Quality factor Q in dissipative anisotropic media

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ABSTRACT

In an isotropic dissipative medium, the attenuation properties of rocks are usually specified by quality factor Q, which is a positive, dimensionless, real-valued, scalar quantity, independent of the direction of wave propagation. We propose a similar, scalar, but direction-dependent quality factor for time-harmonic, homogeneous or inhomogeneous plane waves, propagating in unbounded homogeneous dissipative anisotropic media. We define the quality factor, similarly as in isotropic viscoelastic media, as the ratio of the time-averaged complete stored energy and the dissipated energy, per unit volume. A solution of an algebraic equation of the sixth degree with complex-valued coefficients is necessary for the exact determination of Q. For weakly inhomogeneous plane waves, propagating in arbitrarily anisotropic, weakly dissipative media, we simplify the exact expression for Q considerably using the perturbation method. The solution of the equation of the sixth degree is no longer required. We obtain a simple explicit perturbation expression for quality factor, denoted $\hat{Q}$. We prove that the direction-dependent $\hat{Q}$ is related to the attenuation coefficient α measured along a profile in the direction of the energy-velocity vector (ray direction). $\hat{Q}$ does not depend on the inhomogeneity of the plane wave under consideration, and thus may be used as a convenient measure of the intrinsic dissipative properties of rocks in the ray direction. In all other directions, the quality factor is influenced by the inhomogeneity of the wave under consideration. We illustrate the peculiarities in the behavior of $\hat{Q}$ and its accuracy on a model of anisotropic, weakly dissipative sedimentary rock. Examples show interesting inner loops in polar diagrams of $\hat{Q}$ in regions of S-wave triplications.

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INTRODUCTION

The dissipative properties of rocks are commonly specified by the quality factor Q. In isotropic dissipative media, Q is a positive dimensionless scalar quantity, independent of the direction of wave propagation. Review of definitions, terminology and observations in seismology and seismic exploration, devoted to isotropic dissipative media can be found in Aki and Richards (1981), Johnston and Toksöz (1981), Cormier (1989) and Carcione (2001, 2006, 2007). The most common definition of the quality factor Q of time-harmonic plane P and S waves propagating in homogeneous isotropic weakly dissipative (low-loss) isotropic media is $Q = -\mathrm{Re}(M)/\mathrm{Im}(M)$, where M is the complex-valued P- or S-wave viscoelastic modulus. Alternatively, the definition may take the form $Q = -\mathrm{Re}(v_c^2)/\mathrm{Im}(v_c^2)$, where v_c is the complex-valued P- or S-wave phase velocity (e.g., Johnston and Toksöz, 1981; Moczo et al., 1987). The signs in the expressions for Q depend on the sign convention used for time-harmonic waves. With the sign convention used in this paper (the exponential time factor $\exp(-\mathrm{i}\omega t)$), $\mathrm{Im}(M)$ and $\mathrm{Im}(v_c^2)$ are negative, and expressions for Q thus contain minus signs.

In this paper, we are interested in the direction-dependent quality factor Q for anisotropic dissipative media. We consider only such dissipation mechanisms, which can be described within the framework of linear viscoelasticity. Classically, quality factor Q of time-harmonic plane waves has been defined by the relation $Q^{-1} = W_d/E$ (Buchen, 1971), where W_d is the density of the time-averaged dissipated energy and E is the density of the time-averaged complete stored energy, $E = K + U$. K and U are densities of the time-averaged kinetic and strain energies, respectively. For perfectly elastic media U equals K, but for viscoelastic media U generally differs from K. Modified versions of the definition $Q^{-1} = W_d/E$ have been sometimes used in the seismological literature. For example, Carcione (2001, pp. 58-59) used the definition $Q^{-1} = W_d/2U$. This definition was adopted by Červený and Pšenčík (2006), and also by Zhu and Tsvankin (2006, 2007). In this paper, we consider systematically the classical definition $Q^{-1} = W_d/E$ of Buchen (1971). The definition is applicable to homogeneous and inhomogeneous plane waves propagating in isotropic as well as anisotropic viscoelastic media.

For time-harmonic plane waves, propagating in homogeneous, anisotropic, viscoelastic media, quality factor Q has often been introduced in the same way as for isotropic media, using the definition $Q = -\mathrm{Re}(v_c^2)/\mathrm{Im}(v_c^2)$, where v_c is the complex-valued phase velocity of the plane wave under consideration. Individual approaches to the determination of quality factor Q, based on the above definition, differ in the way of computation of complex-valued phase velocities, in the specification of the complex-valued slowness vector, and in the treatment of inhomogeneous waves (e.g., Carcione, 1994, 2001, 2007; Krebes and Le, 1994; Chichinina et al., 2006). The applicability of the above definition of Q to anisotropic media is, however, problematic, particularly for inhomogeneous plane waves. For example, Krebes and Le (1994) showed that the above definition of Q "gives a generally unreliable measure of anelasticity" for inhomogeneous plane waves propagating in anisotropic viscoelastic media. This might, however, be the consequence of inappropriate parametrization of the slowness vectors of inhomogeneous plane waves using the attenuation angle.

Inhomogeneous waves play an important role in the propagation of waves in viscoelastic, isotropic or anisotropic, media. The slowness vector $\mathbf{p}$ of waves propagating in viscoelastic media is always complex-valued: $\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A}$, where $\mathbf{P}$ is the propagation vector, $\mathbf{A}$ the attenuation vector and i is the imaginary unit. For $\mathbf{A}$ parallel to $\mathbf{P}$ or for $\mathbf{A} = \mathbf{0}$, the wave is called homogeneous; otherwise it is inhomogeneous. The advantage of plane waves propagating in homogeneous viscoelastic media is that their inhomogeneity may be chosen arbitrarily. This is not the case for waves generated by a point source in a viscoelastic medium. The inhomogeneity of these waves results from the solution of the problem under consideration. In most cases, including homogeneous anisotropic viscoelastic media, waves propagating from a point source to a receiver, are inhomogeneous (Červený et al., 2007). Waves generated by a point source are homogeneous only in unbounded homogeneous isotropic viscoelastic media. In general, the waves propagating in viscoelastic anisotropic media are inhomogeneous although their inhomogeneity may be weak. We show that the attenuation of the wave does not depend on the strength of its inhomogeneity only for a certain privileged direction, the direction of energy flux. Inhomogeneous waves play an important role in many branches of physics. Declercq et al. (2005) give an excellent review of ultrasonic inhomogeneous waves, their generation and applications in nondestructive testing. Applications of inhomogeneous waves in seismology and acoustics are studied by Borcherdt and Wennerberg (1985), Borcherdt et al. (1986), Hosten et al. (1987).

In viscoelastic media, the stiffnesses c_{ijkl}, relating the stress and strain tensors, are not real-valued as in perfectly elastic media, but complex-valued. We call them viscoelastic moduli. The imaginary parts of viscoelastic moduli are responsible for the dissipative properties of waves, propagating in viscoelastic media. The sign convention for the imaginary parts of the viscoelastic moduli must correspond to the sign convention used to describe the time-harmonic wave under consideration, particularly to the sign in the exponential time factor $\exp(\pm \mathrm{i}\omega t)$. We use the exponential time factor $\exp(-\mathrm{i}\omega t)$, for which $c_{ijkl} = c^R_{ijkl} - \mathrm{i}c^I_{ijkl}$. To simplify certain relations, we use the density-normalized viscoelastic moduli $a_{ijkl} = c_{ijkl}/\rho$ throughout the paper:

$$a_{ijkl} = a^R_{ijkl} - \mathrm{i}a^I_{ijkl} \ . \tag{1}$$

The density-normalized viscoelastic moduli a_{ijkl} are frequency-dependent in viscoelastic media, but we do not write $a_{ijkl}(\omega)$, as we consider an arbitrary, but fixed $\omega > 0$ in the whole paper. We assume that a_{ijkl} satisfy the symmetry relations,

$$a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij} \ . \tag{2}$$

Due to the symmetry relations 2, the number of density-normalized viscoelastic moduli reduces from 81 to 21. They can be described by the 6×6 symmetric matrix in the Voigt notation, $\bar{\mathbf{A}} = \bar{\mathbf{A}}^R - \mathrm{i}\bar{\mathbf{A}}^I$ with elements $\bar{A}_{\alpha\beta}$ $(\alpha, \beta = 1, 2, ...6)$. The elements can be expressed as $\bar{A}_{\alpha\beta} = \bar{A}^R_{\alpha\beta} - \mathrm{i}\bar{A}^I_{\alpha\beta}$. We use bars in $\bar{\mathbf{A}}$, $\bar{\mathbf{A}}^R$ and $\bar{\mathbf{A}}^I$ to distinguish the matrix $\bar{\mathbf{A}}$ from attenuation vector $\mathbf{A}$. Hereinafter, we call a^I_{ijkl} and $\bar{A}^I_{\alpha\beta}$ the *dissipation moduli*, and the 6×6 matrix $\bar{\mathbf{A}}^I$ the *dissipation matrix*. The so-called $\mathbf{Q}$ matrix with elements $Q_{\alpha\beta} = \bar{A}^R_{\alpha\beta}/\bar{A}^I_{\alpha\beta}$, sometimes used in the literature, is not used in this paper at all. The direct application of dissipation matrix $\bar{\mathbf{A}}^I$ makes all expressions very simple.

We assume that matrix $\bar{\mathbf{A}}^R$ is positive definite (Fedorov, 1968; Auld, 1973; Helbig, 1994; Carcione, 2001, etc.). Similarly, we assume that dissipation matrix $\bar{\mathbf{A}}^I$ is positive definite or zero (Červený and Pšenčík, 2006, p.1302). These assumptions are valid for both sign options of the exponential time factor. In the seismological literature, both signs have been used. Consequently, the reader must be careful in comparing the signs in equations given by different authors.

There are not many studies of the quality factor Q of plane waves propagating in homogeneous viscoelastic anisotropic media. Among them, in the first place, is the important contribution by Carcione and his colleagues (Carcione, 1992, 1994, 2000, 2001, 2007; Carcione and Cavallini, 1993, 1995a). For plane SH waves propagating in the plane of symmetry of an anisotropic medium, we refer to Krebes and Le (1994). For plane P waves propagating in TI and orthorhombic media, we refer to the recent papers by Zhu and Tsvankin (2006, 2007). In the latter references, the authors based their concept of attenuation exclusively on *homogeneous* plane waves. This concept is not used in this paper. The expression for the quality factor Q for homogeneous plane SH waves propagating in homogeneous media of unrestricted viscoelasticity and anisotropy can be found in Carcione (2001, 2007). The expression for Q, valid for homogeneous and inhomogeneous plane waves propagating in media of unrestricted anisotropy and viscoelasticity, can be found in Červený and Pšenčík (2006). Using Carcione's (2001) definition of Q ($Q^{-1} = W_d/2U$) and the so-called mixed specification of the slowness vector, Červený and Pšenčík (2006) derived an exact formula, in which Q was expressed in terms of the viscoelastic moduli and complex-valued slowness and polarization vectors of homogeneous or inhomogeneous waves. To obtain exact Q, it was necessary to solve an algebraic equation of the sixth degree with complex-valued coefficients. Červený and Pšenčík (2008) show that the solution of the algebraic equation of the sixth degree can be avoided, if weakly inhomogeneous plane waves propagating in weakly dissipative anisotropic media are considered and the perturbation approach is applied. We denote the corresponding quality factor $\hat{Q}$, and derive and discuss the relevant perturbation formula for it in this paper.

The first expression for the direction-dependent quality factor for generally anisotropic, weakly dissipative media, mathematically identical with $\hat{Q}$, was given by Gajewski and Pšenčík (1992). To derive $\hat{Q}$, Gajewski and Pšenčík (1992) used a quite different approach, based on the modification of the asymptotic ray-series method. The inhomogeneous waves were not explicitly considered in the derivation. The same expression for $\hat{Q}$ was also derived by Červený (2001, eq. 5.5.21), using the first-order perturbation equation for traveltime. Neither of the mentioned publications did provide answers to two important questions: i) How is the expression derived for $\hat{Q}$ related to the definition $Q^{-1} = W_d/E$? ii) Is the result also valid for inhomogeneous waves, when inhomogeneous waves were not considered in the derivation? We answer these two questions in this paper. We show that the expression for the quality factor, given by Gajewski and Pšenčík (1992) and Červený (2001), is consistent with the definition $Q^{-1} = W_d/E$, and that it is valid not only for homogeneous, but also for weakly inhomogeneous waves. From formulae given by Gajewski and Pšenčík (1992), $\hat{Q}$ for anisotropic weakly dissipative media can be also expressed in terms of a complex-valued energy velocity (ray velocity) $v_r : \hat{Q}^{-1} = -2\mathrm{Im}(v_r)/\mathrm{Re}(v_r)$.

