Perturbation Hamiltonians in heterogeneous anisotropic weakly dissipative media

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Summary

Perturbation Hamiltonians simplify considerably the solution of various travel-time perturbation problems of seismic body waves propagating in heterogeneous, isotropic or anisotropic, weakly dissipative media. In addition to the phase-space coordinates, representing the Cartesian coordinates and Cartesian components of the slowness vector, the perturbation Hamiltonians depend also on one or more perturbation parameters, which can be chosen in different ways. General travel-time perturbation procedures, based on the perturbation Hamiltonians, were proposed by Klimeš (2002). Evaluation of travel-time perturbations of any order does not require the computation of perturbed rays in these procedures, and consists fully in the quadratures along a reference ray, computed in the reference medium. Different forms of perturbation Hamiltonian yield different accuracy and numerical efficiency of computations. In this paper, the special case of perturbation Hamiltonian, called here the difference perturbation Hamiltonian, is used to compute the first- and second-order perturbations of complex-valued travel time and the first-order perturbation of the complex-valued travel-time gradient. Heterogeneous perfectly elastic, isotropic or anisotropic, reference media are considered, in which reference rays can be calculated without problems by well-known methods. The derived perturbed equations are valid very generally. For isotropic reference model, the perturbed medium may be isotropic perfectly elastic, isotropic weakly dissipative, weakly anisotropic perfectly elastic, or weakly anisotropic weakly dissipative. Similarly, for reference anisotropic model, the perturbed medium may be anisotropic perfectly elastic, or anisotropic weakly dissipative. For perturbed weakly dissipative anisotropic media, also expressions for the attenuation vector, the global absorption factor, and the direction-dependent scalar local quality factor are derived. It is shown that the dissipative perturbation influences also the real-valued travel times. Simple expressions are given for such travel-time perturbation corrections, caused fully by dissipation.

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1 Introduction

In computation and inversion of travel-time fields of seismic body waves propagating in heterogeneous, isotropic or anisotropic media, an important role is played by perturbation methods. The perturbation methods are based on the assumption that the travel time in a reference model is known or may be simply computed, and that the perturbed model differs only slightly from the reference model. The travel time in the perturbed model that we wish to compute is then expanded in terms of power series in the deviation of perturbed from reference model. Depending on how many terms we keep, we speak of first-order perturbation, second-order perturbations, etc. Similar power series may be also found for travel-time derivatives, e.g. for the travel-time gradient.

A great attention has been devoted mainly to the first-order and second-order perturbations of the travel time, and to the first-order perturbation of the travel-time gradient. The highest interest in the first-order perturbation of the travel time is not surprizing. In most practical applications, the first-order perturbations of travel times have been used. Moreover, computation of first-order perturbations is relatively simple, in comparison with higher-order perturbations and/or in comparison with the first-order perturbation of travel-time derivatives.

Several approaches have been used in the travel-time perturbation methods. The first approach is based on the Fermat's principle, see Aki & Richards (1980) and Nolet (1987) for isotropic media, and Chapman & Pratt (1992) for anisotropic media. The approach has been used mostly for the computation of first-order travel-time perturbations. The second approach is based on the linearization of eikonal equation and/or on the perturbation of the Hamiltonian. For anisotropic media, see Červený (1982), Červený & Jech (1982), Hanyga (1982), Farra & Madariaga (1987), Nowack & Lutter (1988), Farra (1989), Farra, Virieux & Madariaga (1989), Jech & Pšenčík (1989), Nowack & Pšenčík (1991), Farra & Le Bégat (1995). This approach is again suitable for determination of first-order travel-time perturbation. Supplemented by dynamic ray tracing along the reference ray, however, the method can be also used in the computation of paraxial rays, and, consequently, in the computation of the second-order perturbation of travel times (Farra & Le Bégat, 1995; Farra, 1999; Červený, 2001), and of first-order perturbation of the travel-time gradient (Červený, 2002). Another approach is used on the first-order perturbation of the Hamiltonian, and on the use of the first-order perturbed Hamiltonian for ray tracing and dynamic ray tracing. Travel-time along such first-order rays is of first-order, and can be corrected, by quadratures, to second order (Pšenčík & Farra, 2005, 2007; Farra & Pšenčík, 2008). The third approach is based on the Lagrangian formulation of ray equations, see Snieder & Sambridge (1992), Snieder & Aldridge (1995), etc. Finally, the fourth approach, proposed by Klimeš (2002), is based on the perturbation Hamiltonians. The perturbation Hamiltonian $\mathcal{H}(x_a, p_b, f_\kappa)$ is a function of phase coordinates, consisting of the Cartesian coordinates x_i and Cartesian components of the slowness vector p_j, and of one or several perturbation parameters f_κ, $\kappa = 1, 2....$ The procedure of determination of travel-time perturbations is based on the quadratures along the reference ray. It does not require the computation of perturbed rays. It is quite universal and general, and may be recursively used to calculate the perturbations of any order of travel time and its derivative of any order. Mostly the perturbation Hamiltonians, which are homogeneous functions of degrees $N = 2$ or $N = -1$ in p_b, have been used although arbitrary degree N or even non-homogeneous Hamiltonians may be also used. The method of perturbation Hamiltonian has been successfully used in the investigation of coupling ray theory of S waves in anisotropic media by Klimeš & Bulant (2004). They considered homogeneous perturbation Hamiltonians of the degree $N = -1$ in p_b, which are better suited for travel-time perturbations than more natural $N = 2$, as numerically demonstrated by Bulant & Klimeš (2007). For homogeneous Hamiltonians of arbitrary N, see Klimeš & Bulant (2006).

In all above listed papers, the real-valued travel time was considered. The perturbation Hamiltonian, however, can be used even to compute complex-valued travel times, corresponding to weakly dissipative media. The reference medium is perfectly elastic and thus real-valued, and the reference rays are also real-valued. Consequently, standard well-known methods may be used to compute reference rays, and a weakly dissipative medium is considered as a perturbation of a perfectly elastic medium. The method of perturbation Hamiltonian was first used to investigate the properties of attenuation vector of seismic body waves in heterogeneous anisotropic dissipative media by Červený, Klimeš & Pšenčík (2008), where also the term "perturbation" Hamiltonian was first used. The paper considered homogeneous perturbation Hamiltonian of the degree $N = 2$ in p_b.

The perturbation Hamiltonian may be constructed in different forms. The numerical efficiency of perturbation computations and the accuracy of results depends on the used form of the perturbation Hamiltonian. In this paper, we use a special form of the perturbation Hamiltonian, called here the difference perturbation Hamiltonian, which promises high accuracy of results. Note that the difference perturbation Hamiltonian used here is different from the perturbation Hamiltonian used in Červený, Klimeš & Pšenčík (2008).

The main purpose of this paper is to use the difference perturbation Hamiltonian to derive and discuss the expressions for the first-order and second-order perturbations of complex-valued travel time and for the first-order perturbation of complex-valued travel-time gradient. We consider general Hamiltonians of the N-th degree, where N may be arbitrary integer. In several places, we also specify the results for the case of $N = 2$, which have been mostly used in applications, and for $N = -1$. We also include the discussion of the attenuation vector, of the global absorption factor and of the local quality factor in weakly dissipative media.

The content of this paper is as follows: In Section 2, perturbation Hamiltonians are introduced, and the difference perturbation Hamiltonian is defined. The definition is very general; it includes heterogeneous, isotropic or anisotropic, perfectly elastic, reference medium, and heterogeneous, isotropic or anisotropic, perfectly elastic or weakly dissipative perturbed medium. In Section 3, expressions for the perturbation expansions for complex-valued travel-time and for complex-valued travel time gradient are summarized. In Section 4, all perturbation expansions are expressed in terms of difference perturbation

Hamiltonian. Section 4 offers the main results of the paper. In Section 5, special cases are discussed. In Section 5.1, we discuss the case that the perturbation of the medium is purely imaginary. Consequently, the real-valued parts of the density-normalized viscoelastic moduli are the same in reference and perturbed media, only the imaginary parts differ. The imaginary parts are zero in the reference medium. In Section 5.2, a heterogeneous isotropic reference model is considered. The perturbed model may be then isotropic perfectly elastic, isotropic weakly dissipative, weakly anisotropic perfectly elastic, or weakly anisotropic weakly dissipative. As isotropic media are very important in practical applications, we present complete explicit equations for the case of the reference medium isotropic and perfectly elastic, and for the perturbed medium isotropic, perfectly elastic or weakly dissipative.

