Basic quantities in homogeneous anisotropic viscoelastic media using a perturbation approach

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ABSTRACT

Velocity, attenuation and the quality factor of waves propagating in homogenous media of arbitrary strength of anisotropy and attenuation are calculated in high-frequency asymptotics using a stationary slowness vector - the vector evaluated at the stationary point of the slowness surface. This vector is, in general, complex-valued and inhomogeneous, meaning that the real and imaginary parts of the vector have different directions. The slowness vector can be determined by solving three coupled polynomial equations of the 6^{th} order or by a non–linear inversion. The procedure is simplified, if perturbation theory is applied. The elastic medium is viewed as a background medium, and the attenuation effects are incorporated as perturbations. In the first-order approximation, the phase and ray velocities and their directions remain unchanged being the same as those in the background elastic medium. The perturbation of the slowness vector is calculated by solving a system of three linear equations. The phase attenuation and phase quality factor are linear functions of the perturbation of the slowness vector. Calculating the ray attenuation and ray quality factor is even simpler than calculating the phase quantities, because they are expressed just in terms of perturbations of the medium without the necessity to evaluate the perturbation of the slowness vector. Numerical modeling indicates that the perturbations are highly accurate, the errors being less than 0.3% for a medium with a quality factor of 20 or higher. The accuracy can further be enhanced by a simple modification of the first-order perturbation formulas.

Keywords: anisotropy, attenuation, perturbation theory, ray theory, seismic waves, viscoelasticity

1. INTRODUCTION

The high-frequency asymptotic approximation is frequently used in the wave propagation modeling because it yields a reasonably good accuracy in many seismic applications and requires almost no computer time compared with exact methods. Asymptotic formulas for velocities, attenuations and quality factors of waves propagating in homogeneous anisotropic attenuative media have recently been published by Vavryčuk (2007a,b). The formulas are valid for waves propagating in media of arbitrary strength of anisotropy and attenuation. The wave quantities are calculated using a stationary slowness vector which predicts the complex energy velocity vector parallel to a ray. The stationary slowness vector is, in general, complex-valued and inhomogeneous, and determining it is the most complicated step when calculating the asymptotic wave quantities. The stationary slowness vector can be calculated either by a non-linear inversion for four real-valued angles defining the directions of the real and imaginary parts of the slowness vector or by solving a system of three coupled polynomial equations of the 6^{th} order in three unknown components of the complex-valued slowness vector (Vavryčuk 2006).

The problem is simplified if perturbation theory is applied. The elastic medium is viewed as the background medium and the attenuation effects are calculated as perturbations. The procedure is similar to that for calculating wave quantities in weakly anisotropic elastic media, where weak anisotropy is introduced as a perturbation of the isotropic background (Thomsen 1986; Jech & Pšenčík 1989; Tsvankin & Thomsen 1994; Vavryčuk 1997, 2003; Farra 2001, 2004; Song et al. 2001; Pšenčík & Vavryčuk 2002). The only difference is that the isotropic background is degenerate and the perturbations due to weak anisotropy are real valued. In case of anisotropic media with attenuation, the background medium is non-degenerate and perturbations are complex valued. This approach has been adopted by several authors and applied to studying various aspects of propagation of plane waves in seismic exploration (Gajewski and Pšenčík 1992; Carcione 2000; Chichinina et al. 2006; Zhu and Tsvankin 2006, 2007; Červený and Pšenčík 2007).

In this paper, I apply the first-order perturbation theory to calculating high-frequency asymptotic quantities in anisotropic viscoelastic media. I derive the perturbation formula for the stationary slowness vector and, consequently, for other asymptotic quantities such as ray and phase velocities, attenuations and quality factors. The accuracy of the perturbations is tested using numerical examples.

2. BASIC PERTURBATION FORMULAS

In formulas, the real and imaginary parts of complex-valued quantities are denoted by superscripts *R* and *I*, respectively. A complex-conjugate quantity is denoted by an asterisk. The direction of a complex-valued vector v is calculated as $\mathrm{v}/\sqrt{\mathrm{v}^T\mathrm{v}}$, where the superscript T means the transposition (the normalization condition $\mathrm{v}/\sqrt{\mathrm{v}\cdot\mathrm{v}^*}$ is not used). The magnitude of complex-valued vector v is complex valued and is calculated as $\sqrt{\mathrm{v}^T\mathrm{v}}$. If any complex-valued vector is defined by a real-valued direction, it is called homogeneous, and if defined by a complex-valued direction, it is called inhomogeneous. The type of wave (P, S1 and S2) is denoted by superscripts (1), (2) and (3). The background quantity is denoted by superscript 0. The perturbation of f is denoted by Δf. The symbol $\Delta_a f$ denotes the perturbation of $f(a_{ijkl}, p_i)$ due to the perturbations of medium parameters a_{ijkl}. The symbol $\Delta_p f$ denotes the perturbation of $f(a_{ijkl}, p_i)$ due to the perturbation of slowness vector p. In formulas, the Einstein summation convention is used for repeated subscripts. Besides the standard four-index notation for viscoelastic parameters a_{ijkl} and quality parameters q_{ijkl}, also the two-index Voigt notation A_{MN} and Q_{MN} is alternatively used. The Voigt notation reduces pairs of indices *i,j* or *k,l* into a single index *M* or *N* using the following rules: $11 \to 1, 22 \to 2, 33 \to 3, 23 \to 4, 13 \to 5$, and $12 \to 6$.