In this paper, we investigate only the quality factor of plane waves propagating in unbounded homogeneous dissipative anisotropic media. Vavryčuk (2007a,b) investigates the attenuation of waves generated by a point source in a similar medium. He defines the so-called ray-quality factor by the relation $Q^{ray} = -\mathrm{Re}(v_r^2)/\mathrm{Im}(v_r^2)$, where

v_r is again the complex-valued energy velocity (Vavryčuk, 2007b, eq. 24). For weakly dissipative anisotropic media, the latter relation immediately yields the expression $\hat{Q}^{-1} = -2\mathrm{Im}(v_r)/\mathrm{Re}(v_r)$, given by Gajewski and Pšenčík (1992), valid locally even for heterogeneous media. A uniform treatment of the quality factor in weakly dissipative anisotropic media, valid for both plane waves and waves generated by point sources, can be found in Červený et al. (2007).

We use the following terminology. We speak of velocity isotropy or anisotropy when the reference perfectly elastic medium, specified by the moduli a^R_{ijkl}, is isotropic or anisotropic. We speak of Q-factor isotropy or anisotropy when the Q factor is or is not directionally dependent. For nonvanishing dissipation moduli a^I_{ijkl}, the velocity anisotropy generally implies Q-factor anisotropy. Only for a special choice of dissipation matrix $\bar{\mathbf{A}}^I$ related to $\bar{\mathbf{A}}^R$ by the relation $\bar{\mathbf{A}}^I = \bar{\mathbf{A}}^R/q$, where q is a real-valued positive constant, we do obtain strict Q-factor isotropy.

The paper begins with a brief review of the theory of time-harmonic, homogeneous and inhomogeneous plane waves, propagating in general viscoelastic anisotropic media. We use the mixed specification of slowness vector $\mathbf{p}$ (Červený and Pšenčík, 2005a), which reduces the problem of determining $\mathbf{p}$ to the solution of an algebraic equation of the sixth degree with complex-valued coefficients. Once the equation is solved, the exact expression for quality factor Q can be evaluated for all P, S1 and S2 homogeneous and inhomogeneous plane waves. For weakly inhomogeneous plane waves propagating in weakly dissipative anisotropic media, we substitute the numerical solution of the algebraic equation by an approximate, but simple, first-order perturbation formula. The expression for quality factor then simplifies considerably. We show that the perturbation formula for quality factor $\hat{Q}$, which follows from Buchen's (1971) definition $Q^{-1} = W_d/E$, is explicitly related to the direction of the energy flux and is independent of the inhomogeneity of the considered plane wave. It is inversely proportional to the intrinsic attenuation factor $\mathcal{A}^{in}$; $\hat{Q}^{-1} = \mathcal{A}^{in}$, where $\mathcal{A}^{in}$ is a scalar quantity, characterizing intrinsic dissipative properties of the medium. Finally, we study the relation of quality factor $\hat{Q}$ to attenuation coefficient α. We show that both quality factor $\hat{Q}$ and attenuation coefficient α are independent of the inhomogeneity of the plane wave under consideration only if they are considered along the ray direction. We illustrate the behavior of Q^{-1} and $\hat{Q}^{-1}$ on numerical examples for a model of a viscoelastic anisotropic medium. Appendix A contains explicit formulae for $\hat{Q}$ for weakly dissipative media of higher-symmetry anisotropy, both for arbitrarily strong anisotropy as well as for the weak-anisotropy approximation.

In the whole paper, we use Cartesian coordinates x_i and time t. The lower-case italic indices $i, j, ...$ take the values of 1,2,3, the Greek indices $\alpha, \beta, ...$ the values of 1,2,...,6. The Einstein summation convention over repeated indices is used. For time-harmonic waves, we consider the exponential time factor $\exp(-\mathrm{i}\omega t)$, where ω is a fixed, real-valued, positive, circular frequency. The dots above letters denote partial derivatives with respect to time ($\ddot{u}_i = \partial^2 u_i/\partial t^2$) and the index following the comma in the subscript indicates the partial derivative with respect to the relevant Cartesian coordinate ($u_{i,j} = \partial u_i/\partial x_j$).

TIME-HARMONIC PLANE WAVES IN VISCOELASTIC ANISOTROPIC MEDIA

Time-harmonic plane waves propagating in an unbounded homogeneous viscoelastic anisotropic medium can be described by the expression

$$u_i(x_j,t) = aU_i \exp[-\mathrm{i}\omega(t - p_n x_n)] \ , \tag{3}$$

where ω is a circular frequency, a is the scalar amplitude, u_i, U_i and p_i are Cartesian components of the complex-valued displacement vector $\mathbf{u}$, the normalized polarization vector $\mathbf{U}$, ($\mathbf{U}\cdot\mathbf{U} = 1$), and slowness vector $\mathbf{p}$, respectively. The normalized polarization vector $\mathbf{U}$, slowness vector $\mathbf{p}$ and complex-valued scalar amplitude a are independent of $\mathbf{x}$ and t. Displacement vector $\mathbf{u}$ satisfies the equation of motion $\tau_{ij,j} = \rho\ddot{u}_i$, where ρ is the density and τ_{ij} is the stress tensor, related to the infinitesimal strain tensor $e_{ij} = (u_{i,j} + u_{j,i})/2$ by the generalized Hooke's law $\tau_{ij} = c_{ijkl}e_{kl}$. Quantities c_{ijkl} are complex-valued viscoelastic moduli.

It follows from the equation of motion that equation 3 represents a plane wave if, and only if, U_1, U_2 and U_3 satisfy the system of three linear equations

$$(\Gamma_{ik} - \delta_{ik})U_k = 0, \qquad i = 1,2,3, \tag{4}$$

where Γ_{ik} is the generalized, complex-valued Christoffel matrix, given by the relation

$$\Gamma_{ik} = a_{ijkl}p_j p_l \tag{5}$$

and δ_{ik} is the Kronecker symbol. System 4 can be solved for U_k only if

$$\det(\Gamma_{ik} - \delta_{ik}) = 0 \ . \tag{6}$$

This equation represents a constraint relation for the complex-valued slowness vector $\mathbf{p}$.

We express slowness vector $\mathbf{p}$ using the so-called mixed specification, proposed by Červený (2004); see also Červený and Pšenčík (2005a):

$$\mathbf{p} = \sigma\mathbf{n} + \mathrm{i}D\mathbf{m} \ , \quad \mathbf{n}\cdot\mathbf{m} = 0 \ . \tag{7}$$

The two real-valued, mutually orthogonal unit vectors $\mathbf{n}$ and $\mathbf{m}$, and the real-valued scalar quantity D uniquely specify the plane wave under consideration, and can be chosen arbitrarily. Vector $\mathbf{n}$ specifies the direction of vector Re$\mathbf{p}$ and vector $\mathbf{m}$ is perpendicular to $\mathbf{n}$. Vectors $\mathbf{n}$ and $\mathbf{m}$ specify the propagation-attenuation plane. The real-valued scalar quantity D is called the inhomogeneity parameter since it controls the inhomogeneity of the plane wave. For $D = 0$, the plane wave is homogeneous, for $D \neq 0$ it is inhomogeneous, and for $|D|$ small it is weakly inhomogeneous. The complex-valued quantity σ in equation 7 can be determined from the constraint relation 6, which, after inserting equation 7 takes the form (Červený and Pšenčík, 2005a):

$$\det[a_{ijkl}(\sigma n_j + \mathrm{i}Dm_j)(\sigma n_l + \mathrm{i}Dm_l) - \delta_{ik}] = 0 \ . \tag{8}$$

Equation 8 is an algebraic equation of the sixth degree for σ, with complex-valued coefficients. It has six complex-valued roots, corresponding to P, S1 and S2 plane waves, propagating in the directions of $\mathbf{n}$ and $-\mathbf{n}$. Quantity σ is a function of a_{ijkl}, $\mathbf{n}$, $\mathbf{m}$ and D.

For certain simple cases, the algebraic equation of the sixth degree can be factorized to two equations; one of the fourth degree, the other of the second degree. Quantity σ of a plane wave propagating in an isotropic viscoelastic medium or of an SH wave propagating in a plane of symmetry of a monoclinic (orthorhombic, hexagonal) medium is the solution of the quadratic equation resulting from factorizing equation 8. If we know σ, formula 7 for the slowness vector can be expressed in a closed analytical form.

It is often useful to express slowness vector $\mathbf{p}$ as:

$$\mathbf{p} = \mathbf{P} + \mathrm{i}\mathbf{A} \ , \tag{9}$$

where $\mathbf{P}$ and $\mathbf{A}$ are real-valued vectors, called the propagation and attenuation vector, respectively. Vector $\mathbf{P}$ is perpendicular to the wavefront defined by equation $P_i x_i = const.$ Both $\mathbf{P}$ and $\mathbf{A}$ are situated in the propagation-attenuation plane formed by unit vectors $\mathbf{n}$ and $\mathbf{m}$. It follows from 7 and 9 that $\mathbf{P}$ and $\mathbf{A}$ are given by the relations:

$$\mathbf{P} = \mathbf{n}\mathrm{Re}\sigma \ , \qquad \mathbf{A} = \mathbf{n}\mathrm{Im}\sigma + D\mathbf{m} \ . \tag{10}$$

The real-valued phase-velocity $\mathcal{C}$ in the direction of $\mathbf{n}$ and attenuation angle γ (the angle between $\mathbf{P}$ and $\mathbf{A}$) are then expressed as follows:

$$\mathcal{C} = 1/|\mathbf{P}| = 1/|\mathrm{Re}\sigma| \ , \quad \cos\gamma = \epsilon\,\mathrm{Im}\sigma/[(\mathrm{Im}\sigma)^2 + D^2]^{1/2} \ , \tag{11}$$

where $\epsilon = \mathrm{Re}\sigma/|\mathrm{Re}\sigma| = \pm 1$. Once σ has been found, we can also simply determine the complex-valued polarization vectors $\mathbf{U}$ from equation 4 and from the normalization condition $\mathbf{U} \cdot \mathbf{U} = 1$.

In the whole paper, we use inhomogeneity parameter D to specify the inhomogeneity of the wave, and eq. 7 to describe slowness vector $\mathbf{p}$. We call the inhomogeneous plane waves with small $|D|$ (not with small γ) weakly inhomogeneous plane waves. Červený and Pšenčík (2005a,b) showed that attenuation angle γ is not a suitable parameter for characterizing the inhomogeneity of plane waves, particularly in weakly dissipative media. The two main reasons are: 1) In weakly dissipative media, with $|a^I_{ijkl}|$ small, attenuation angle γ is very unstable. It can take any value between 0^o and γ_{max}, where γ_{max} depends on velocity anisotropy. It is usually close to 90^o, and may also exceed it. For example, in a perfectly elastic isotropic medium, attenuation angle γ is always 90^o. However, weakly inhomogeneous waves, including homogeneous waves ($\gamma = 0^o$), can propagate in a weakly dissipative isotropic medium (arbitrarily close to a perfectly elastic isotropic medium). These observations simply follow from eq. 11, considering $\mathrm{Im}\sigma$ small (weak dissipation). 2) If attenuation angle γ is used as a parameter characterizing the inhomogeneity of plane waves, so-called forbidden directions can be generated. In these directions, the square of the phase velocity $\mathcal{C}^2$ is negative (Krebes and Le, 1994; Carcione and Cavallini, 1995b; Carcione, 2007), which is, of course, physically unacceptable. Červený and Pšenčík (2005a,b) discuss and explain this phenomenon in detail.