We denote Cartesian coordinates x_i and time t. We consider high-frequency time-harmonic seismic body waves, with the exponential factor $\exp(-\mathrm{i}\omega t)$, where ω is fixed, real-valued circular frequency. The lower-case Roman indices take the values 1, 2, 3, the upper-case 1, 2. The Greek indices take any positive integer value. The Einstein summation convention over repeating Roman indices is used.

2 Hamiltonians

2.1 Reference Hamiltonians

The travel time $\tau(x_i)$ of any high-frequency seismic body wave propagating in smoothly heterogeneous, isotropic or anisotropic medium is fully described by the eikonal equation and by relevant initial conditions. The eikonal equation can be derived from the equation of motion by high-frequency asymptotic methods. It is common to express the eikonal equation in a Hamiltonian form:

$$\mathcal{H}(x_a, p_b) = c \ , \tag{1}$$

where $\mathcal{H}$ is the Hamiltonian, x_a and p_b are phase-space coordinates, consisting of the Cartesian coordinates and the Cartesian components of the slowness vector, $p_b = \partial\tau/\partial x_b$, and c is a real-valued constant. For perfectly elastic media, the Hamiltonian is defined in a real-valued six-dimensional phase-space x_a, p_b $(a, b = 1, 2, 3)$, where x_a and p_b are real valued. For viscoelastic media, complex-valued x_a and p_b should be considered, and the six-dimensional phase space is complex-valued (Thomson, 1997).

We use the following notations for first and second partial derivatives of Hamiltonian $\mathcal{H}(x_a, p_b)$:

$$\begin{aligned} &\mathcal{H}^{,i} = \partial\mathcal{H}/\partial p_i \ , && \mathcal{H}_{,i} = \partial\mathcal{H}/\partial x_i \ , \\ &\mathcal{H}^{,ij} = \partial^2\mathcal{H}/\partial p_i \partial p_j \ , && \mathcal{H}^{,i}_{,j} = \partial^2\mathcal{H}/\partial p_i \partial x_j \ , \quad \mathcal{H}_{,ij} = \partial^2\mathcal{H}/\partial x_i \partial x_j \ . \end{aligned} \tag{2}$$

Here the index i in the superscript denotes the partial derivative of the Hamiltonian with respect to p_i, and the index i in the subscript the partial derivative of the Hamiltonian with respect to x_i.

Equation (1) is a non-linear partial differential equation of the first order. We consider here mostly Hamiltonians, which are homogeneous functions of the N-th degree in p_b. Actually, it would be possible to consider arbitrary Hamiltonians, but we limit ourselves to the simplest case of homogeneous Hamiltonians, to make the treatment most transparent. For anisotropic heterogeneous media, we can use the Hamiltonian of the N-th degree in the form

$$\mathcal{H}(x_a, p_b) = \tfrac{1}{N}[G_m(x_a, p_b)]^{N/2} \ . \tag{3}$$

Here $G_m(x_a, p_b)$ is one of the three eigenvalues of the 3×3 Christoffel matrix Γ_{ik}:

$$\Gamma_{ik}(x_a, p_b) = a_{ijkl}(x_n) p_j p_l \ . \tag{4}$$

Here $a_{ijkl}(x_n)$ are spatially variable density-normalized elastic or viscoelastic moduli, satisfying the symmetry relations:

$$a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij} \ . \tag{5}$$

The viscoelastic moduli are frequency dependent. We, however, consider an arbitrarily selected, fixed ω. The subscript m ($m = 1, 2, 3$) in $G_m(x_a, p_b)$ indicates the mode of the wave under consideration (P, S1, S2). We also denote $\mathbf{g}^{(m)}(x_a, p_b)$ the relevant eigenvector of the Christoffel matrix (4), corresponding to the eigenvalue $G_m(x_a, p_b)$. For a selected wave mode (P, S1 or S2), the eigenvector $\mathbf{g}^{(m)}$ is a function of $\mathbf{x}$ and $\mathbf{p}$. The Cartesian components of $\mathbf{g}^{(m)}$ satisfy the system of three linear equations:

$$\Gamma_{ik} g_k^{(m)} = g_i^{(m)} \ , \tag{6}$$

and the normalization condition,

$$\mathbf{g}^{(m)T} \mathbf{g}^{(m)} = 1 \tag{7}$$

(no summation over m). Consequently, the eigenvector $\mathbf{g}^{(m)}(x_a, p_b)$ is a real-valued or complex-valued vector of unit length. Then we can express the eigenvalue $G_m(x_a, p_b)$ in the following form:

$$G_m(x_a, p_b) = \Gamma_{ik}(x_a, p_b) g_i^{(m)} g_k^{(m)} = a_{ijkl}(x_n) p_j p_l g_i^{(m)} g_k^{(m)} \ , \tag{8}$$

(no summation over m). It follows from equation of motion that the eigenvalue (8) $G_m(x_a, p_b)$ of any selected wave mode satisfies in the phase space the relation (Červený, 2001, eq. 3.6.2):

$$G_m(x_a, p_b) = a_{ijkl}(x_n) p_j p_l g_i^{(m)} g_k^{(m)} = 1 \tag{9}$$

(no summation over m). For formulae for partial derivatives of $G_m(x_a, p_b)$ and $\mathcal{H}(x_a, p_b)$ in anisotropic media see Červený (2001, sec. 4.14.1), and Klimeš (2006b).

Using the expression (3) for Hamiltonian, we can compute any ray. The ray tracing equations read

$$\mathrm{d}x_i / \mathrm{d}\gamma_3 = \mathcal{H}^{,i} \ , \qquad \mathrm{d}p_i / \mathrm{d}\gamma_3 = -\mathcal{H}_{,i} \ . \tag{10}$$

$\mathcal{H}^{,i}$ and $\mathcal{H}_{,i}$ are partial derivatives of Hamiltonian with respect to p_i and x_i, see (2), and γ_3 is the variable along the ray. The variable γ_3 represents the travel time along the ray.

For perfectly elastic media, the variable γ_3 along the ray is real-valued, and the computation of rays is a well-understood problem, both for isotropic and anisotropic media

(Červený, 2001). For viscoelastic media, the position vector $\mathbf{x}$, slowness vector $\mathbf{p}$ and the variable γ_3 are complex-valued, and the ray tracing is considerably more complicated than in perfectly elastic media (Thomson, 1997). This applies particularly to the practically important case of two-point ray tracing, with the point source and receiver situated in the real space.

In this paper, we consider only weakly dissipative media. We use the perturbation approach, and consider the weakly-dissipative medium as a small perturbation of a perfectly elastic medium. Then we do not need to use the Hamiltonian (1) and ray tracing system (10) in a complex-valued six-dimensional phase space with complex-valued x_i, p_i and γ_3. It is sufficient to perform standard ray tracing in a reference perfectly elastic medium, which yields real-valued rays. The computation of perturbed complex-valued rays (Červený & Pšenčík, 2002) is not required. We also introduce a real-valued vector $\boldsymbol{\mathcal{U}}$, tangent to the real-valued rays, with Cartesian components $\mathcal{H}^{,i}$, and call it the ray velocity vector. Along real-valued rays, however, we need to perform dynamic ray tracing and perform some additional quadratures. To perform dynamic ray tracing, we specify each real-valued ray by ray parameters γ_1, γ_2, which may represent the take-off angles at a point source, or Gaussian coordinates along an initial surface. We introduce the 3×3 matrices $\mathbf{Q}$ and $\mathbf{P}$, with components Q_{ia} and P_{ia} by relations:

$$Q_{ia} = \partial x_i / \partial \gamma_a \ , \qquad P_{ia} = \partial p_i / \partial \gamma_a \ . \tag{11}$$

These matrices play a very important role in the perturbation method described in this paper. They can be computed by dynamic ray tracing along the known ray (or together with ray tracing equations):

$$\mathrm{d}Q_{ia}/\mathrm{d}\gamma_3 = \mathcal{H}^{,i}_{,j} Q_{ja} + \mathcal{H}^{,ij} P_{ja} \ , \quad \mathrm{d}P_{ia}/\mathrm{d}\gamma_3 = -\mathcal{H}_{,ij} Q_{ja} - \mathcal{H}^{,j}_{,i} P_{ja} \ . \tag{12}$$

Here $\mathcal{H}^{,i}_{,j}$, $\mathcal{H}^{,ij}$ and $\mathcal{H}_{,ij}$ denote the second partial derivatives of the Hamiltonian, given by (2).

