameters are first-order quantities, the expression (C-3) simplifies:

$$Q_P^{-1} = \mathcal{A}_P^{in} = 2\Big(\epsilon_x^I n_1^4 + \epsilon_y^I n_2^4 + \epsilon_z^I n_3^4 + \delta_x^I n_1^2 n_3^2 + \delta_y^I n_2^2 n_3^2 + \delta_z^I n_1^2 n_2^2 + 2[(\epsilon_{16}^I n_2 + \epsilon_{15}^I n_3) n_1^3$$
$$+(\epsilon_{24}^I n_3 + \epsilon_{26}^I n_1) n_2^3 + (\epsilon_{35}^I n_1 + \epsilon_{34}^I n_2) n_3^3 + (\chi_x^I n_1 + \chi_y^I n_2 + \chi_z^I n_3) n_1 n_2 n_3]\Big) \ . \qquad (C-6)$$

In the case of transverse isotropy with vertical axis of symmetry (VTI), we have

$$\chi_x^I = \chi_y^I = \chi_z^I = \epsilon_{15}^I = \epsilon_{16}^I = \epsilon_{24}^I = \epsilon_{26}^I = \epsilon_{34}^I = \epsilon_{35}^I = 0 \ ,$$

$$\epsilon_x^I = \epsilon_y^I = \frac{1}{2}\delta_z^I, \qquad \delta_x^I = \delta_y^I \ . \qquad (C-7)$$

The expression for P-wave quality factor Q_P reduces then to:

$$Q_P^{-1} = 2[\epsilon_x^I (1 - n_3^2)^2 + \epsilon_z^I n_3^4 + \delta_x^I (1 - n_3^2) n_3^2] \ . \qquad (C-8)$$

Appendix D

Attenuation coefficients in various directions

Eq. (31), can be also used to investigate the attenuation coefficients in other directions $\mathbf{l}$ than the direction of the energy-velocity vector $\boldsymbol{\mathcal{U}}^0$. We present here two important examples.

i) **Profile along the wavefront**. We introduce $\mathbf{l} = \mathbf{m}'$, where $\mathbf{m}'$ is a unit vector perpendicular to $\mathbf{n}$ (i.e., $\mathbf{m}'$ is tangent to the wavefront). We then obtain from (31):

$$\alpha(\mathbf{m}') = \omega D(\mathbf{m} \cdot \mathbf{m}') \ . \qquad (D-1)$$

The attenuation coefficient $\alpha(\mathbf{m}')$ along the profile $\mathbf{m}'$ does not depend on intrinsic dissipation parameters, but only on $(\mathbf{m} \cdot \mathbf{m}')$ and on the inhomogeneity parameter D. For $\mathbf{m}' = \mathbf{m}$, we have $\alpha(\mathbf{m}') = \omega D$. Similarly, for $\mathbf{m}' = -\mathbf{m}$, we have $\alpha(\mathbf{m}') = -\omega D$. For $\mathbf{l} = \mathbf{m}'$, where $\mathbf{m}'$ is perpendicular to $\mathbf{m}$, that is for $\mathbf{m}'$ perpendicular to the propagation-attenuation plane, coefficient of attenuation is zero.

ii) **Profile in the direction perpendicular to the wavefront**. In this case, $\mathbf{l} = \mathbf{n}$, and (31) yields

$$\alpha(\mathbf{n}) = \omega(\mathcal{C}^0)^{-1}[1/2Q - D(\boldsymbol{\mathcal{U}}^0 \cdot \mathbf{m})] \ . \qquad (D-2)$$

In anisotropic media, $\boldsymbol{\mathcal{U}}^0 \cdot \mathbf{m}$ is generally different from zero, so that the attenuation coefficient $\alpha(\mathbf{n})$, measured along the profile $\mathbf{l} = \mathbf{n}$, is practically always influenced by the inhomogeneity parameter D of the wave under consideration. For certain $\mathbf{n}$ ($\mathbf{n}$ specifies the orientation of the slowness vector), the attenuation coefficient $\alpha(\mathbf{n})$ may be even negative. This case was also briefly discussed by Červený and Pšenčík (2005). Thus, the profile $\mathbf{l} = \mathbf{n}$ is not suitable for measurements of intrinsic dissipation in anisotropic media.

If we wish to determine the quality factor Q of inhomogeneous plane waves from the attenuation coefficient α, it is most suitable to use the profile parallel to the energy-velocity vector $\boldsymbol{\mathcal{U}}^0$. Then we obtain $Q^{-1} = 2\mathcal{U}^0\alpha/\omega$. Alternatively, we can use the profile perpendicular to wavefronts. In order to get Q independent of the inhomogeneity parameter D, however, we have to consider homogeneous plane waves ($D = 0$) only, see (D-2). Then we obtain $Q^{-1} = 2\mathcal{C}^0\alpha(\mathbf{n})/\omega$, where $\mathcal{C}^0$ is the phase velocity, given by (21). Both approaches yield the same Q, given by (24).

The mentioned two alternatives are, however, only applicable to the determination of the quality factor Q of plane waves, propagating in *homogeneous* anisotropic weakly dissipative media. For a point-source-receiver configuration in *inhomogeneous* anisotropic weakly dissipative media, the quality factor is calculated always along rays, i.e., along the direction of the energy-velocity vector, see the last section. Moreover, in the point-source-receiver configuration in anisotropic dissipative media, we always deal with inhomogeneous waves so that $\alpha(\mathbf{n})$ cannot be measured.