EXACT EXPRESSIONS FOR QUALITY FACTOR Q

We use the classical definition of quality factor Q, given by Buchen (1971):

$$Q^{-1} = W_d/E \ . \tag{12}$$

If σ and $\mathbf{U}$ are known, we can compute the densities of the time-averaged energy related quantities, e.g., the kinetic K, strain U, complete $E = K + U$ and dissipated W_d energies, the complex-valued Poynting vector $\mathbf{F}$ (also called the complex-valued energy flux), and the real-valued Poynting vector $\mathbf{S} = \mathrm{Re}\mathbf{F}$ (also called the energy flux). The most important is the expression for the complex-valued time-averaged Poynting vector $\mathbf{F}$. Its Cartesian components F_i for plane wave 3 are given by the relation (Červený and Pšenčík, 2006, eq. 24):

$$F_i = c \ a_{ijkl} p_l U_j^* U_k \ , \tag{13}$$

where

$$c = \frac{1}{2}\rho\omega^2 |a| \exp(-2\omega A_n x_n) \ . \tag{14}$$

In equation 14, ρ is the density and a is the scalar amplitude. Consequently, c is a positive real-valued constant, and the real-valued time-averaged Poynting vector $\mathbf{S}$ has the form:

$$S_i = c \ \mathrm{Re}(a_{ijkl} p_l U_j^* U_k) \ . \tag{15}$$

In equation 12, we can use the relations

$$E = \mathbf{P} \cdot \mathbf{S} \ , \qquad W_d = 2\mathbf{A} \cdot \mathbf{S}, \tag{16}$$

derived by Carcione and Cavallini (1993); see also Červený and Pšenčík (2006). In equation 16, $\mathbf{P}$ and $\mathbf{A}$ are given by equation 10 and $\mathbf{S}$ by equation 15. Inserting equations 16 into equation 12, we obtain the expression for Q:

$$Q^{-1} = 2\mathbf{A} \cdot \mathbf{S}/\mathbf{P} \cdot \mathbf{S} \ . \tag{17}$$

Equation 17 for Q, with equation 10 for $\mathbf{P}$ and $\mathbf{A}$ and equation 15 for $\mathbf{S}$, is exact, valid for homogeneous and inhomogeneous plane waves propagating in isotropic or anisotropic, perfectly elastic or viscoelastic media. No approximation has been used in its derivation, and no restrictions have been imposed on the anisotropy or viscoelasticity of the medium and on the inhomogeneity of the plane wave. It is obvious from equation 17 that Q does not depend on constant c, specified in equation 14.

We show a simple schematic diagram illustrating most important quantities related to the propagation of an inhomogeneous plane wave in a homogeneous viscoelastic anisotropic medium in Figure 1.

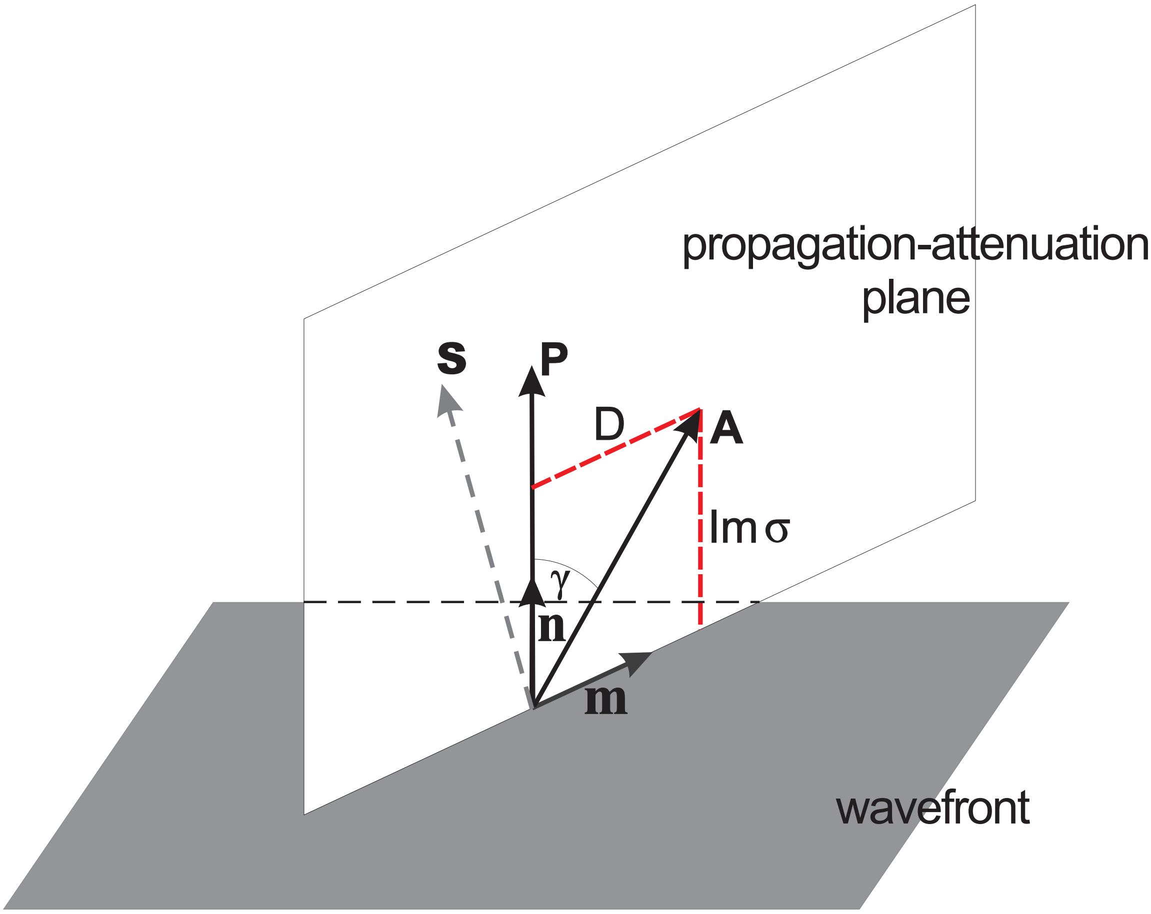

Figure 1 Schematic diagram showing most important quantities related to inhomogeneous plane-wave propagation in anisotropic dissipative media. Two real-valued, mutually orthogonal unit vectors **n** and **m** are used to define complex-valued slowness vector **p**; see 7. Vector **n** is perpendicular to the wavefront and specifies the direction of propagation vector **P**. **P** and the attenuation vector **A** define propagation-attenuation plane. Vectors **P** and **A** make an attenuation angle γ. Projections of **A** into **n** and **m** are Imσ and inhomogeneity parameter D, respectively. As indicated, the real-valued time-averaged Poynting vector **S**, given by 15, may point out of the propagation-attenuation plane.

THE QUALITY FACTOR FOR WEAKLY INHOMOGENEOUS PLANE WAVES IN ANISOTROPIC, WEAKLY DISSIPATIVE MEDIA

In this section, we use the first-order perturbation method to derive an approximate expression for the quality factor $\hat{Q}$ of a weakly inhomogeneous plane wave propagating in an anisotropic, weakly dissipative medium ($|a^I_{ijkl}|$ small). In practical problems of seismology and seismic exploration, the dissipation is usually small, and the inhomogeneity of the considered waves is also weak. In such a case, the numerical solution of the algebraic equation 8 of the sixth degree can be substituted by its approximate explicit solution, and the quality factor can be expressed in a simple analytical form.

As usual in perturbation methods, we use the word "weakly" in the terms "weakly inhomogeneous plane waves" and "weakly dissipative media" only as a qualitative description. The accuracy of perturbation equations increases with decreasing inhomogeneity of the studied plane waves and with decreasing dissipation of the medium. Quantitative estimation of the error of the perturbation equations would not be simple. The simplest estimation is to compare the results of exact and perturbation equations for several test situations, as we do in the section on numerical examples. This is one of the reasons for presenting exact equations.

The problem of weakly inhomogeneous plane waves propagating in weakly dissipative media was treated by Červený and Pšenčík (2008) who used the first-order perturbation method to solve it. As a reference (unperturbed) case, a homogeneous plane wave with a real-valued slowness vector ($D = 0$) propagating in a perfectly elastic medium ($a^I_{ijkl} = 0$) was considered. The quantities corresponding to this reference case are denoted below by upper indices 0. Therefore, $a^0_{ijkl} = a^R_{ijkl}$.

The solutions for the reference case are well-known. For σ^0, $\mathbf{P}^0$ and $\mathbf{A}^0$, we have

$$\sigma^0 = 1/\mathcal{C}^0 \ , \quad \mathbf{P}^0 = \mathbf{n}/\mathcal{C}^0 \ , \quad \mathbf{A}^0 = \mathbf{0} \ . \tag{18}$$

Symbol $\mathcal{C}^0$ denotes the phase velocity in the direction of $\mathbf{n}$ in the reference medium. The polarization vector $\mathbf{U}^0$ of the reference homogeneous wave can be determined from equation 4 specified for the reference, perfectly elastic medium:

$$(\Gamma^0_{ik} - \delta_{ik})U^0_k = 0, \qquad i = 1, 2, 3. \tag{19}$$

Similarly as in equation 4, we consider $\mathbf{U}^0$ to be normalized, $U^0_i U^0_i = 1$. The real-valued energy-velocity vector $\boldsymbol{\mathcal{U}}^0$ of a homogeneous plane wave propagating in the reference, perfectly elastic medium is given by the relation:

$$\mathcal{U}^0_i = a^0_{ijkl} p^0_l U^0_j U^0_k \ . \tag{20}$$

Multiplying 19 by U^0_i, we obtain the relation $a^0_{ijkl} p^0_j p^0_l U^0_i U^0_k = 1$, and the relations:

$$\mathcal{U}^0_i p^0_i = 1 \ , \qquad \mathcal{U}^0_i n_i = \mathcal{C}^0 \ . \tag{21}$$

The inhomogeneity of the studied plane wave (specified by D) and the imaginary parts of the viscoelastic moduli (a^I_{ijkl}) are considered to be small perturbations. Unit vectors

$\mathbf{n}$ and $\mathbf{m}$ remain fixed and are not perturbed. Under these assumptions, Červený and Pšenčík (2008) derived the following approximate expression for quantity σ:

$$\sigma = (\mathcal{C}^0)^{-1}[1 + \mathrm{i}\mathcal{A}^{in}/2 - \mathrm{i}D(\boldsymbol{\mathcal{U}}^0 \cdot \mathbf{m})] \ . \qquad (22)$$

We call quantity $\mathcal{A}^{in}$ the *intrinsic attenuation factor* since it depends only on the intrinsic dissipation of the medium, not on D. It is given by the relation

$$\mathcal{A}^{in} = a^I_{ijkl} p^0_j p^0_l U^0_i U^0_k \ . \qquad (23)$$

For the perturbed propagation vector $\mathbf{P}$ and attenuation vector $\mathbf{A}$, we get from equation 10:

$$\mathbf{P} = \mathbf{n}/\mathcal{C}^0 \ , \quad \mathbf{A} = \mathbf{n}\mathcal{A}^{in}/2\mathcal{C}^0 + D[\mathbf{m} - \mathbf{n}(\boldsymbol{\mathcal{U}}^0 \cdot \mathbf{m})/\mathcal{C}^0] \ . \qquad (24)$$

Thus, the perturbed propagation vector $\mathbf{P}$ is the same as the unperturbed, $\mathbf{P} = \mathbf{P}^0$. The perturbed attenuation vector $\mathbf{A}$, however, differs from the unperturbed significantly. The unperturbed attenuation vector $\mathbf{A}^0$ is zero (equation 18), but the perturbed, $\mathbf{A}$, depends on dissipation moduli a^I_{ijkl} as well as on inhomogeneity parameter D.