It is useful to express the 3×3 matrix $\mathbf{Q}$ in terms of three real-valued contravariant basis vectors $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3$ connected with the reference ray Ω:

$$\mathbf{Q} \equiv [\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3 \equiv \boldsymbol{\mathcal{U}}] \ . \tag{13}$$

Here $\boldsymbol{\mathcal{U}}$ is the ray-velocity vector, tangent to the reference ray Ω. Vectors $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are tangent to the wavefront in the reference perfectly elastic medium. The Cartesian components of the vectors $\mathbf{Q}_I$ are Q_{iI}, and the Cartesian components of $\boldsymbol{\mathcal{U}}$ are Q_{i3}.

The initial point of the ray may represent a point source, or may be situated on an initial surface, on an initial line, or in a smooth medium. The relevant initial conditions for ray tracing system (10) and for dynamic ray tracing system (12) may be found in Červený and Moser (2008). For more details on ray tracing and dynamic ray tracing in perfectly elastic media, see Červený (2001), Červený et al. (2007).

In the following, we consider an arbitrary perfectly elastic medium, specified by real-valued, smoothly varying, density-normalized elastic moduli a_{ijkl}, and call it the **reference** medium, and the relevant Hamiltonian the reference Hamiltonian. We further consider an arbitrary, real-valued ray Ω in the reference medium and call it the reference ray Ω. We assume that the real-valued slowness vector $\mathbf{p}$, eigenvector $\mathbf{g}$, ray-velocity vector $\boldsymbol{\mathcal{U}}$, 3×3 matrices $\mathbf{Q}$ and $\mathbf{P}$, and the partial derivatives $\mathcal{H}^{,i}$, $\mathcal{H}_{,i}$, $\mathcal{H}^{,ij}$, $\mathcal{H}^{,i}_{,j}$, $\mathcal{H}^{,j}_{,i}$, $\mathcal{H}_{,ij}$ of the reference Hamiltonian, are known along the ray Ω, or can be simply calculated along it.

Now we consider another medium, which differs only weakly from the reference one, and call it the **perturbed medium**. We describe the density-normalized moduli in the perturbed medium by the expression

$$a_{ijkl}(x_n) + b_{ijkl}(x_n) \ , \tag{14}$$

where $b_{ijkl}(x_n)$ are smoothly varying perturbations of the density-normalized elastic moduli. The perturbations $b_{ijkl}(x_n)$ should satisfy symmetry relations analogous to (5). They are real-valued when both reference and perturbed media are perfectly elastic. If $b_{ijkl}(x_n)$ are purely imaginary, the reference medium is perfectly elastic and the perturbed medium dissipative. For complex-valued $b_{ijkl}(x_n)$, the perturbations include both the real-value (perfectly elastic) and imaginary-valued (dissipative) perturbations of $a_{ijkl}(x_n)$. The perturbed medium is dissipative and its elastic part differs from the reference medium.

For complex-valued b_{ijkl}, we use notation

$$b_{ijkl} = b^R_{ijkl} - ib^I_{ijkl} \ . \tag{15}$$

The minus sign in (15) is related to the minus sign in the exponential factor $\exp(-i\omega t)$ of the wave under consideration. We consider only such "physical" media, for which matrices a_{ijkl} and $a_{ijkl} + b^R_{ijkl}$ in the Voigt notation are positive definite (Fedorov, 1968; Musgrave, 1970; Helbig, 1994), and b^I_{ijkl} are positive-definite or zero (Červený & Pšenčík, 2005). We call b^I_{ijkl} the dissipation moduli. The case of $b^I_{ijkl} = 0$ corresponds to the perfectly elastic perturbed model.

2.2 Perturbation Hamiltonians

The perturbation Hamiltonian, introduced to seismic travel-time perturbation theory by Klimeš (2002), reads

$$\mathcal{H} = \mathcal{H}(x_a, p_b, f_\kappa) \ . \tag{16}$$

For fixed f_κ, the perturbation Hamiltonian is defined in a six-dimensional phase space x_a, p_b $(a, b = 1, 2, 3)$, where x_a are real-valued Cartesian coordinates and p_b real-valued components of the slowness vector, both along the reference ray in the reference medium. The perturbation Hamiltonian contains, in addition to x_a and p_b, one or several perturbation parameters f_κ $(\kappa = 1, 2, ...)$. The perturbation parameters are real valued. Actually, $\mathcal{H}(x_a, p_b, f_\kappa)$ represents a κ-parametric family of Hamiltonians.

The perturbation Hamiltonian can be used to compute the perturbation of any order of the travel time $\tau(x_i)$ and of the partial derivatives of travel time of any order. All the quantities in the perturbed medium are obtained just by quadratures along the reference ray, computed in the reference medium. No computation of rays in the perturbed medium is required. This is very important, particularly if the perturbed medium is dissipative (b_{ijkl} complex-valued). The complex-valued ray tracing would complicate the algorithm considerably. The perturbation method is, of course, only approximate, and valid for weakly dissipative media only. The dissipation in the Earth's interior is, however, usually weak, and thus the perturbation method can be used practically universally.

The perturbation Hamiltonian can be constructed in various ways. We shall consider here perturbation Hamiltonians specified by one perturbation parameter f_α only. Two simple forms of such perturbation Hamiltonians are as follows:

The first suitable form of the perturbation Hamiltonian, valid for smoothly variable anisotropic, perfectly elastic or viscoelastic media, for $N = 2$, is given by the relation

$$\mathcal{H}(x_a, p_b, f_\alpha) = \tfrac{1}{2}(a_{ijkl} + f_\alpha b_{ijkl}) p_j p_l g_i g_k \ . \tag{17}$$

This homogeneous perturbation Hamiltonian of the second degree ($N = 2$), with imaginary valued b_{ijkl}, was used to study the attenuation vector in heterogeneous anisotropic weakly dissipative media by Červený, Klimeš and Pšenčík (2008), and yielded compact perturbation expressions for the attenuation vector and for the complex-valued ray-velocity vector along the reference ray.

The second form of the perturbation Hamiltonian, valid also for smoothly variable anisotropic perfectly elastic or viscoelastic media and for arbitrary N, is given by the relation

$$\mathcal{H}(x_a, p_b, f_\alpha) = \mathcal{H}^0(x_a, p_b) + f_\alpha \Delta\mathcal{H}(x_a, p_b) \ , \tag{18}$$

where

$$\Delta\mathcal{H}(x_a, p_b) = \tilde{\mathcal{H}}(x_a, p_b) - \mathcal{H}^0(x_a, p_b) \ , \tag{19}$$

with

$$\mathcal{H}^0(x_a, p_b) = \tfrac{1}{N}[a_{ijkl} p_j p_l g_i g_k]^{N/2} \ , \tag{20}$$

$$\tilde{\mathcal{H}}(x_a, p_b) = \tfrac{1}{N}[(a_{ijkl} + b_{ijkl}) p_j p_l \tilde{g}_i \tilde{g}_k]^{N/2} \ . \tag{21}$$

Here $\mathcal{H}^0(x_a, p_b)$ is the reference Hamiltonian, $\tilde{\mathbf{g}}$ is one of the eigenvectors of the perturbed Christoffel matrix $\tilde{\Gamma}_{ik}(x_a, p_b)$, given by the relation:

$$\tilde{\Gamma}_{ik}(x_a, p_b) = (a_{ijkl} + b_{ijkl}) p_j p_l \ . \tag{22}$$

For real-valued b_{ijkl}, the perturbed Christoffel matrix $\tilde{\Gamma}_{ik}$ is real valued, and the eigenvectors $\tilde{\mathbf{g}}$ are also real-valued. For complex-valued b_{ijkl}, the perturbed Christoffel matrix $\tilde{\Gamma}_{ik}$ is complex-valued, and eigenvectors $\tilde{\mathbf{g}}$ are also complex-valued. We call (18) the *difference perturbation Hamiltonian* in this paper. The term "difference" emphasizes that the perturbation term in (18) is the difference of two Hamiltonians, see (19).

The perturbation Hamiltonians (17) and (18) coincide if $N = 2$ and $\tilde{\mathbf{g}} = \mathbf{g}$ is used in (20) and (21). The results corresponding to the difference perturbation Hamiltonians for different specifications, however, may be more general and more accurate than the results corresponding to the perturbation Hamiltonian (17).