2.1. Definition of the medium

A viscoelastic medium is defined by density-normalized stiffness parameters $a_{ijkl} = c_{ijkl}/\rho$ which are, in general, frequency-dependent and complex-valued. The real and imaginary parts of a_{ijkl},

$$a_{ijkl}(\omega) = a^R_{ijkl} + i\, a^I_{ijkl}, \tag{1}$$

define elastic and viscous properties of the medium, and their ratio, called the matrix of quality parameters,

$$q_{ijkl}(\omega) = -\frac{a^R_{ijkl}}{a^I_{ijkl}} \text{ (no summation over repeated indices),} \tag{2}$$

quantifies how attenuative the medium is. The sign in formula 2 depends on the definition of the Fourier transform used for calculating the viscoelastic parameters in the frequency domain. Here I use the forward Fourier transform with the exponential term $\exp(i\omega t)$, hence the quality parameters are defined with the minus sign in formula 2. When using the Fourier transform with the exponential term $\exp(-i\omega t)$, the minus sign must be omitted.

Let us assume that a_{ijkl} satisfy symmetry relations

$$a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij}, \tag{3}$$

and that viscous parameters a^I_{ijkl} are small with respect to elastic parameters a^R_{ijkl}. The viscoelastic medium can then be viewed as a medium obtained by a small perturbation of an elastic background,

$$a_{ijkl} = a^0_{ijkl} + \Delta a_{ijkl}, \tag{4}$$

where a^0_{ijkl} defines the background medium and Δa_{ijkl} its perturbation,

$$a^0_{ijkl} = a^R_{ijkl},\ \Delta a_{ijkl} = i a^I_{ijkl}\ . \tag{5}$$

2.2. Eigenvalues and eigenvectors of the Christoffel tensor

The Christoffel tensor in the viscoelastic medium is defined alternatively in terms of slowness direction n,

$$\Gamma_{jk}(\mathrm{n}) = a_{ijkl} n_i n_l, \tag{6}$$

or slowness vector p,

$$\Gamma_{jk}(\mathrm{p}) = a_{ijkl} p_i p_l , \tag{7}$$

where a_{ijkl} are the complex-valued viscoelastic parameters. The slowness direction n and the slowness magnitude $p = \sqrt{p_j p_j}$ are generally complex valued. The Christoffel tensor Γ_{jk} has three eigenvalues and three eigenvectors, which define properties of the P, S1 and S2 waves. Eigenvalues $G(\mathrm{n})$ and $G(\mathrm{p})$ read

$$G(\mathrm{n}) = a_{ijkl} n_i n_l g_j g_k = c^2 , \tag{8}$$

$$G(\mathrm{p}) = a_{ijkl} p_i p_l g_j g_k = 1 , \tag{9}$$

where $c = 1/p$ is the complex phase velocity. The eigenvectors of Γ_{jk} define the polarization vectors g.

If not specified in the next, the Christoffel tensor Γ_{jk} and eigenvalues G will be assumed to be functions of the slowness vector p, $\Gamma_{jk} = \Gamma_{jk}(\mathrm{p})$ and $G = G(\mathrm{p})$. Using perturbation theory, the Christoffel tensor Γ_{jk} is decomposed as

$$\Gamma_{jk} = \Gamma^0_{jk} + \Delta\Gamma_{jk} , \tag{10}$$

where Γ^0_{jk} is the Christoffel tensor in the background medium,

$$\Gamma^0_{jk} = a^0_{ijkl} p^0_i p^0_l , \tag{11}$$

and $\Delta\Gamma_{jk}$ is its perturbation controlled by perturbations of elastic parameters, Δa_{ijkl}, and by perturbation of the slowness vector, $\Delta\mathrm{p}$,

$$\Delta\Gamma_{jk} = \Delta a_{ijkl} p^0_i p^0_l + a^0_{ijkl}\left(p^0_i \Delta p_l + p^0_l \Delta p_i\right) . \tag{12}$$