Now we take the exact relation 17 for Q^{-1} and use it to derive the first-order perturbation relation for $\hat{Q}^{-1}$. As $\mathbf{A}^0 = 0$, we can substitute $\mathbf{S}$ and $\mathbf{P}$ in equation 17 by their values $\mathbf{S}^0$ and $\mathbf{P}^0$ in the reference medium. Equation 15 yields

$$S^0_i = c\mathrm{Re}(a^0_{ijkl} p^0_l U^0_j U^0_k) = c\, \mathcal{U}^0_i \ , \qquad (25)$$

where c is a real-valued positive constant given in 14, and $\boldsymbol{\mathcal{U}}^0$ is the energy-velocity vector in the reference medium (equation 20). Multiplying equation 25 successively by $\mathbf{A}$ and $\mathbf{P} = \mathbf{P}^0$, we obtain $A_j S^0_j = cA_j \mathcal{U}^0_j$ and $P^0_j S^0_j = cp^0_j \mathcal{U}^0_j = c$. We insert this into relation 17, and get the simple first-order perturbation formula:

$$\hat{Q}^{-1} = 2\mathbf{A} \cdot \boldsymbol{\mathcal{U}}^0 \ . \qquad (26)$$

We use the circumflex over the letter Q in order to emphasize that $\hat{Q}$ is derived by the first-order perturbation method, and is valid for weakly inhomogeneous plane waves propagating in weakly dissipative media only. Inserting $\mathbf{A}$ from equation 24 into 26, we get the final expression for $\hat{Q}^{-1}$:

$$\hat{Q}^{-1} = \mathcal{A}^{in} = a^I_{ijkl} p^0_j p^0_l U^0_i U^0_k \ . \qquad (27)$$

The term depending on D in the expression for $\mathbf{A}$ in equation 24 vanished when multiplied by $\boldsymbol{\mathcal{U}}^0$. Thus, the *quality factor $\hat{Q}$ in equation 27 does not depend on D*, regardless of whether inhomogeneous plane waves (with $D \neq 0$) are considered.

We have thus shown that for weakly inhomogeneous plane waves, propagating in weakly dissipative anisotropic media, the exact quality factor $Q = E/W_d$ reduces to the approximate form $\hat{Q}$ given by equation 27. It does not depend on the inhomogeneity parameter D of the plane wave under consideration; it depends only on the intrinsic dissipation parameters of the medium. In the next section, we will show that we should associate quality factor $\hat{Q}$ with the direction of unit vector $\mathbf{t} = \boldsymbol{\mathcal{U}}^0/|\boldsymbol{\mathcal{U}}^0|$. Then $\hat{Q}$ is inversely proportional to attenuation coefficient $\alpha(\mathbf{t})$, which does not depend on inhomogeneity parameter D. The plane wave under consideration can be inhomogeneous, with its attenuation vector $\mathbf{A}$ depending on D; see equation 24.

The equation for quality factor, mathematically identical to 27, was derived by Gajewski and Pšenčík (1992, eq. 7). They used a modification of the asymptotic ray-series method, in which they assumed that dissipation moduli a^I_{ijkl} were small quantities of the order of ω^{-1} for $\omega \to \infty$. The method is valid locally for high-frequency elementary waves propagating in smoothly heterogeneous, weakly dissipative, generally anisotropic media. The resulting direction-dependent quality factor corresponds to the attenuation along the reference ray direction. Consequently, it corresponds to the attenuation along energy-velocity vector $\boldsymbol{\mathcal{U}}^0$ (not along propagation vector $\mathbf{P}^0$). Although Gajewski and Pšenčík (1992) did consider neither Buchen's (1971) definition of Q nor inhomogeneity of the wave in their derivation, their eq. 7 for the quality factor is fully consistent with our equation 27. Thus, it is also consistent with definition 12 of Buchen (1971), and it is valid even for weakly inhomogeneous waves. The equation for quality factor, identical to 27, was also derived by Červený (2001, eq.5.5.28) using the first-order perturbation equation for traveltime.

The intrinsic attenuation factor $\mathcal{A}^{in}$ is equivalent to $\hat{Q}^{-1}$. Consequently, using classical terminology, it could also be called the loss factor. We, however, prefer to keep the term intrinsic attenuation factor throughout this paper, as we wish to emphasize the independence of $\mathcal{A}^{in}$ on inhomogeneity parameter D and its strict definition by equation 23.

For some special cases, equation 27 for quality factor $\hat{Q}$ simplifies. For example, we can consider the 6×6 matrix $\bar{\mathbf{A}}$ of viscoelastic moduli given by relation

$$\bar{\mathbf{A}} = \bar{\mathbf{A}}^R - \mathrm{i}\bar{\mathbf{A}}^I = \bar{\mathbf{A}}^R(1 - \mathrm{i}/q) \ , \tag{28}$$

where q is an arbitrary real-valued positive constant. This means that we can express any a^I_{ijkl} as $a^I_{ijkl} = a^R_{ijkl}/q$. Equation 27 then yields

$$\hat{Q}^{-1} = a^R_{ijkl} p^0_j p^0_l U^0_i U^0_k / q = 1/q \ . \tag{29}$$

Equation 29 is the consequence of relations 20 and 21. As Gajewski and Pšenčík (1992), who obtained a similar result, we can conclude that under condition 28, quality factor $\hat{Q}$ is direction-independent. We can thus speak of *Q-factor isotropy.*

In anisotropic dissipative media, $\hat{Q}$ is generally direction-dependent if condition 28 is not satisfied. $\hat{Q}$ is then direction-dependent even if dissipation moduli a^I_{ijkl} have an isotropic structure:

$$a^I_{ijkl} = (\lambda^I/\rho)\delta_{ij}\delta_{kl} + (\mu^I/\rho)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \ , \tag{30}$$

where $-\lambda^I$ and $-\mu^I$ are the imaginary parts of Lamé's viscoelastic moduli. Inserting 30 into 27 yields

$$\hat{Q}^{-1} = \frac{\lambda^I + \mu^I}{\rho}(p^0_i U^0_i)(p^0_j U^0_j) + \frac{\mu^I}{\rho}(p^0_i p^0_i) \ . \tag{31}$$

If the reference, perfectly elastic medium is anisotropic, i.e. if we deal with velocity anisotropy, $\hat{Q}$ is direction dependent (even if the matrix of dissipation moduli has an isotropic structure 30). In this case, we can speak of *Q-factor anisotropy.* For dissipation moduli given in 30, $\hat{Q}$ becomes approximately direction-independent in weakly anisotropic media with weak anisotropy parameters of the same order as the weak dissipation parameters; see equations A-7 to A-9.

If the reference medium is isotropic, equation 31 yields the well-known results for isotropic viscoelastic media. Namely $\hat{Q}^{-1} = (\lambda^I + 2\mu^I)/(\lambda^R + 2\mu^R) = -\mathrm{Im}(v_P^2)/\mathrm{Re}(v_P^2)$ for P waves, and $\hat{Q}^{-1} = \mu^I/\mu^R = -\mathrm{Im}(v_S^2)/\mathrm{Re}(v_S^2)$ for S waves, where v_P and v_S are the complex-valued P- and S-wave phase velocities and λ^R, μ^R and $-\lambda^I$, $-\mu^I$ are real and imaginary parts of complex-valued Lamé's viscoelastic moduli. Moreover, the unit vectors $\mathbf{n}$ and $\mathbf{t}$ coincide in the reference isotropic medium. Thus, equation 27 is fully consistent with the most commonly used definition of the quality factor in isotropic dissipative media, and represents its natural generalization for anisotropic dissipative media.

The specifications of formula 27 for anisotropic media of higher symmetry and for weakly anisotropic media are given in Appendix A.

RELATION OF QUALITY FACTOR $\hat{Q}$ TO ATTENUATION COEFFICIENT α

Attenuation coefficient α is another important parameter characterizing dissipation. In this section, we study its relation to quality factor $\hat{Q}$. We discuss attenuation coefficient α of the arbitrary time-harmonic homogeneous or inhomogeneous plane wave 3, specified by vectors $\mathbf{n}$, $\mathbf{m}$ and inhomogeneity parameter D, along an arbitrarily oriented straight-line profile in a homogeneous viscoelastic anisotropic medium. Using equation 9 in formula 3 for the plane wave, we can alter the formula to the following form:

$$u_i(x_j, t) = aU_i \exp[-\mathrm{i}\omega(t - P_n x_n)] \exp[-\omega A_n x_n] \ . \tag{32}$$

We now introduce the plane wave dissipation filter $\mathcal{D}(\omega)$ by the relation

$$\mathcal{D}(\omega) = \exp[-\omega A_n x_n] \ . \tag{33}$$

Equation 32 can then be expressed as:

$$u_i(x_j, t) = aU_i \mathcal{D}(\omega) \exp[-\mathrm{i}\omega(t - P_n x_n)] \ . \tag{34}$$

The physical meaning of the plane-wave dissipation filter is obvious. It describes the variation of amplitudes along an arbitrarily chosen straight profile. We specify the profile under consideration by the unit real-valued vector $\mathbf{l}$. We further introduce distance $s - s_0$, measured along profile $\mathbf{l}$, with an arbitrarily selected initial point s_0 on the profile so that $s - s_0$ is positive in the direction of $\mathbf{l}$. The plane-wave dissipation filter 33 along profile $\mathbf{l}$ is then given by

$$\mathcal{D}(\omega) = \exp[-\omega(s - s_0)\mathbf{A} \cdot \mathbf{l}] = \exp[-\alpha(\mathbf{l})(s - s_0)] \ , \tag{35}$$

where we have introduced quantity $\alpha(\mathbf{l})$

$$\alpha(\mathbf{l}) = \omega \mathbf{A} \cdot \mathbf{l} \ . \tag{36}$$

We call $\alpha(\mathbf{l})$ the *attenuation coefficient* (or absorption coefficient) of the studied homogeneous or inhomogeneous plane wave along profile $\mathbf{l}$. Attenuation coefficient $\alpha(\mathbf{l})$ is real-valued, and its dimension is $[length]^{-1}$. For certain directions of $\mathbf{l}$, the attenuation coefficient may be negative, and may thus actually represent the exponential growth of amplitude (for example, if profile $\mathbf{l}$ is oriented against the direction of propagation).

Inserting $\mathbf{A}$ from equation 10 into equation 36, we obtain a formally simple expression for attenuation coefficient $\alpha(\mathbf{l})$ along profile $\mathbf{l}$:

$$\alpha(\mathbf{l}) = \omega[(\mathbf{n}\cdot\mathbf{l})\mathrm{Im}\sigma + (\mathbf{m}\cdot\mathbf{l})D] \ . \tag{37}$$

This expression for $\alpha(\mathbf{l})$ is exact and may be used for any, homogeneous or inhomogeneous, plane wave (P, S1, S2) propagating in an arbitrary, perfectly elastic or viscoelastic, isotropic or anisotropic medium along any profile $\mathbf{l}$. The plane wave is specified by unit vectors $\mathbf{n},\mathbf{m}$, and by inhomogeneity parameter D. To determine $\alpha(\mathbf{l})$ exactly using equation 37, we have to solve algebraic equation 8 of the sixth degree in σ. For viscoelastic isotropic media, the algebraic equation of the sixth degree factorizes into quadratic algebraic equations for plane P and S waves.

For weakly inhomogeneous plane waves propagating in weakly dissipative media, we can simplify equation 37. We can use the first-order perturbation formula 24 for attenuation vector $\mathbf{A}$ and insert it into equation 37. We obtain

$$\alpha(\mathbf{l}) = (\omega/\mathcal{C}^0)\Big(\mathbf{n}\cdot\mathbf{l}\ \mathcal{A}^{in}/2 + D[\mathcal{C}^0(\mathbf{m}\cdot\mathbf{l}) - (\mathbf{n}\cdot\mathbf{l})(\boldsymbol{\mathcal{U}}^0\cdot\mathbf{m})]\Big) \ . \tag{38}$$

In equation 38, we use the intrinsic attenuation factor $\mathcal{A}^{in}$ given by equation 23. Expression 38 is approximate, but otherwise it is quite general. We assume, of course, dissipation moduli a^I_{ijkl} and inhomogeneity parameter D to be small. The effects of dissipation moduli a^I_{ijkl} and of inhomogeneity parameter D are coupled in equation 38 even for weakly inhomogeneous waves propagating in weakly dissipative media.

In seismological structural studies, in seismic exploration or in nondestructive testing of materials by ultrasonic techniques, researchers are mostly interested in the "intrinsic" dissipation, caused exclusively by a^I_{ijkl}. The attenuation coefficients, related fully or partly to inhomogeneity parameter D, are usually of no interest, as they are not related to the properties of the tested rocks only, but also to the properties of the wave used for testing. Therefore, a very important question arises, whether there exists a profile $\mathbf{l}$, along which attenuation coefficient $\alpha(\mathbf{l})$ does not depend on D.