3 Perturbation expansions

The travel time τ in the perturbed medium is a function of spatial coordinates x_i and of perturbation parameter f_α,

$$\tau = \tau(x_m, f_\alpha) \ . \tag{23}$$

Similar expression can be written for partial derivative of τ of any order. For example, for travel time gradient $\tau_{,i} = \partial\tau/\partial x_i$, we have

$$\tau_{,i} = \tau_{,i}(x_m, f_\alpha) \ . \tag{24}$$

Expressions (23) and (24) can be expanded in terms of f_α. We are interested here only in the second-order perturbation expansion of the travel time $\tau(x_m, f_\alpha)$, and in the first-order perturbation expansion of the travel-time gradient $\tau_{,i}(x_m, f_\alpha)$. These perturbation expansions play the most important role in seismic applications. They read:

$$\begin{aligned} \tau(x_m, f_\alpha) &\approx \tau(x_m) + f_\alpha \tau_{,\alpha}(x_m) + \tfrac{1}{2}(f_\alpha)^2 \tau_{,\alpha\alpha}(x_m) \ , & (25) \\ \tau_{,i}(x_m, f_\alpha) &\approx \tau_{,i}(x_m) + f_\alpha \tau_{,i\alpha}(x_m) \ . & (26) \end{aligned}$$

Here $\tau_{,\alpha}(x_m)$, $\tau_{,\alpha\alpha}(x_m)$ and $\tau_{,i\alpha}(x_m)$ denote the perturbation derivatives at $f_\alpha = 0$, given by expressions:

$$\begin{aligned} \tau_{,\alpha}(x_m) &= [\partial\tau(x_m, f_\alpha)/\partial f_\alpha]_{f_\alpha=0} \ , \\ \tau_{,i\alpha}(x_m) &= [\partial^2\tau(x_m, f_\alpha)/\partial x_i \partial f_\alpha]_{f_\alpha=0} \ , \\ \tau_{,\alpha\alpha}(x_m) &= [\partial^2\tau(x_m, f_\alpha)/\partial f_\alpha^2]_{f_\alpha=0} \ . \end{aligned} \tag{27}$$

The procedures how to compute the perturbation derivatives of the travel time or of any order of a partial derivative of the travel time of any order were proposed by Klimeš (2002). The perturbation derivatives can be determined by quadratures along the reference ray, using the perturbation Hamiltonian $\mathcal{H}(x_a, p_b, f_\alpha)$ and its derivatives, and the 3×3 matrices $\mathbf{Q}, \mathbf{P}$, known from dynamic ray tracing, see (11)–(12).

The algorithm proposed by Klimeš (2002) is very general, and can be used to compute the perturbation expansions of travel time of any order. The algorithm is recursive. When computing the higher order spatial and perturbation derivatives, we have to know the partial derivatives of lower orders. Once the perturbation derivatives (27) are computed, it is easy to determine the perturbation expansions (25) and (26), valid also in some f_α-vicinity of the real-valued reference ray. For the complex-valued perturbations b_{ijkl} of the density-normalized elastic moduli, the f_α-vicinity is complex-valued.

As we need to know the perturbation derivatives only along the reference ray, we can parameterize them by γ_3, which uniquely specifies any point on the ray, including the real-valued slowness vector $\mathbf{p}$ at that point. We also assume that $\tau_{,\alpha}(\gamma_3^0) = 0$ and $\tau_{,\alpha\alpha}(\gamma_3^0) = 0$, expressing the fact that the initial travel time is not perturbed. Note, however, that $\tau_{,i\alpha}(\gamma_3^0)$ is generally non-zero. The Klimeš's algorithm for evaluation of the perturbation derivatives $\tau_{,\alpha}(\gamma_3)$, $\tau_{,i\alpha}(\gamma_3)$ and $\tau_{,\alpha\alpha}(\gamma_3)$ is then simple. Without derivation, we present here the required results (Klimeš, 2002):

$$\begin{aligned} \tau_{,\alpha}(\gamma_3) &= T_\alpha(\gamma_3) \ , \qquad \tau_{,\alpha\alpha}(\gamma_3) = T_{\alpha\alpha}(\gamma_3) \ , & (28) \\ \tau_{,i\alpha}(\gamma_3) &= T_{a\alpha}(\gamma_3) Q_{ai}^{-1}(\gamma_3) \ , & (29) \end{aligned}$$

Here $T_\alpha(\gamma_3)$, $T_{\alpha\alpha}(\gamma_3)$ and $T_{\alpha\alpha}(\gamma_3)$ are given by the relations:

$$
\begin{aligned}
T_\alpha(\gamma_3) &= -\int_{\gamma_3^0}^{\gamma_3} \mathcal{H}_{,\alpha} \mathrm{d}\gamma_3' \\
T_{K\alpha}(\gamma_3) &= T_{K\alpha}(\gamma_3^0) - \int_{\gamma_3^0}^{\gamma_3} \left[\mathcal{H}_{,i\alpha} Q_{iK} + \mathcal{H}^{,i}_{,\alpha} P_{iK}\right] \mathrm{d}\gamma_3' \ , \\
T_{3\alpha}(\gamma_3) &= -\mathcal{H}_{,\alpha}(\gamma_3) \ . \\
T_{\alpha\alpha}(\gamma_3) &= -\int_{\gamma_3^0}^{\gamma_3} \left[\mathcal{H}_{,\alpha\alpha} + 2\mathcal{H}^{,i}_{,\alpha}\tau_{,i\alpha} + 2\mathcal{H}^{,ij}\tau_{,i\alpha}\tau_{,i\beta}\right] \mathrm{d}\gamma_3' \ .
\end{aligned}
\tag{30}
$$

Note that $T_\alpha(\gamma_3^0) = T_{\alpha\alpha}(\gamma_3^0) = 0$. The quantities $\mathcal{H}_{,\alpha}$, $\mathcal{H}_{,i\alpha}$, $\mathcal{H}^{,i}_{,\alpha}$ and $\mathcal{H}_{,\alpha\alpha}$ denote the perturbation derivatives of the perturbation Hamiltonian:

$$
\begin{aligned}
\mathcal{H}_{,\alpha} = \partial\mathcal{H}/\partial f_\alpha \ , \qquad & \mathcal{H}_{,\alpha\alpha} = \partial^2\mathcal{H}/\partial f_\alpha^2 \ , \\
\mathcal{H}^{,i}_{,\alpha} = \partial^2\mathcal{H}/\partial p_i \partial f_\alpha \ , \quad & \mathcal{H}_{,i\alpha} = \partial^2\mathcal{H}/\partial x_i \partial f_\alpha \ .
\end{aligned}
\tag{31}
$$

They are again considered as functions of γ_3, and are taken for $f_\alpha = 0$.

For $b^I_{ijkl} \neq 0$, the perturbed medium is dissipative. The perturbation expansions (25) and (26) for travel times $\tau(x_m, f_\alpha)$ and travel time gradient $\tau_{,i}(x_m, f_\alpha)$ are then complex-valued:

$$
\tau(x_m, f_\alpha) = \mathrm{Re}\tau(x_m, f_\alpha) + \mathrm{i}\mathrm{Im}\tau(x_m, f_\alpha) \ , \tag{32}
$$

$$
\tau_{,i}(x_m, f_\alpha) = \mathrm{Re}\tau_{,i}(x_m, f_\alpha) + \mathrm{i}\mathrm{Im}\tau_{,i}(x_m, f_\alpha) \ . \tag{33}
$$

Note that the real-valued vector $\mathbf{P}$ with components $P_i = \mathrm{Re}\tau_{,i}$ is often called the *propagation vector*, and the real-valued vector $\mathbf{A}$ with components $A_i = \mathrm{Im}\tau_{,i}$ is called the *attenuation vector*. The wave for which $\mathbf{P}$ is parallel to $\mathbf{A}$ at a point is usually called homogeneous at that point. The wave for which $\mathbf{P}$ has different direction than $\mathbf{A}$ is called inhomogeneous at a given point. This terminology is taken from the theory of plane waves in homogeneous viscoelastic media, see Declereq et al. (2005), Červený & Pšenčík (2005).

In the perturbed dissipative medium, we are mostly interested in the first-order perturbation expansions of τ and $\tau_{,i}$ only. For this reason, we introduce the terminology related to first-order perturbation expansion. From (32), (33) and (25) (26), we obtain

$$
\mathrm{Im}\tau(x_m, f_\alpha) \approx f_\alpha \mathrm{Im}\tau_{,\alpha}(x_m) \ , \tag{34}
$$

$$
\mathrm{Im}\tau_{,i}(x_m, f_\alpha) \approx f_\alpha \mathrm{Im}\tau_{,i\alpha}(x_m) \ . \tag{35}
$$

Note that the first terms of RHS's of (25) and (26), $\tau(x_m)$ and $\tau_{,i}(x_m)$, are real-valued and thus do not appear in (34) and (35).

We use the following notation. The tilde's above symbols denote that only the first-order perturbation is considered, and that the factor f_α is chosen $f_\alpha = 1$. We introduce the attenuation vector $\tilde{A}$, the global absorption factor $\tilde{t}^*(\gamma_3, \gamma_3^0)$, and the local quality factor $\tilde{Q}$ as follows:

The **attenuation vector $\tilde{\mathbf{A}}$** is introduced by the relation:

$$
\tilde{A}_i(\gamma_3) = \mathrm{Im}\tau_{,i\alpha}(\gamma_3) \ , \tag{36}
$$

where the perturbation derivative $\tau_{,i\alpha}(\gamma_3)$ is given by (29). The attenuation vector $\tilde{\mathbf{A}}$ was introduced in the above way by Červený, Klimeš & Pšenčík (2008), who used the perturbation Hamiltonian (17). The properties of the attenuation vector $\tilde{\mathbf{A}}$ in heterogeneous anisotropic weakly dissipative medium were thoroughfully investigated there. The above authors specified their results also for plane waves and for waves generated by a point source in homogeneous anisotropic weakly dissipative media. In this paper, we generalize the obtained results also for difference perturbation Hamiltonian (18).

Equations (29) and (36) yield

$$Q_{ia}(\gamma_3)\tilde{A}_i = \mathrm{Im}T_{a\alpha}(\gamma_3) \ . \tag{37}$$

Equation (37) can be rewritten as follows:

$$Q_{iK}(\gamma_3)\tilde{A}_i = \mathrm{Im}T_{K\alpha}(\gamma_3) \ , \quad Q_{i3}\tilde{A}_i = \mathrm{Im}T_{3\alpha}(\gamma_3) \ . \tag{38}$$

The first equation of (38) shows that the real-valued scalars $\mathrm{Im}T_{1\alpha}(\gamma_3)$ and $\mathrm{Im}T_{2\alpha}(\gamma_3)$ represent projections of the attenuation vector $\tilde{\mathbf{A}}$ into the plane tangent to the wavefront at its intersection with the reference ray Ω. In heterogeneous weakly dissipative media, $\mathrm{Im}T_{1\alpha}(\gamma_3)$ and $\mathrm{Im}T_{2\alpha}(\gamma_3)$ are, in general, non-vanishing. Consequently, the waves propagating in heterogeneous weakly dissipative media are, in general, inhomogeneous. For anisotropic weakly dissipative media, the exceptions are very rare. Only the waves generated by a point source in isotropic homogeneous weakly dissipative media, are homogeneous. The plane waves in homogeneous media may be homogeneous or inhomogeneous, depending on how initial conditions are chosen.

If we consider eq. (30), the second equation of (38) yields:

$$\mathcal{U}_i(\gamma_3)\tilde{A}_i(\gamma_3) = \mathrm{Im}T_{3\alpha}(\gamma_3) = -\mathrm{Im}\mathcal{H}_{,\alpha}(\gamma_3) \ . \tag{39}$$

The **global absorption factor** $\hat{t}^*(\gamma_3, \gamma_3^0)$ is given by the relation

$$\tilde{t}^*(\gamma_3, \gamma_3^0) = 2\mathrm{Im}\tau_{,\alpha}(\gamma_3) = -2\mathrm{Im}\int_{\gamma_3^0}^{\gamma_3} \mathcal{H}_{,\alpha}(\gamma_3')\mathrm{d}\gamma_3' \ , \tag{40}$$

where the perturbation derivative $\tau_{,\alpha}(\gamma_3)$ is given by (28), with the first equation of (30). In seismology, the global absorption factor is often called "t-star" factor. The global absorption factor $\tilde{t}^*(\gamma_3, \gamma_3^0)$ gives the complete dissipative decay of amplitudes of seismic body waves propagating in a weakly dissipative anisotropic (or isotropic) heterogeneous medium along the reference ray Ω from γ_3^0 to γ_3. The dimension of $\tilde{t}^*$ is time.

The **local quality factor** $\tilde{Q}(\gamma_3)$ is given by the relation

$$1/\tilde{Q}(\gamma_3) = 2\mathrm{Im}T_{3\alpha}(\gamma_3) = -2\mathrm{Im}\mathcal{H}_{,\alpha}(\gamma_3) \ . \tag{41}$$

Here $\mathcal{H}_{,\alpha}(\gamma_3)$ is the perturbation derivative of the perturbation Hamiltonian $\mathcal{H}(x_a, p_b, f_\alpha)$. As the determination of $\mathcal{H}_{,\alpha}$ from $\mathcal{H}(x_a, p_b, f_\alpha)$ is extremely simple, the computation of $\tilde{Q}(\gamma_3)$ does not cause any difficulty. $\tilde{Q}(\gamma_3)$ is a local quantity, and does not require any integration along the reference ray Ω.

The local quality factor $\tilde{Q}(\gamma_3)$ is a positive, dimensionless real-valued scalar quantity. In isotropic medium, it is direction independent. In anisotropic medium, however, it depends on the direction of the wave propagation, specified by the ray-velocity vector $\boldsymbol{\mathcal{U}}$ (tangent to the reference ray Ω). The local quality factor $\tilde{Q}(\gamma_3)$ may be used as a convenient measure of the local intrinsic dissipative properties of rocks.

As a simple example, let us consider the perturbation Hamiltonian (17). Then the local quality factor $\tilde{Q}(\gamma_3)$ in anisotropic weakly dissipative heterogeneous medium reads

$$1/\tilde{Q}(\gamma_3) = b^I_{ijkl}\, p_j p_l g_i g_k \; . \tag{42}$$

We can see that $1/\tilde{Q}(\gamma_3)$, given by (41), is an integrand of the integral expression (40) for the global absorption factor $\tilde{t}^*(\gamma_3, \gamma_3^0)$. The expression (42) for the local quality factor $\tilde{Q}$ in heterogeneous anisotropic weakly dissipative medium has been derived by Gajewski & Pšenčík (1992), see also Červený (2001, sec. 5.5). Another known relation for $\tilde{Q}(\gamma_3)$ is obtained from (39):

$$1/\tilde{Q}(\gamma_3) = 2\mathcal{U}_i(\gamma_3)\tilde{A}_i(\gamma_3) \; . \tag{43}$$

This equation shows that $1/\tilde{Q}(\gamma_3)$ is twice the scalar product of the attenuation vector $\tilde{\mathbf{A}}$ with the ray-velocity vector $\boldsymbol{\mathcal{U}}$. Equation (42) and (43) were also derived, in a different way, for plane waves propagating in homogeneous weakly dissipative anisotropic media by Červený & Pšenčík (2008). It was shown there that this expression does not depend on the inhomogeneity of the wave under consideration. See also Červený, Klimeš & Pšenčík (2008).

Of course, different forms of the perturbation Hamiltonian $\mathcal{H}(x_a, p_b, f_\alpha)$ give different expressions for the local quality factor $\tilde{Q}$, which differ in the accuracy. See a more detailed study for isotropic media in Section 5.2.

4 Perturbation expansion in terms of difference perturbation Hamiltonian

In this section, we consider the difference perturbation Hamiltonian introduced by (18). We use it to derive the second-order perturbation expansion for travel time $\tau(\gamma_3, f_\alpha)$ and the first-order perturbation expansions for the travel-time gradient $\tau_{,i}(\gamma_3, f_\alpha)$. The reference ray is constructed in a reference, perfectly elastic, isotropic or anisotropic, heterogeneous medium. The general perturbation expansions for these quantities for an arbitrary form of the perturbation Hamiltonian $\mathcal{H}(x_a, p_b, f_\alpha)$ are presented in Section 3.