An exact answer would require studying equation 37 numerically because $\mathrm{Im}\sigma$ in equation 37 also depends on D. However, for weakly inhomogeneous plane waves, propagating in weakly dissipative anisotropic media, we can use equation 38 and obtain the answer immediately: attenuation coefficient $\alpha(\mathbf{l})$, given by equation 38, does not depend on D if

$$\mathcal{C}^0(\mathbf{m}\cdot\mathbf{l}) - (\mathbf{n}\cdot\mathbf{l})(\boldsymbol{\mathcal{U}}^0\cdot\mathbf{m}) = 0 \ . \tag{39}$$

This equation is satisfied only if unit vector $\mathbf{l}$ is taken parallel to $\boldsymbol{\mathcal{U}}^0$,

$$\mathbf{l} \parallel \boldsymbol{\mathcal{U}}^0 \ . \tag{40}$$

Thus attenuation coefficient α does not depend on inhomogeneity parameter D if profile $\mathbf{l}$ is parallel to the unperturbed energy-velocity vector $\boldsymbol{\mathcal{U}}^0$ (the ray direction in the reference medium). Hereinafter, we consider the attenuation coefficient $\alpha(\mathbf{t})$ along profile $\mathbf{t}$, where

$\mathbf{t}$ is the real-valued unit vector along $\boldsymbol{\mathcal{U}}^0$, $\mathbf{t} = \boldsymbol{\mathcal{U}}^0/|\boldsymbol{\mathcal{U}}^0|$. Consequently, for profile $\mathbf{l}$ taken along $\mathbf{t}$, attenuation coefficient $\alpha(\mathbf{t})$ is given by the simple relation

$$\alpha(\mathbf{t}) = \omega/(2\hat{Q}\mathcal{U}^0) \ , \tag{41}$$

where $\hat{Q}$ is the direction-dependent quality factor given by equation 27.

Equation 41 shows that the attenuation coefficient measured along the direction of $\boldsymbol{\mathcal{U}}^0$ depends only on intrinsic attenuation parameters a^I_{ijkl}, and does not depend on inhomogeneity parameter D of the studied plane wave. Attenuation coefficient $\alpha(\mathbf{t})$ is inversely proportional to $\hat{Q}$. We obtained this result with the use of the first-order perturbation method. Consequently, the result is approximate and valid only for weakly inhomogeneous plane waves propagating in weakly dissipative media. A similar conclusion was first drawn by Deschamps and Assouline (2000) in the numerical study of plane waves in orthorhombic anisotropic media.

Using equation 41, we can define quality factor $\hat{Q}$ in terms of attenuation coefficient $\alpha(\mathbf{t})$ along profile $\mathbf{l} = \mathbf{t}$:

$$\hat{Q}^{-1} = \alpha(\mathbf{t})\mathcal{U}^0/\pi f \ , \tag{42}$$

where f is the frequency (in Hz). This relation for $\hat{Q}$ is formally the same as the relation known for isotropic viscoelastic media, namely $Q^{-1} = \alpha V/\pi f$, where V is the velocity of the relevant plane P or S wave (Johnston and Toksöz, 1981, eq.36). In homogeneous isotropic viscoelastic media, attenuation coefficient α, velocity V and quality factor Q are direction-independent. In homogeneous anisotropic viscoelastic media, attenuation coefficient $\alpha(\mathbf{t})$, energy velocity $\mathcal{U}^0$ and quality factor $\hat{Q}$ depend on the direction of the real-valued unit vector $\mathbf{t}$. In this way, equation 42 represents a generalization of the relation $Q^{-1} = \alpha V/\pi f$ for anisotropic viscoelastic media.

In order to calculate $\hat{Q}$ corresponding to the direction of $\mathbf{t}$, using equation 27, one has to know energy-velocity vector $\boldsymbol{\mathcal{U}}^0$ and the corresponding slowness vector $\mathbf{p}^0$, both in the reference medium. Both real-valued vectors, $\mathbf{p}^0$ and $\boldsymbol{\mathcal{U}}^0$, are mutually related; see equation 20. We can proceed in two ways:

a) The first possibility is to specify $\mathbf{p}^0$, and to calculate the corresponding $\boldsymbol{\mathcal{U}}^0$ from equation 20. This straightforward procedure is extremely simple. S-wave singularities and the multivaluedness of S-wave surfaces cause no problems. It is typical for ray tracing in heterogeneous, anisotropic, weakly dissipative media (Gajewski and Pšenčík, 1992). A certain disadvantage is that $\boldsymbol{\mathcal{U}}^0$ and thus $\mathbf{t}$ is not known in advance, but has to be computed. Consequently, we cannot choose the profile of measurements in advance. The situation simplifies considerably for computations in any plane of symmetry.

b) The second possibility consists in the direct computation of $\hat{Q}$ for a given $\mathbf{t}$. In this case, $\mathbf{p}^0$ appearing in equation 27 is not known and should be calculated from a given $\boldsymbol{\mathcal{U}}^0$. This procedure is considerably more complicated than the previous one. It is very sensitive to S-wave singular regions and to the multivaluedness of the wave surfaces when several values of $\mathbf{p}^0$ can be obtained for one given $\boldsymbol{\mathcal{U}}^0$. The procedure resembles the two-point ray tracing procedure. To solve the problem, $\mathbf{p}^0$ is iteratively sought so that it yields the given $\boldsymbol{\mathcal{U}}^0$. Alternatively, the method based on the numerical solution of a system of polynomial equations, proposed by Vavryčuk (2006), can be used.

If we use profile $\mathbf{l}$ along energy-velocity vector $\boldsymbol{\mathcal{U}}^0$ in 35 and 41, the final expression for the dissipation filter in weakly dissipative anisotropic media is as follows:

$$\mathcal{D}(\omega) = \exp[-\omega(s - s_0)/2\hat{Q}\mathcal{U}^0] \ , \tag{43}$$

where $\hat{Q}$ is given by equation 27, and $\mathcal{U}^0 = |\boldsymbol{\mathcal{U}}^0|$ by equation 20. For weakly inhomogeneous plane waves, attenuation coefficient $\alpha(\mathbf{t})$, quality factor $\hat{Q}$, and dissipation filter $\mathcal{D}(\omega)$ do not depend on inhomogeneity parameter D of the wave. Equations 41 and 43 remain valid also for isotropic media, where $\mathcal{U}^0$ and $\hat{Q}$ are direction-independent.

Equation 38 can be used to investigate attenuation coefficients $\alpha(\mathbf{l})$ also in directions $\mathbf{l}$ other than the direction of energy-velocity vector $\boldsymbol{\mathcal{U}}^0$. In all these directions, however, attenuation coefficient $\alpha(\mathbf{l})$ depends on the inhomogeneity of the considered plane wave. As an example, let us consider the case of the profile in the direction of vector $\mathbf{n}$, perpendicular to the wavefront in the reference medium, and oriented in the direction of wave propagation. Equation (38) then yields

$$\alpha(\mathbf{n}) = (\omega/\mathcal{C}^0)[\mathcal{A}^{in}/2 - D(\boldsymbol{\mathcal{U}}^0 \cdot \mathbf{m})] \ . \tag{44}$$

In anisotropic media, $\boldsymbol{\mathcal{U}}^0 \cdot \mathbf{m}$ is generally different from zero so that attenuation coefficient $\alpha(\mathbf{n})$ is practically always influenced by the inhomogeneity parameter D of the considered wave. For certain $\mathbf{n}$, attenuation coefficient $\alpha(\mathbf{n})$ may be even negative (Červený and Pšenčík, 2005b). Consequently, profile $\mathbf{n}$ (perpendicular to the wavefront in the reference medium) is not suitable for measurements of attenuation coefficient and of quality factor in inhomogeneous anisotropic media.

NUMERICAL EXAMPLES

In this section, we illustrate the behavior of quality factor $\hat{Q}$ in a model of a sedimentary rock, and estimate the accuracy of the perturbation formula for $\hat{Q}^{-1} = \mathcal{A}^{in}$ by comparing its results with the exact results for Q^{-1}. The model used is a modification of the model used by Zhu and Tsvankin (2006) and Vavryčuk (2007b). The 6×6 matrix $\bar{\mathbf{A}}$ of complex-valued, density-normalized, viscoelastic moduli $\bar{A}_{\alpha\beta}$ in the Voigt notation, measured in $(\mathrm{km/s})^2$ is as follows:

$$\bar{\mathbf{A}} = \bar{\mathbf{A}}^R - \mathrm{i}\bar{\mathbf{A}}^I \ , \tag{45}$$

where

$$\bar{\mathbf{A}}^R = \begin{pmatrix} 14.40 & 4.40 & 4.50 & 0. & 0. & 0. \\ & 14.40 & 4.50 & 0. & 0. & 0. \\ & & 9.00 & 0. & 0. & 0. \\ & & & 2.25 & 0. & 0. \\ & & & & 2.25 & 0. \\ & & & & & 5.00 \end{pmatrix} \tag{46}$$

and

$$\bar{\mathbf{A}}^I = \begin{pmatrix} 0.48 & 0.15 & 0.30 & 0. & 0. & 0. \\ & 0.48 & 0.30 & 0. & 0. & 0. \\ & & 0.45 & 0. & 0. & 0. \\ & & & 0.15 & 0. & 0. \\ & & & & 0.15 & 0. \\ & & & & & 0.66 \end{pmatrix}. \tag{47}$$

Both matrices $\bar{\mathbf{A}}^R$ and $\bar{\mathbf{A}}^I$ are positive definite. Matrix $\bar{\mathbf{A}}^R$ corresponds to an anisotropic medium of hexagonal symmetry with a vertical axis of symmetry.

Although the formulae derived in previous sections can be used quite generally, we concentrate, for simplicity of illustrations, on a symmetry plane of the above medium. We can choose it so that it is situated in the (x_1, x_3) plane. In this case, the components of vectors $\mathbf{n}$ (unit vector parallel to the propagation vector) and $\mathbf{t}$ (unit vector parallel to the energy-velocity vector) can be expressed as

$$n_1 = \sin i_{ph}, \quad n_2 = 0, \quad n_3 = \cos i_{ph}, \qquad t_1 = \sin i_{en}, \quad t_2 = 0, \quad t_3 = \cos i_{en}. \tag{48}$$

We call angle i_{ph} the *phase angle* and i_{en} the *ray angle*. In the following figures, we show polar diagrams parameterized by phase and ray angles, with $i = 0^{\circ}$ upwards and $i = 90^{\circ}$ to the right. The polar diagrams in Figures 2-5 are plotted in the form used by Carcione (1992).

In Figure 2 we show the polar diagrams of the exact phase velocity $\mathcal{C}$ (km/s) (left column) and of the exact energy velocity $\mathcal{U}$ (km/s) (right column) in the plane of symmetry of medium 46, 47. The phase-velocity diagrams are parameterized by the phase angle, the energy-velocity diagrams by the ray angle. The red color is used for the fastest wave (P wave). Black and blue colors are used for S waves: black corresponds to the S wave with higher phase velocity, blue to the slower S wave. We can see that different S waves (SH or SV) are faster in different directions. From the polarization diagrams, we can identify the egg-shaped curves as curves corresponding to the SH waves. The velocity diagrams are shown for two values of inhomogeneity parameter D, namely for $D = 0$ s/km (homogeneous wave) and for $D = 0.15$ s/km. We shall later prove that the value $D = 0.15$ s/km corresponds to rather strongly inhomogeneous waves. Due to strong S-wave velocity anisotropy, loops are formed on the SV-wave energy-velocity diagrams, even for $D = 0$. The size of the loops increases with increasing inhomogeneity of the wave. Figure 2 shows that the SV- and SH-wave phase velocity curves intersect each other close to the vertical axis and close to phase angles $i_{ph} = \pi/4$, $3\pi/4$, ... They also show the mutual "attraction" of P- and SV-wave phase-velocity curves for increasing D. These interesting phenomena have already been described by Červený and Pšenčík (2005b). The analogous plots of the phase and energy velocities for a perfectly elastic reference medium $\bar{\mathbf{A}}^R$ from 46 are effectively identical with the plots shown in Figure 2.