We remind the reader that the vector $\tilde{\mathbf{g}}$ is introduced as the eigenvector of the perturbed Christoffel matrix $\tilde{\Gamma}_{ik}$ given by (22). It is real-valued if the perturbed medium is perfectly elastic, and complex-valued if the perturbed medium is dissipative. The Hamiltonians $\mathcal{H}^0$ and $\tilde{\mathcal{H}}$ are introduced by (20) and (21). The Hamiltonian $\tilde{\mathcal{H}}$ can be simply computed at any point of the reference ray as soon as the eigenvector $\tilde{\mathbf{g}}$ is determined.

The perturbation derivatives of the difference perturbation Hamiltonian (18) are then given by extremely simple expressions:

$$\begin{array}{ll} \mathcal{H}_{,\alpha} = \tilde{\mathcal{H}} - \mathcal{H}^0 \ , & \mathcal{H}^{,i}_{,\alpha} = \tilde{\mathcal{H}}^{,i} - \mathcal{H}^{0,i} \ , \\ \mathcal{H}_{,\alpha\alpha} = 0 \ , & \mathcal{H}_{,i\alpha} = \tilde{\mathcal{H}}_{,i} - \tilde{\mathcal{H}}^0_{,i} \ . \end{array} \tag{44}$$

The perturbation derivatives of travel time and of travel-time gradient are again given by (28) and (29), where $T_\alpha(\gamma_3)$, $T_{a\alpha}(\gamma_3)$ and $T_{\alpha\alpha}(\gamma_3)$ are given by relations:

$$\begin{aligned} T_\alpha(\gamma_3) &= -\int_{\gamma_3^0}^{\gamma_3} (\tilde{\mathcal{H}} - \mathcal{H}^0)\mathrm{d}\gamma_3' \ , \\ T_{K\alpha}(\gamma_3) &= T_{K\alpha}(\gamma_3^0) - \int_{\gamma_3^0}^{\gamma_3} \left[(\tilde{\mathcal{H}}_{,i} - \mathcal{H}^0_{,i})Q_{iK} - (\tilde{\mathcal{H}}^{,i} - \mathcal{H}^{0,i})P_{iK}\right]\mathrm{d}\gamma_3' \ , \\ T_{3\alpha}(\gamma_3) &= \mathcal{H}^0 - \tilde{\mathcal{H}} \ , \\ T_{\alpha\alpha}(\gamma_3) &= -2\int_{\gamma_3^0}^{\gamma_3} \tau_{,i\alpha}\left[(\tilde{\mathcal{H}}^{,i} - \mathcal{H}^{0,i}) + \mathcal{H}^{,ij}\tau_{,j\alpha}\right]\mathrm{d}\gamma_3' \ . \end{aligned} \tag{45}$$

The local quality factor $\hat{Q}(\gamma_3)$, defined by (41), is now expressed as follows:

$$1/\hat{Q}(\gamma_3) = -2\mathrm{Im}\tilde{\mathcal{H}}(\gamma_3) \ , \tag{46}$$

and the global absorption factor $\hat{t}^*(\gamma_3, \gamma_3^0)$ reads

$$\hat{t}^*(\gamma_3, \gamma_3^0) = -2\mathrm{Im}\int_{\gamma_3^0}^{\gamma_3} \tilde{\mathcal{H}}(\gamma_3')\mathrm{d}\gamma_3' \ . \tag{47}$$

The attenuation vector $\hat{\mathbf{A}}$ is again given by (36). It again satisfies equations (37)–(39).

Inserting (28) and (29) with (45), into perturbation expansions (25) and (26), we obtain the final solution of our problem. Resulting equations are valid quite generally, for isotropic and anisotropic, homogeneous and inhomogeneous, perfectly elastic and weakly dissipative, media.

For real-valued perturbations ($b^I_{ijkl} = 0$), the difference perturbation Hamiltonian (18) and the expressions for first- and second-order perturbations of travel times were used in the investigation of coupling ray theory of S waves by Klimeš & Bulant (2004, 2006) and by Bulant and Klimeš (2007). For isotropic reference media, the first- and second-order perturbations of the travel time, from an isotropic reference ray to any other ray (anisotropic, common anisotropic, isotropic) were computed by Klimeš and Bulant (2004). Accuracy of the algorithm was studied in detail. Klimeš and Bulant (2006) extended the approach to arbitrary reference ray in reference anisotropic medium. Bulant and Klimeš (2007) discussed first- and second-order perturbations from a common anisotropic reference ray to any other ray, and discussed numerical examples and accuracy of the approach. Klimeš (2006a) also presented algorithm for the computation of amplitudes using the difference perturbation Hamiltonian.

The studies of Klimeš and Bulant suggest that the method based on the difference perturbation Hamiltonian is highly accurate and universal (at least for $b^I_{ijkl} = 0$). Its accuracy clearly exceeds the accuracy of the method based on the perturbation Hamiltonian (17).

Even if we use only the first-order perturbation derivative $\tau_{,\alpha}$, the accuracy highly exceeds the accuracy of other one-step perturbation methods. The reason consists mainly in the exact computation of the eigenvector $\tilde{\mathbf{g}}$ from the perturbed Christoffel matrix $\tilde{\Gamma}(x_a, p_b)$, see (22). The first-order perturbation of eigenvectors is singular in the vicinity of shear wave singularities, see Jech & Pšenčík (1989) or Červený, Klimeš & Pšenčík (2008).

The method based on the difference perturbation Hamiltonian, applied to complex-valued perturbations b_{ijkl}, corresponds to weakly dissipative media. It is expected to give higher accuracy for the quality factor $\tilde{Q}$ than (42). See Section 5.1.

5 Special cases

In this section, we specify general equations presented in Section 4, based on the difference perturbation Hamiltonian (18), to two special cases.

5.1 Perturbation from perfectly elastic to weakly dissipative media

We consider the perturbed medium specified by density normalized viscoelastic moduli

$$a_{ijkl} - \mathrm{i}b^I_{ijkl} \; . \tag{48}$$

Consequently, $b^R_{ijkl} = 0$. The real-valued parts of density normalized viscoelastic moduli a_{ijkl} are the same in both reference and perturbed media; only the imaginary parts (dissipation moduli) differ.

This case was studied in detail by Červený, Klimeš & Pšenčík (2008), using the perturbation Hamiltonian (17). Here we will use the difference perturbation Hamiltonian (18). We do not discuss all equations in detail. Actually, all equations can be simply obtained from those presented in Sec. 4, inserting $b^R_{ijkl} = 0$. In the following, we discuss only the first-order perturbations of travel time $\tau_{,\alpha}(\gamma_3)$, which will clearly show the differences between the approaches based on the perturbation Hamiltonians (17) and (18).

The perturbation Hamiltonian (17) used in (29) and (30) yields,

$$\tau_{,\alpha}(\gamma_3) = \tfrac{\mathrm{i}}{2} \int_{\gamma_3^0}^{\gamma_3} b^I_{ijkl} p_j p_l g_i g_k \mathrm{d}\gamma_3' \; . \tag{49}$$

Consequently, $\tau_{,\alpha}(\gamma_3)$ is purely imaginary, and contains only the real-valued reference eigenvectors $\mathbf{g}$. It does not yield any change in the real-valued travel time. Using (40), we obtain the well-known expression (42) for $\hat{Q}$ from (49). Equations (49) and (42) contain the reference polarization vector $\mathbf{g}$ along the reference ray Ω.

Using the difference perturbation Hamiltonian (18), discussed in this paper, we obtain from (28) and (45)

$$\tau_{,\alpha}(\gamma_3) = - \int_{\gamma_3^0}^{\gamma_3} (\tilde{\mathcal{H}} - \mathcal{H}^0) \mathrm{d}\gamma_3' \; , \tag{50}$$

where $\mathcal{H}^0$ and $\tilde{\mathcal{H}}$ are given by (20) and (21). In (21), $\tilde{\mathbf{g}}$ is an exact eigenvector of the Christoffel matrix (22). The expression (50) yields more accurate results than the expression (49), because it solves exactly the eigenvalue problem for a 3×3 complex-valued Christoffel matrix (22). The main difference between (49) and (50) is that $\tau_{,\alpha}(\gamma_3)$ in (49) is purely imaginary and $\tau_{,\alpha}(\gamma_3)$ in (50) is complex-valued. As (50) is complex-valued, it includes also the real-valued corrections of the travel time. Thus, the introduced dissipation implies even corrections of real-valued travel times. These real-valued travel-time corrections are given by relation

$$\mathrm{Re}(\tau_{,\alpha}(\gamma_3)) = -\mathrm{Re}\int_{\gamma_3^0}^{\gamma_3} (\tilde{\mathcal{H}} - \mathcal{H}^0)\mathrm{d}\gamma_3^{'} \ . \tag{51}$$

It is interesting that these travel time corrections due to dissipation are obtained in the framework of the first-order perturbations.