In Figure 3 we show the polar diagrams of $\hat{Q}^{-1}$ calculated from formula 27 as a function of the ray angle (specified by the energy-velocity vector in the reference medium). The simpler plot on the left corresponds to the P wave, the more complicated plots on the right to both S waves. The egg-shaped curve corresponds to the SH wave, the curve with internal loops corresponds to the SV wave. Due to low P-wave attenuation anisotropy, the behavior of $\hat{Q}^{-1}$ for the P wave is, in comparison with the SV wave, rather simple.

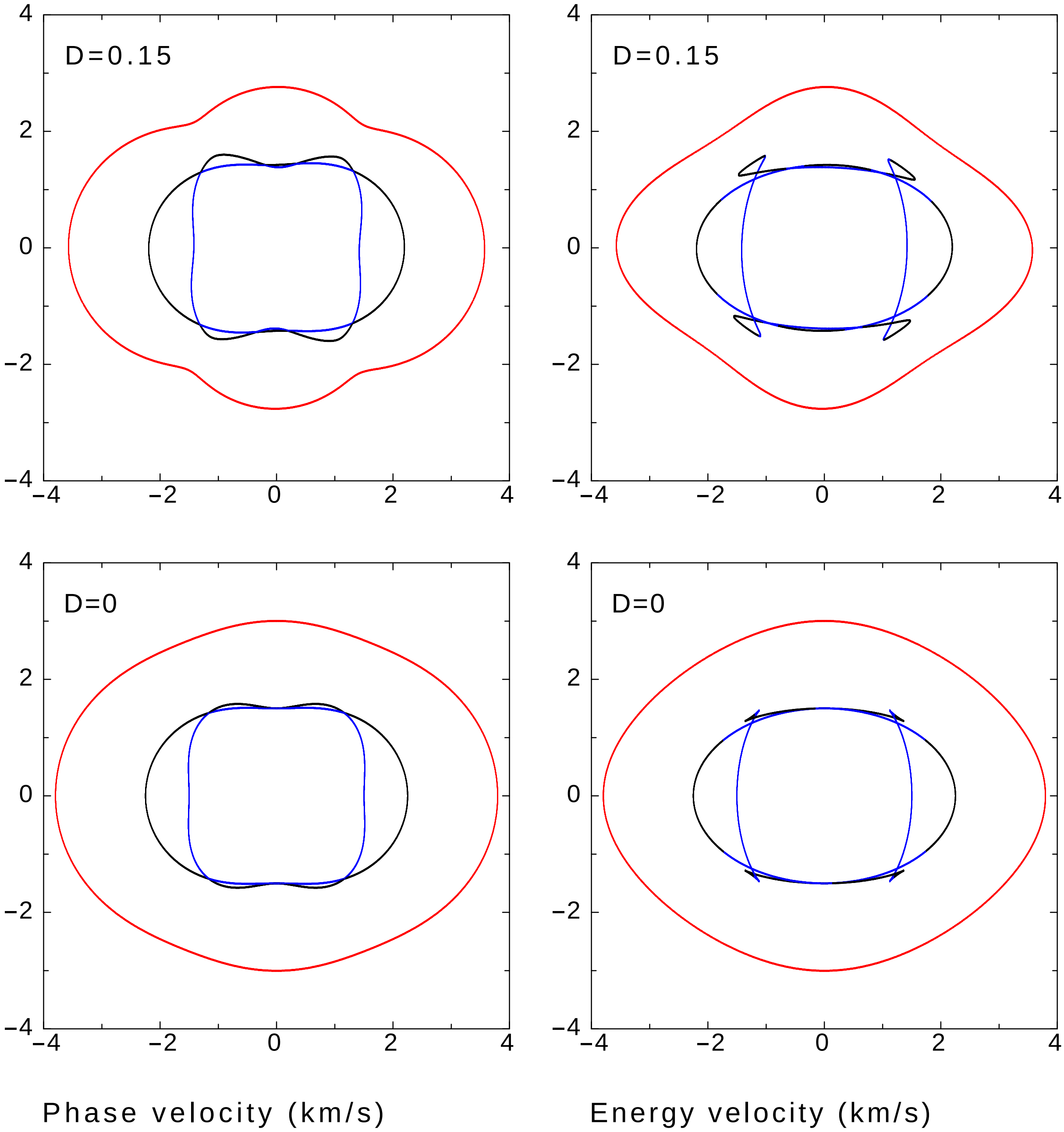

Figure 2 Polar diagrams of phase velocity $\mathcal{C}$ (km/s) (left) and of energy velocity $\mathcal{U}$ (km/s) (right) in a plane of symmetry of medium 46, 47 for $D = 0$ s/km and $D = 0.15$ s/km: red, the fastest wave (P wave); black, S wave with intermediate phase velocity; blue, the slower S wave. The velocity diagrams of the SH wave are egg-shaped. The SH wave is faster than the SV wave in the directions around the horizontal, and, for $D = 0.15$ (s/km), very close to the vertical.

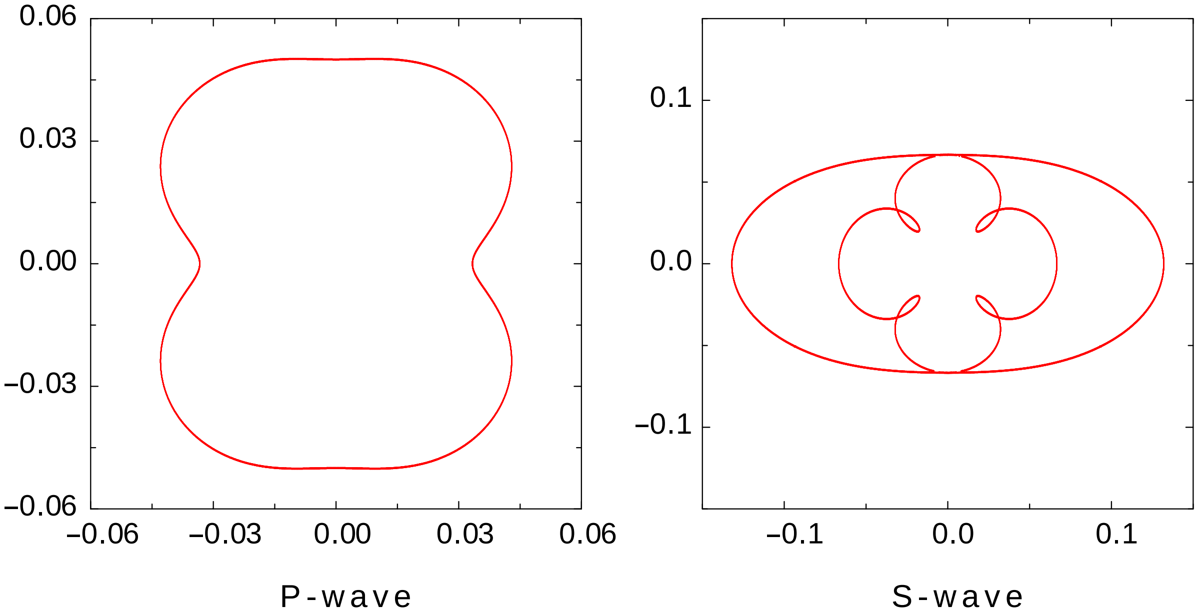

Figure 3 Polar diagrams of the inverse value of quality factor $\hat{Q}$ (dimensionless; $\hat{Q}^{-1} = \mathcal{A}^{in}$) calculated from equation 27 as a function of the *ray angle*. P wave (left) and S waves (right) in a plane of symmetry of medium 46, 47. Note the inner smooth loops on the curve corresponding to the SV wave. The loops are situated at the same ray angles as the triplications on the SV-wave energy-velocity curves; see Figure 2. The loops correspond to the lowest attenuation.

The P wave is attenuated most in directions close to diagonals, the SH wave is attenuated most in directions close to the horizontal. The SV wave is attenuated most in directions close to both the vertical and the horizontal. It displays the strongest directivity of all three waves.

The smooth inner loops on the curve, corresponding to the SV wave, in the S-wave diagram are the most remarkable phenomenon in Figure 3. The loops are situated at exactly the same ray angles, at which we can observe the outer loops on the SV-wave energy-velocity curves in Figure 2. At these loops, the values of $\hat{Q}^{-1}$ attain minimum values. Consequently, the attenuation of the SV waves is minimum in the regions of the loops. This important phenomenon, which we have also observed in other models, certainly deserves a more detailed investigation.

As the profile of measurement corresponds to the direction of energy-velocity vector in the reference medium $\mathbf{l} = \mathbf{t}$, both quality factor $\hat{Q}$ and attenuation coefficient α are independent of inhomogeneity parameter D of the wave under consideration and are related by equation 41. Consequently, they are both suitable for measuring intrinsic dissipative properties of anisotropic media.

In Figure 4 we compare the polar diagrams of $\hat{Q}^{-1} = \mathcal{A}^{in}$, shown in red, with the polar diagrams of exact Q^{-1} shown in black. The diagrams are again calculated in the plane of symmetry of the above medium. The polar diagrams are displayed as functions of phase angle i_{ph}, i.e. as functions of the unit vector $\mathbf{n}$ parallel to the propagation vector. The left column shows the plots corresponding to the P wave, the right column to the S waves. In both columns, plots are shown for inhomogeneity parameters $D = 0$, 0.1 and 0.15 s/km. Quality factor $\hat{Q}$ (calculated using equation 27) is independent of D, but the exact values of Q (calculated using equation 17) depend on D. Only for $D = 0$ s/km are $\hat{Q}$ and Q the same. The difference between them increases with increasing D. As $D = 0.15$ s/km corresponds to a rather strong inhomogeneity of the wave under consideration (see Figure 6), we can conclude that $\hat{Q}^{-1}$ is a good approximation of the exact Q^{-1} for weakly inhomogeneous waves. The red curves, corresponding to $\hat{Q}^{-1}$, are plotted over the black curves, so that the red color dominates in case of exact fit. Consequently, it would be possible to use $\hat{Q}$ as a good approximation of Q in the direction of $\mathbf{n}$, $Q(\mathbf{n})$. The problem, however, is that attenuation coefficient $\alpha(\mathbf{n})$ along the profile in the direction of $\mathbf{n}$ is not intrinsic and depends on D; see equation 44. Consequently, profile $\mathbf{l} = \mathbf{n}$ is not suitable for measuring the intrinsic dissipative properties of rocks in anisotropic media.

In Figure 5 we show the same comparison as Figure 4; the difference being the use of the ray angle i_{en} in Figure 5 instead of the phase angle i_{ph} in Figure 4. Consequently, both quality factor $\hat{Q}$ and attenuation coefficient α are independent of D, and are suitable for measuring the intrinsic dissipative properties of anisotropic media. The system of plots and the use of colors is the same as in Figure 4. There is a remarkable difference of behavior of SV waves between Figures 4 and 5. The most remarkable phenomenon are the above-mentioned inner smooth loops on the SV-wave curves. We can see that their position is the same for the exact and perturbation results. The width of the exactly calculated loops increases slightly with increasing inhomogeneity of the wave, whereas the perturbation result does not depend on the inhomogeneity of the wave.

In order to estimate the strength of the inhomogeneity of the studied waves, we present

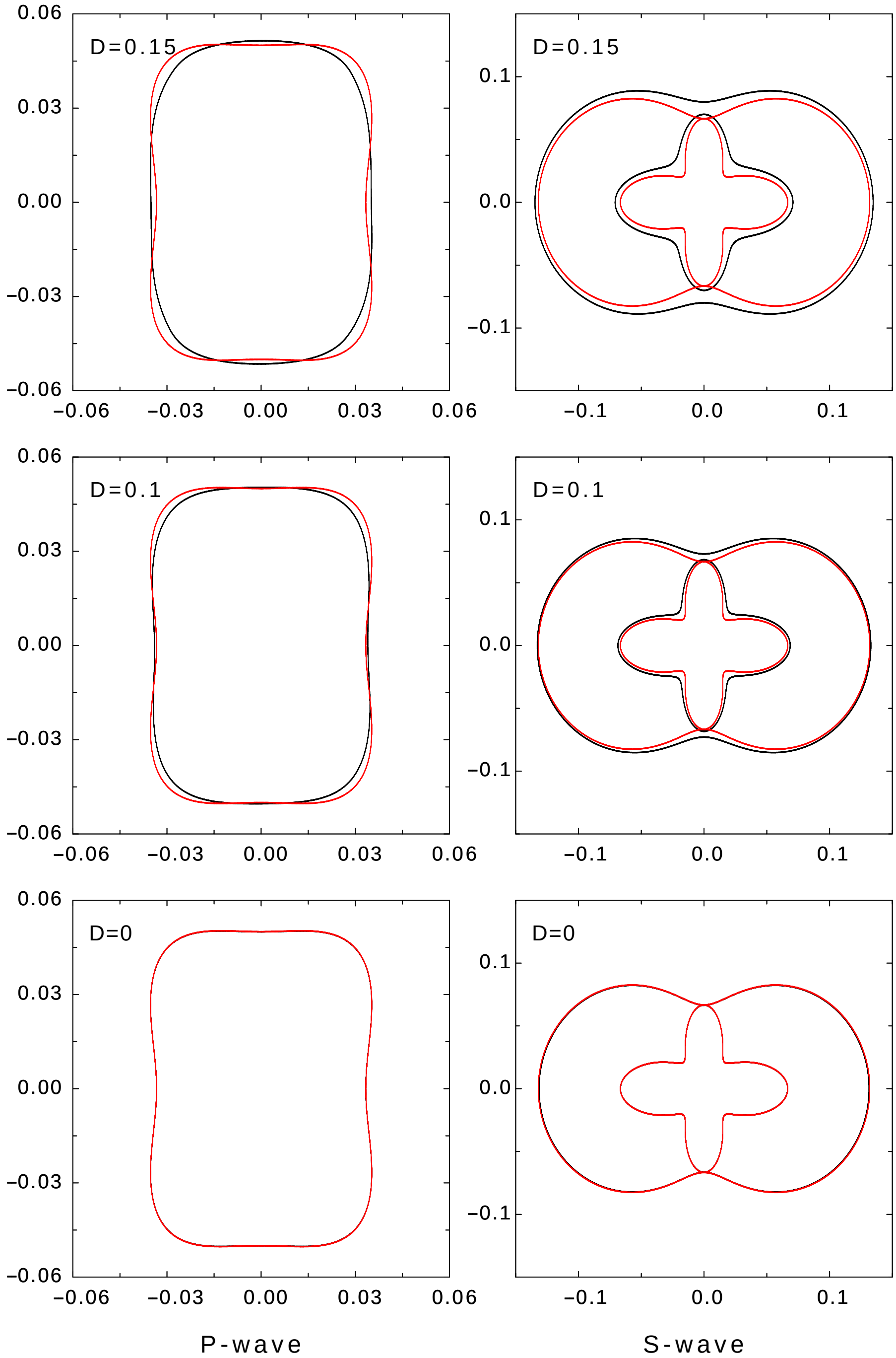

Figure 4 Comparison of polar diagrams of $\hat{Q}^{-1}$ ($\hat{Q}^{-1} = \mathcal{A}^{in}$), calculated from equation 27 (red) with polar diagrams of exact Q^{-1} calculated from equation 17 (black) as a function of the *phase angle*. $\hat{Q}$ and Q are dimensionless. P wave (left) and S waves (right) in a plane of symmetry of medium 46, 47 for $D = 0$, 0.1 and 0.15 s/km. $\hat{Q}^{-1}$ (red) does not depend on inhomogeneity parameter D, but attenuation coefficient α does; see 44.

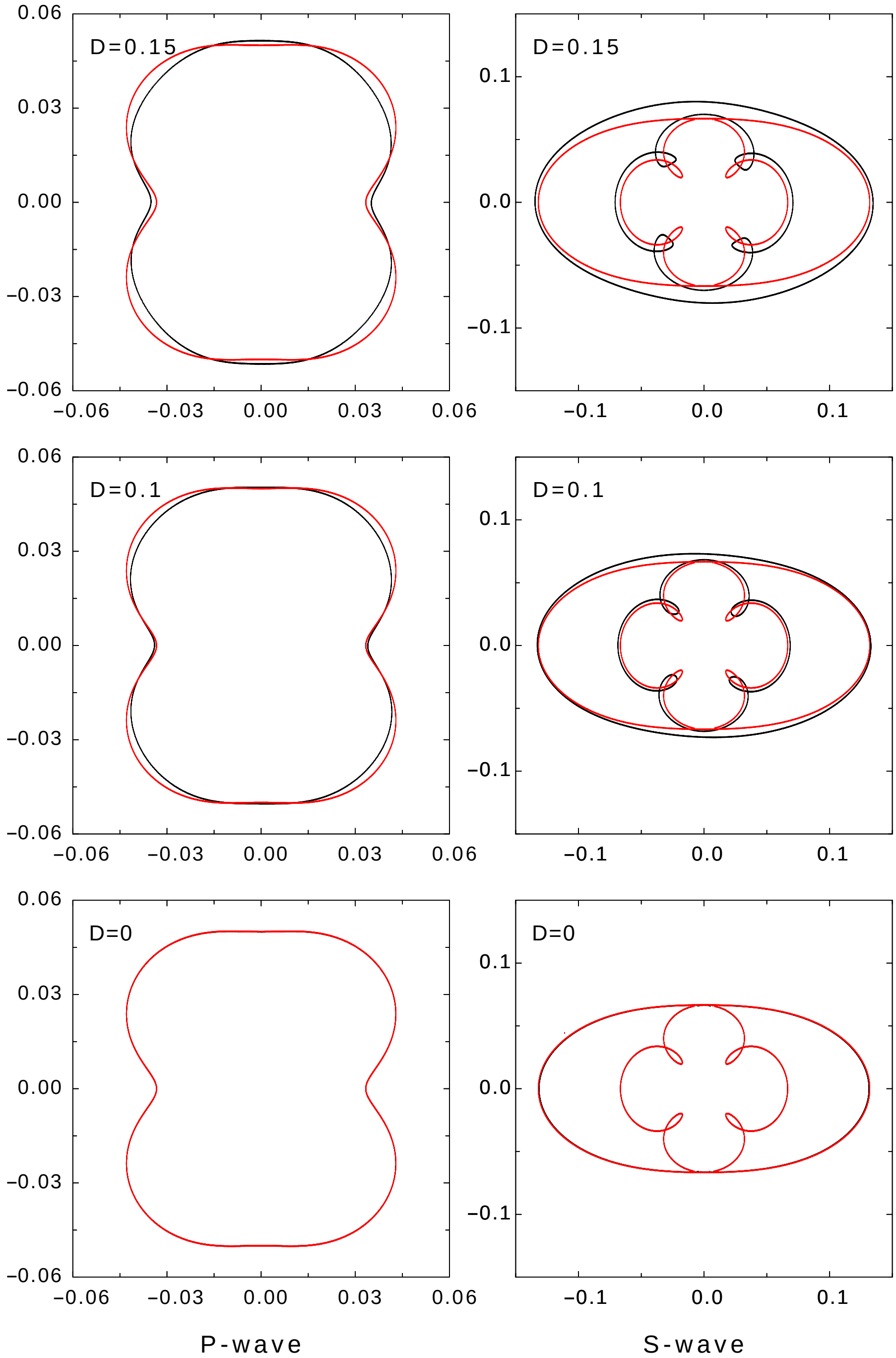

Figure 5 Comparison of polar diagrams of $\hat{Q}^{-1}$ ($\hat{Q}^{-1} = \mathcal{A}^{in}$), calculated from equation 27 (red) with exact polar giagrams of Q^{-1} calculated from equation 17 (black) as a function of the *ray angle*. $\hat{Q}$ and Q are dimensionless. P wave (left) and S waves (right) in a plane of symmetry of medium 46, 47 for $D = 0$, 0.1 and 0.15 s/km. Neither $\hat{Q}^{-1}$ (red) nor α do depend on inhomogeneity parameter D; see 41.

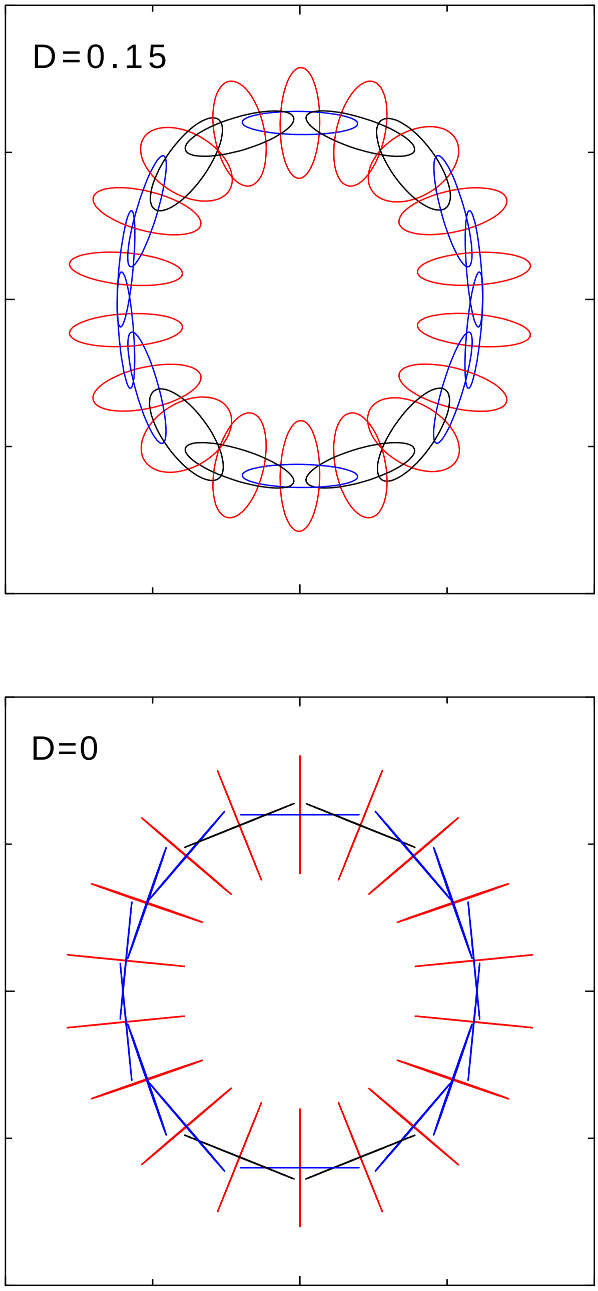

Figure 6   Particle motion polar diagrams in the plane of symmetry of medium 46, 47 for $D = 0$ and 0.15 s/km. The colors are the same as in Figure 2. The SH wave is polarized perpendicularly to the plane of the plots. Note the effective linearity and strong ellipticity of the particle motion of the P and SV waves for $D = 0$ and 0.15 s/km, respectively. The ellipticity is caused primarily by the inhomogeneity of the considered waves. The value of $D = 0.15$ s/km thus corresponds to rather strongly inhomogeneous waves.

the particle motion diagrams for $D = 0$ (homogeneous wave) and for $D = 0.15$ s/km in the plane of symmetry of the above medium in Figure 6. The colors are again the same as in Figure 2. In Figure 6 we show that for $D = 0$ s/km, the polarization of all waves is effectively linear. The SH wave is polarized linearly in both plots. Its polarization is perpendicular to the planes of the plots. In contrast to the homogeneous waves, the polarization of the inhomogeneous P and SV waves for $D = 0.15$ s/km is strongly elliptical, which is caused primarily by the inhomogeneity of the waves. This indicates that the value $D = 0.15$ s/km corresponds to the rather strong inhomogeneity of the wave. Good fit of $\hat{Q}$ and Q in Figure 5 thus implies that the applicability of perturbation formula 27 is rather broad. It may yield sufficiently accurate results even for waves of relatively strong inhomogeneity.

The above tests seem to indicate that equation 27 can be applied to inhomogeneous waves of considerable strength, propagating in anisotropic media of arbitrary symmetry and strength, and of rather strong Q-factor anisotropy.