Using the relation (41), we can obtain a more accurate estimation of the local quality factor $\hat{Q}$ than that given in (42):

$$1/\hat{Q}(\gamma_3) = -2\mathrm{Im}\tilde{\mathcal{H}}(\gamma_3) \ . \tag{52}$$

It can be clearly seen that (52) yields (42) if $N = 2$ and if $\tilde{\mathbf{g}} = \mathbf{g}$.

5.2 Isotropic reference model

The perturbation Hamiltonians (17) and (18) can be also used if the reference medium is isotropic perfectly elastic, and perturbed medium weakly anisotropic and weakly dissipative. For perfectly elastic isotropic reference medium, the density normalized elastic moduli are given by the relation,

$$a_{ijkl} = (V_P^2 - 2V_S^2)\delta_{ij}\delta_{kl} + V_S^2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \ , \tag{53}$$

where V_P and V_S are the velocities of P and S waves. Then the reference Christoffel matrix reads:

$$\Gamma_{ik}(x_a, p_b) = (V_P^2 - V_S^2)p_i p_k + \delta_{ik}V_S^2 p_l p_l \ . \tag{54}$$

The eigenvalues of the Christoffel matrix (54) are given by the relation

$$G(x_a, p_b) = V^2(x_n)p_l p_l \ . \tag{55}$$

For P waves, $V = V_P$, and the eigenvector is a unit vector tangent to the slowness vector $\mathbf{p}$ (perpendicular to the wavefront). For S waves, $V = V_S$, and the eigenvectors are two mutually perpendicular unit vectors, perpendicular to the slowness vector $\mathbf{p}$. Consequently, the homogeneous perturbation Hamiltonian (17) of the second degree ($N = 2$) for isotropic reference medium reads

$$\mathcal{H}(x_a, p_j, f_\alpha) = \tfrac{1}{2}V^2 p_i p_i + \tfrac{1}{2}f_\alpha b_{ijkl}p_i p_l g_j g_k \ , \tag{56}$$

The difference perturbation Hamiltonian (18) of the N-th degree is given by the relation (18) with (19) and (21), and with the reference Hamiltonian $\mathcal{H}^0(x_j, p_b)$ given by the relation:

$$\mathcal{H}^0(x_a, p_b) = \tfrac{1}{N}(V^2 p_k p_k)^{N/2} \ . \tag{57}$$

The eigenvector $\tilde{\mathbf{g}}$ is the appropriate eigenvector of the perturbed Christoffel matrix $\tilde{\Gamma}_{ik}$, given by the relation

$$\tilde{\Gamma}_{ik} = (V_P^2 - V_S^2)p_i p_k + \delta_{ik} V_S^2 p_n p_n + b_{ijkl} p_j p_l \ . \tag{58}$$

The difference between the first and second form of the perturbation Hamiltonian consists in the use of the eigenvectors $\mathbf{g}$ and $\tilde{\mathbf{g}}$ and in the freedom to choose the degree of homogeneity N of the Hamiltonian under consideration.

The perturbations b_{ijkl} in (56) and (58) are defined as the differences between the density-normalized viscoelastic moduli in the perturbed anisotropic medium and the density-normalized elastic moduli a_{ijkl} given by (53) in reference perfectly elastic isotropic medium. The perturbed medium may be weakly anisotropic weakly dissipative (b_{ijkl} complex-valued) or weakly anisotropic perfectly elastic (b_{ijkl} real-valued). For purely imaginary b_{ijkl}, the perturbed medium is characterized by anisotropic attenuation. However, it also includes weak anisotropic changes of real-valued travel time due to dissipation.

The idea to consider weakly anisotropic media as a perturbation of isotropic media has a long tradition in seismology. This includes the computation of phase and ray velocities of waves propagating in homogeneous weakly anisotropic perfectly elastic media by Backus (1965), the calculation of travel times in heterogeneous media along rays computed in reference isotropic medium (Červený & Jech, 1982; Červený & Firbas, 1984; Jech & Pšenčík, 1989; Nowack & Pšenčík, 1991; Bulant & Klimeš, 2002). Klimeš & Bulant (2004) calculated the first- and second-order travel time perturbations in weakly anisotropic perfectly elastic media along reference rays in isotropic medium, using the difference perturbation Hamiltonian. Pšenčík & Farra (2005, 2007) and Farra & Pšenčík (2008) proposed a procedure, in which first-order rays are computed directly in a studied medium. The travel times along the rays are first-order travel times. By quadratures along the first-order rays, second-order travel-time perturbations can be simply obtained.

The results considerably simplify if also the perturbed medium is isotropic. Moczo, Bard & Pšenčík (1987) studied such a case. Since they considered weakly dissipative media, they assumed an imaginary valued perturbation. Then the eigenvalue of the Christoffel matrix in the perturbed medium can be again expressed in the form of (55), where V^2 is replaced by a complex-valued function $b(x_i)$. The difference perturbation Hamiltonian is then given by (18) and (19), where $\mathcal{H}^0$ and $\tilde{\mathcal{H}}$ are given by expressions:

$$\mathcal{H}^0 = \tfrac{1}{N}[V^2 p_k p_k]^{N/2} \ , \quad \tilde{\mathcal{H}} = \tfrac{1}{N}[(V^2 + b)p_k p_k]^{N/2} \ . \tag{59}$$

The relations for $\tau_{,\alpha}(\gamma_3)$, $\tau_{,i\alpha}(\gamma_3)$ and $\tau_{,\alpha\alpha}(\gamma_3)$ remain the same as in (28)–(29) with (45), where we use

$$\begin{aligned} \mathcal{H}^{0,i} &= [V^2 p_k p_k]^{N/2-1} V^2 p_i \ , & \tilde{\mathcal{H}}^{,i} &= [(V^2 + b)p_k p_k]^{N/2-1}(V^2 + b)p_i \ , \\ \mathcal{H}^0_{,i} &= \tfrac{1}{2}[V^2 p_k p_k]^{N/2-1}(V^2)_{,i} p_n p_n \ , & \tilde{\mathcal{H}}_{,i} &= \tfrac{1}{2}[(V^2 + b)p_k p_k]^{N/2-1}(V^2 + b)_{,i} p_n p_n \ . \end{aligned} \tag{60}$$

In (59) and (60), we can, of course, insert $p_k p_k = V^{-2}$. Note that the resulting expressions are not influenced by the eigenvectors of the Christoffel matrices. Consequently, the differences between $\tilde{\mathbf{g}}$ and $\mathbf{g}$ do not play any role in isotropic media.

Let us now discuss, in a greater detail, the most important expression for $\tau_{,\alpha}(\gamma_3)$. From (28) with (45), we obtain a simple expression for $\tau_{,\alpha}(\gamma_3)$, valid for any N:

$$\tau_{,\alpha}(\gamma_3) = -\tfrac{1}{N}\int_{\gamma_3^0}^{\gamma_3} \left\{ \left[(V^2+b)p_kp_k\right]^{N/2} - \left[V^2p_kp_k\right]^{N/2} \right\} \mathrm{d}\gamma_3' \ . \tag{61}$$

For the commonly used case of $N=2$, which corresponds to the perturbation Hamiltonian (17) in isotropic media, the expression (61) simplifies

$$\tau_{,\alpha}(\gamma_3) = -\tfrac{1}{2}\int_{\gamma_3^0}^{\gamma_3} bp_kp_k\mathrm{d}\gamma_3' \ . \tag{62}$$

For real-valued b, the formula (62) corresponds to the well-known travel-time perturbation formula for isotropic inhomogeneous media (Aki & Richards, 1981). Another important case is $N=-1$:

$$\tau_{,\alpha}(\gamma_3) = \int_{\gamma_3^0}^{\gamma_3} \left\{ \left[(V^2+b)p_kp_k\right]^{-1/2} - \left[V^2p_kp_k\right]^{-1/2} \right\} \mathrm{d}\gamma_3' \ . \tag{63}$$

In (61)–(63), we can also insert $p_kp_k = V^{-2}$.

It is simple to see that (61) and (63) approach (62) for $bV^{-2} \rightarrow 0$. In general, however, the accuracy of the expression (61) for $\tau_{,\alpha}(\gamma_3)$ depends on N. It was shown by Bulant & Klimeš (2007) that the most accurate results for real-valued travel times are obtained for $N=-1$.