CONCLUSIONS

We have derived the simple perturbation formula 27 for quality factor $\hat{Q}$. It has several interesting and important properties:

Expression 27 for $\hat{Q}$ is valid for homogeneous and weakly inhomogeneous time-harmonic P, S1 and S2 plane waves propagating in homogeneous, isotropic or generally anisotropic, weakly dissipative media. Quality factor $\hat{Q}$ is a positive scalar quantity, both in isotropic and anisotropic dissipative media. It does not depend on inhomogeneity parameter D of the considered plane wave although its attenuation vector $\mathbf{A}$ might depend on D. Consequently, $\hat{Q}$ describes the intrinsic dissipative properties of the medium.

In anisotropic media, expression 27 for $\hat{Q}$ is fully consistent with the most common definitions of Q used in isotropic dissipative media, and represents their natural generalization. It is generally direction-dependent; the appropriate direction being specified by the direction of the energy-velocity vector (ray direction), not by the direction of the propagation vector. This is due to $\hat{Q}$ being inversely proportional to attenuation coefficient $\alpha(\mathbf{t})$, which does not depend on the inhomogeneity of the wave and corresponds to the profile parallel to the energy-velocity vector. $\hat{Q}$ is direction-dependent even if dissipation matrix $\bar{\mathbf{A}}^I$ in 45 has a structure, which corresponds to an isotropic medium, i.e. Q-factor anisotropy is always related to velocity anisotropy. Q-factor isotropy can exist strictly only in anisotropic media with property 28 or approximately in weakly anisotropic, weakly dissipative media. Expression 27 for $\hat{Q}$ can be easily specified for an anisotropy of higher symmetry (orthorhombic, hexagonal) as well as for weak anisotropy. The computation of $\hat{Q}$ in the vicinity of S-wave singularities or in the case of the multivaluedness of wave surfaces does not cause any difficulties, as shown by the examples presented. The examples display certain interesting phenomena, especially in the regions of triplication of the S-wave wavefront, not known from previous studies.

Expression 27 for $\hat{Q}$ remains valid locally even for high-frequency elementary waves generated by point sources and propagating in heterogeneous, isotropic or anisotropic, weakly dissipative media (Gajewski and Pšenčík, 1992; Červený, 2001; Červený et al., 2007).

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APPENDIX A

QUALITY FACTOR $\hat{Q}$ FOR HIGHER-ANISOTROPY SYMMETRIES AND FOR WEAK ANISOTROPY

Equation 27 for $\hat{Q}^{-1}$ holds for all three waves, P, S1 and S2, propagating in weakly dissipative anisotropic media of arbitrary symmetry and strength. Equations for the individual waves can be obtained from equation 27 by specifying the appropriate polarization vector $\mathbf{U}^0$ in it. Although it is expressed in terms of the unit vector $\mathbf{n}$, perpendicular to the wavefront in the reference, perfectly elastic medium, the quality factor $\hat{Q}$ in equation 27 corresponds to the direction of reference ray $\mathbf{t}$, $\mathbf{t} = \boldsymbol{\mathcal{U}}^0/|\boldsymbol{\mathcal{U}}^0|$. Consequently, equation 27 must always be supplemented by equation 20, which provides the relation for the determination of $\mathbf{t}$ from $\mathbf{n}$. Equation 27 can be used without problems even in the case of wavefront triplication.

For models with higher material symmetry, it is more suitable to express dissipation moduli a^I_{ijkl} in Voigt notation $\bar{A}^I_{\alpha\beta}$. Here $\bar{A}^I_{\alpha\beta}$ are elements of the 6×6 real-valued, symmetric, positive definite, dissipation matrix $\bar{\mathbf{A}}^I$. The material symmetry of dissipation matrix $\bar{\mathbf{A}}^I$ may, in general, differ from the material symmetry of matrix $\bar{\mathbf{A}}^R$. For simplicity, we are considering the material symmetry of $\bar{\mathbf{A}}^I$ to be equal to or higher than the symmetry of $\bar{\mathbf{A}}^R$.

Let us consider orthorhombic symmetry with its axes of symmetry coinciding with the axes of the Cartesian coordinate system, in which the x_1 and x_2 axes are horizontal and the x_3 axis is vertical, positive downwards. The coordinate system is right-handed. In this case, dissipation matrix $\bar{\mathbf{A}}^I$ (similarly as matrix $\bar{\mathbf{A}}^R$) is specified by nine non-zero dissipation moduli $\bar{A}^I_{11}$, $\bar{A}^I_{22}$, $\bar{A}^I_{33}$, $\bar{A}^I_{44}$, $\bar{A}^I_{55}$, $\bar{A}^I_{66}$, $\bar{A}^I_{12}$, $\bar{A}^I_{13}$ and $\bar{A}^I_{23}$. We can then express equation 27 for $\hat{Q}^{-1}$ as follows:

$$\hat{Q}^{-1} = \mathcal{A}^{in} = \bar{A}^I_{11}(p^0_1U^0_1)^2 + \bar{A}^I_{22}(p^0_2U^0_2)^2 + \bar{A}^I_{33}(p^0_3U^0_3)^2$$
$$+\bar{A}^I_{44}(p^0_2U^0_3 + p^0_3U^0_2)^2 + \bar{A}^I_{55}(p^0_1U^0_3 + p^0_3U^0_1)^2 + \bar{A}^I_{66}(p^0_1U^0_2 + p^0_2U^0_1)^2$$
$$+2\bar{A}^I_{12}p^0_1p^0_2U^0_1U^0_2 + 2\bar{A}^I_{13}p^0_1p^0_3U^0_1U^0_3 + 2\bar{A}^I_{23}p^0_2p^0_3U^0_2U^0_3 \ . \qquad (A-1)$$

For transverse isotropy with a vertical axis of symmetry (VTI), we have $\bar{A}^I_{11} = \bar{A}^I_{22}$, $\bar{A}^I_{44} = \bar{A}^I_{55}$, $\bar{A}^I_{13} = \bar{A}^I_{23}$ and $\bar{A}^I_{12} = \bar{A}^I_{11} - 2\bar{A}^I_{66}$. For $\hat{Q}^{-1}$ A-1 then yields

$$\hat{Q}^{-1} = \mathcal{A}^{in} = \bar{A}^I_{11}(p^0_1U^0_1 + p^0_2U^0_2)^2 + \bar{A}^I_{33}(p^0_3U^0_3)^2$$
$$+\bar{A}^I_{44}[(p^0_2U^0_3 + p^0_3U^0_2)^2 + (p^0_1U^0_3 + p^0_3U^0_1)^2]$$
$$+\bar{A}^I_{66}(p^0_1U^0_2 - p^0_2U^0_1)^2 + 2\bar{A}^I_{13}p^0_3U^0_3(p^0_1U^0_1 + p^0_2U^0_2) \ . \qquad (A-2)$$

Equations A-1 and A-2 hold for velocity anisotropy of unlimited strength and for all three waves: P, S1 and S2. By using the polarization vector $\mathbf{U}^0$ of one of the waves in A-1 or A-2, $\hat{Q}$ can be specified for that wave.

Equation 27 simplifies considerably if weak velocity anisotropy is considered. In this case, we can substitute the polarization vectors by simply determinable unit vectors (Farra and Pšenčík, 2003). Let us, for example, consider a P wave. For the P wave, we have $U^0_i \approx n_i$, where n_i is the unit vector perpendicular to the wavefront in the reference medium. Equation 27 is then given by the relation

$$\hat{Q}^{-1}_P = \mathcal{A}^{in}_P = (\mathcal{C}^0_P)^{-2} a^I_{ijkl} n_j n_l n_i n_k \ . \qquad (A-3)$$

In A-3, $\mathcal{C}^0_P$ is the first-order P-wave phase velocity in the direction of vector $\mathbf{n}$. For P waves in a weakly orthorhombic medium, A-1 yields

$$\hat{Q}^{-1}_P = \mathcal{A}^{in}_P = (\mathcal{C}^0_P)^{-2}[\bar{A}^I_{11}n^4_1 + \bar{A}^I_{22}n^4_2 + \bar{A}^I_{33}n^4_3$$
$$+2(\bar{A}^I_{12} + 2\bar{A}^I_{66})n^2_1n^2_2 + 2(\bar{A}^I_{13} + 2\bar{A}^I_{55})n^2_1n^2_3 + 2(\bar{A}^I_{23} + 2\bar{A}^I_{44})n^2_2n^2_3] \ . \qquad (A-4)$$

Similarly, for P waves in a weakly VTI medium, A-2 yields

$$\hat{Q}^{-1}_P = \mathcal{A}^{in}_P = (\mathcal{C}^0_P)^{-2}[\bar{A}^I_{11}(1-n^2_3)^2 + \bar{A}^I_{33}n^4_3 + 2(\bar{A}^I_{13} + 2\bar{A}^I_{55})n^2_3(1-n^2_3)^2] \ . \qquad (A-5)$$

We can see that the velocity anisotropy, expressed by direction-dependent $\mathcal{C}^0_P$, always affects the Q-factor anisotropy expressed by direction-dependent $\hat{Q}_P$.

We can express the square of first-order P-wave phase velocity $\mathcal{C}^0_P$ as

$$(\mathcal{C}^0_P)^2 = \bar{\alpha}^2[1 + 2Y(n_i)] \ , \qquad (A-6)$$

where $Y(n_i)$ stands for the linear combination of weak-anisotropy parameters, with coefficients containing powers of n_i. By symbol $\bar{\alpha}$ we denote a formal parameter with the dimension of velocity, used to define dimensionless weak-anisotropy parameters. We use a bar over $\bar{\alpha}$ in order to distinguish this parameter from attenuation coefficient α. We can choose the nonzero parameter $\bar{\alpha}$ arbitrarily. For the explicit expression for $Y(n_i)$ for P waves and for more details refer to Pšenčík and Farra (2005; eq. 12). If the weak-anisotropy parameters are of the same order as the dissipation parameters, we can simplify equations A-4 and A-5 further. We can substitute the phase velocity $\mathcal{C}^0_P$ in A-4 and A-5 by $\bar{\alpha}$. We can then express A-4 as follows:

$$\hat{Q}^{-1}_P = \mathcal{A}^{in}_P = (Q^{iso}_P)^{-1}[1 + 2(\epsilon^I_x n^4_1 + \epsilon^I_y n^4_2 + \epsilon^I_z n^4_3 + \delta^I_x n^2_1 n^2_3 + \delta^I_y n^2_2 n^2_3 + \delta^I_z n^2_1 n^2_2)] \ , \quad (A-7)$$

and A-5 as

$$\hat{Q}^{-1}_P = (Q^{iso}_P)^{-1}\Big(1 + 2[\epsilon^I_x(1-n^2_3)^2 + \epsilon^I_z n^4_3 + \delta^I_x(1-n^2_3)n^2_3]\Big) \ . \qquad (A-8)$$

We can see that in this case the velocity anisotropy does not affect Q-factor anisotropy.

We define the coefficients in A-7 and A-8 in a way similar to weak-anisotropy parameters (Pšenčík and Farra, 2005):

$$\epsilon_x^I = (\bar{A}_{11}^I - \bar{\alpha}^2 (Q_P^{iso})^{-1})/2\bar{\alpha}^2, \quad \epsilon_y^I = (\bar{A}_{22}^I - \bar{\alpha}^2 (Q_P^{iso})^{-1})/2\bar{\alpha}^2, \quad \epsilon_z^I = (\bar{A}_{33}^I - \bar{\alpha}^2 (Q_P^{iso})^{-1})/2\bar{\alpha}^2,$$

$$\delta_x^I = (\bar{A}_{13}^I + 2\bar{A}_{55}^I - \bar{\alpha}^2 (Q_P^{iso})^{-1})/\bar{\alpha}^2, \quad \delta_y^I = (\bar{A}_{23}^I + 2\bar{A}_{44}^I - \bar{\alpha}^2 (Q_P^{iso})^{-1})/\bar{\alpha}^2,$$

$$\delta_z^I = (\bar{A}_{12}^I + 2\bar{A}_{66}^I - \bar{\alpha}^2 (Q_P^{iso})^{-1})/\bar{\alpha}^2 \ . \qquad (A-9)$$

Q_P^{iso} is a quality factor corresponding to an isotropic dissipative medium. $\hat{Q}_P$ reduces to Q_P^{iso} if parameters A-9 are zero. By inserting A-9 into A-7 or A-8, we can easily prove that $\hat{Q}_P$ does not depend on Q_P^{iso} if parameters A-9 are nonzero.

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