To make the expressions for $\tau_{,\alpha}(\gamma_3)$ more transparent, we express the perturbation term b in terms of Lamé's viscoelastic complex-valued moduli λ and μ. According to the definition, b is the perturbation of the density-normalized elastic modulus corresponding to the wave under consideration. It is possible to discuss b independently for P and S waves. For simplicity, however, we consider only P waves; for S waves the results would be quite analogous. The exact expressions for the density normalized viscoelastic complex-valued modulus of P waves read

$$\frac{\lambda+2\mu}{\rho} = \alpha_P^2 = \mathrm{Re}\,\alpha_P^2 - \mathrm{i}\mathrm{Im}\,\alpha_P^2 \ . \tag{64}$$

In the reference, perfectly elastic medium, $\alpha_P^2 = \mathrm{Re}(\lambda+2\mu)/\rho = V_P^2$, where V_P is the real-valued velocity of P waves. In the perturbed medium, however, we have

$$\tilde{\alpha}_P^2 = \mathrm{Re}\frac{\tilde{\lambda}+2\tilde{\mu}}{\tilde{\rho}} - \mathrm{i}\mathrm{Im}\frac{\tilde{\lambda}+2\tilde{\mu}}{\tilde{\rho}} = \tilde{V}_P^2(1-\mathrm{i}\tilde{\delta}) \ , \tag{65}$$

where

$$\tilde{\delta} = \mathrm{Im}(\tilde{\lambda}+2\tilde{\mu})/\mathrm{Re}(\tilde{\lambda}+2\tilde{\mu}) \ . \tag{66}$$

The signs "–" in front of the imaginary units in (64) and (65) correspond to the sign convention used in this paper, see (15). The tilde above the letters in (65) and (66) means "perturbed medium". Consequently, the perturbation term b is given by the relation

$$b = \tilde{\alpha}_P^2 - \alpha_P^2 = \tilde{V}_P^2(1-\mathrm{i}\tilde{\delta}) - V_P^2 \ . \tag{67}$$

Similarly, we obtain

$$V_P^2 + b = \tilde{V}_P^2(1 - \mathrm{i}\tilde{\delta}) \ . \tag{68}$$

Inserting (67) and (68) into (61)–(63), and taking into account that $p_k p_k = 1/V_P^2$, we obtain for general N:

$$\tau_{,\alpha}(\gamma_3) = -\tfrac{1}{N}\int_{\gamma_3^0}^{\gamma_3} \left\{[(1-\mathrm{i}\tilde{\delta})\tilde{V}_p^2/V_p^2]^{N/2} - 1\right\} \mathrm{d}\gamma_3' \ , \tag{69}$$

for $N = 2$,

$$\tau_{,\alpha}(\gamma_3) = -\tfrac{1}{2}\int_{\gamma_3^0}^{\gamma_3} \left[(1-\mathrm{i}\tilde{\delta})\tilde{V}_P^2/V_P^2 - 1\right] \mathrm{d}\gamma_3' \ , \tag{70}$$

and for $N = -1$,

$$\tau_{,\alpha}(\gamma_3) = \int_{\gamma_3^0}^{\gamma_3} \left(\frac{1}{\tilde{V}_P\sqrt{1-\mathrm{i}\tilde{\delta}}} - \frac{1}{V_P}\right) V_P \mathrm{d}\gamma_3' \ . \tag{71}$$

Note that $V_P \mathrm{d}\gamma_3' = \mathrm{d}s$, where s is the arclength along the ray. Thus, the integration variable in (71) may be interpreted as the arclength, and does not depend on the travel time.

We now determine the local quality factor $\tilde{Q}(\gamma_3)$. For arbitrary N, we obtain it from (69) by removing the integration sign and taking twice the imaginary part of it. As V_P is real-valued, we obtain:

$$1/\tilde{Q} = -\tfrac{2}{N}\mathrm{Im}\left[(1-\mathrm{i}\tilde{\delta})(\tilde{V}_P/V_P)^2\right]^{N/2} \ . \tag{72}$$

The expression (72) for $\tilde{Q}$ can be also written as follows:

$$1/\tilde{Q} = -\tfrac{2}{N}\mathrm{Im}\left[(1-\mathrm{i}\tilde{\delta})\left(1 + \frac{\tilde{V}_P^2 - V_P^2}{V_P^2}\right)\right]^{N/2} \ . \tag{73}$$

This is a very general expression, valid for arbitrary N, and containing both the structural perturbations $(\tilde{V}_P^2 - V_P^2)/V_P^2$, and the dissipative perturbations $\tilde{\delta}$.

If both these perturbations are small enough, we can neglect the quadratic term of their product, and obtain,

$$1/\tilde{Q} \approx -\tfrac{2}{N}\mathrm{Im}[(1 - \mathrm{i}\tilde{\delta} + (\tilde{V}_P^2 - V_P^2)/V_P^2]^{N/2} \ . \tag{74}$$

Linear expansion of (74) then yields:

$$1/\tilde{Q} \approx \mathrm{Im}[\mathrm{i}\tilde{\delta} - (\tilde{V}_P^2 - V_P^2)/V_P^2] \ . \tag{75}$$

This finally gives

$$1/\tilde{Q} \approx \tilde{\delta} = \mathrm{Im}(\tilde{\lambda} + 2\tilde{\mu})/\mathrm{Re}(\tilde{\lambda} + 2\tilde{\mu}) \ . \tag{76}$$

Thus, the linearization of (73) yields the decoupling of structural and dissipative perturbations, and a simple expression (76) for $1/\tilde{Q}$, independent of N.

In homogeneous viscoelastic isotropic medium, the quality factor Q of plane waves has been mostly defined by the well known relation, corresponding to (76) (Aki & Richards, 1980; Johnston & Toksöz, 1981; etc.). Thus, the linearization of (73) yields the same relation for $1/\tilde{Q}$, valid even for any seismic body wave, propagating in inhomogeneous isotropic weakly dissipative medium. The general expression (73), however, predicts also some higher-order effects resulting from coupling of dissipative and structural perturbations in inhomogeneous media.

6 Conclusions

Perturbation Hamiltonians play an important role in the computation of perturbations of any order of the travel time and of its partial derivatives of any order. The travel time under consideration may be complex-valued. This is very important, as also weakly dissipative media may be considered as a small perturbation of perfectly elastic media. The application of perturbation Hamiltonians does not require ray tracing of perturbed rays, and is fully based on the quadratures along the reference real-valued rays. Consequently, the complex-valued ray tracing in a dissipative medium is not required, even though the dissipative media are considered.

Various forms of perturbation Hamiltonians can be used, which differ in the accuracy and numerical efficiency of computations. In this paper, a suitable perturbation Hamiltonian (18) is used, called here the difference perturbation Hamiltonian. This Hamiltonian offers simple expressions for perturbation derivatives, simple algorithms and high-accuracy computations. We have explicitly derived the expressions for the first- and second-order perturbations of complex-valued travel times, and for first-order perturbations of travel-time gradient. These perturbations play the most important role in practical applications. For weakly dissipative perturbed media, the general expressions for the attenuation vector $\hat{\mathbf{A}}$, for the local quality factor $\hat{Q}$ and for global absorption factor $\hat{t}^*$ are given.

For the weakly dissipative perturbed medium, with the same real-valued parts of the density normalized elastic moduli as in reference medium, the model perturbations are purely imaginary. In this case, the approach based on the difference perturbation Hamiltonian predicts also the perturbation corrections of real-valued travel times. Consequently, we obtain travel-time correction caused by dissipation. The relevant travel-time corrections are involved both in first-order, but also in second order travel-time perturbations.

All equations are valid for any heterogeneous, isotropic or anisotropic, perfectly elastic reference media, and for heterogeneous, isotropic or anisotropic, perfectly elastic or weakly dissipative perturbed media. For isotropic reference medium, the perturbed medium may be isotropic perfectly elastic, isotropic weakly dissipative, anisotropic perfectly elastic, or anisotropic weakly dissipative. Thus, the effects of the weak anisotropy and weak dissipation may be treated together.

In our future investigations, we wish to concentrate on numerical studies of accuracy of derived equations. Further, we wish to derive approximate expressions for the high-frequency ray theory Green function for heterogeneous anisotropic weakly dissipative media.

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