If the Christoffel tensor Γ^0_{jk} has three different eigenvalues $G^{0(m)}$, m=1,2,3, it is called non-degenerate, and standard perturbation formulas for calculating its eigenvalues and eigenvectors can be applied (Korn and Korn 2000, Section 15.4-11). The perturbations of the P-wave eigenvalue $G^{(1)}$ and of the P-wave eigenvector $\mathrm{g}^{(1)}$ of Γ_{jk} are expressed as follows:

$$\Delta G^{(1)} = \Delta\Gamma_{jk} g_j^{0(1)} g_k^{0(1)},$$

(13)

$$\Delta g_i^{(1)} = \frac{\Delta G^{(12)}}{G^{0(1)} - G^{0(2)}} g_i^{0(2)} + \frac{\Delta G^{(13)}}{G^{0(1)} - G^{0(3)}} g_i^{0(3)}, \tag{14}$$

where

$$\Delta G^{(rs)} = \Delta\Gamma_{jk} g_j^{0(r)} g_k^{0(s)}. \tag{15}$$

Formulas for the perturbations of eigenvalues, $\Delta G^{(2)}$ and $\Delta G^{(3)}$, and of eigenvectors, $\Delta\mathrm{g}^{(2)}$ and $\Delta\mathrm{g}^{(3)}$, are analogous.

2.3. Energy velocity

The perturbation of energy velocity v (called the group velocity in elastic media),

$$v_i = \frac{1}{2}\frac{\partial G}{\partial p_i} = a_{ijkl} p_l g_j g_k, \tag{16}$$

is expressed as

$$\Delta v_i = \Delta a_{ijkl} p_l^0 g_j^0 g_k^0 + a_{ijkl}^0 p_l^0 \left(g_j^0 \Delta g_k + g_k^0 \Delta g_j\right) + a_{ijkl}^0 \Delta p_l g_j^0 g_k^0. \tag{17}$$

Taking into account equation 14 we get for the P-wave

$$\Delta v_i^{(1)} = \Delta_a v_i^{(1)} + \Delta_p v_i^{(1)}, \tag{18}$$

$$\Delta_a v_i^{(1)} = \Delta a_{ijkl} p_l^{0(1)} g_j^{0(1)} g_k^{0(1)} + \frac{v_i^{0(12)} \Delta_a G^{(12)}}{G^{0(1)} - G^{0(2)}} + \frac{v_i^{0(13)} \Delta_a G^{(13)}}{G^{0(1)} - G^{0(3)}}, \tag{19}$$

$$\Delta_p v_i^{(1)} = a_{ijkl}^0 \Delta p_l^{(1)} g_j^{0(1)} g_k^{0(1)} + \frac{v_i^{0(12)} \Delta_p G^{(12)}}{G^{0(1)} - G^{0(2)}} + \frac{v_i^{0(13)} \Delta_p G^{(13)}}{G^{0(1)} - G^{0(3)}}, \tag{20}$$

where

$$v_i^{0(rs)} = a_{ijkl}^0 p_l^{0(1)} \left(g_j^{0(r)} g_k^{0(s)} + g_k^{0(r)} g_j^{0(s)} \right), \tag{21}$$

$$\Delta_a G^{(rs)} = \Delta a_{ijkl} p_i^{0(1)} p_l^{0(1)} g_j^{0(r)} g_k^{0(s)}, \tag{22}$$

$$\Delta_p G^{(rs)} = a_{ijkl}^0 \left(p_l^{0(1)} \Delta p_i^{(1)} + p_i^{0(1)} \Delta p_l^{(1)} \right) g_j^{0(r)} g_k^{0(s)}. \tag{23}$$

Perturbation formulas for the S1- and S2-waves are analogous.

3. STATIONARY SLOWNESS VECTOR

In this section, the perturbation approach is applied to asymptotic wave fields. The asymptotic quantities describe high-frequency waves observed at large distances from the source and are calculated for the slowness vector taken at a stationary point on the slowness surface. The stationary point is a point for which the complex-valued energy velocity vector v is homogeneous and its direction is parallel to the ray (for details, see Vavryčuk 2007a). The stationary slowness vector p is, in general, complex valued and inhomogeneous, and can be calculated by a non-linear inversion or by solving a system of three coupled polynomial equations of the 6th order. Once the stationary vector p is found, all asymptotic wave quantities can readily be calculated.

If the viscoelastic medium is obtained by a small perturbation of an elastic background medium (see equations 4 and 5), the viscoelastic wave quantities can be calculated as perturbations of the elastic ones using the general perturbation formulas

derived in the previous section. However, the perturbation formulas must further be modified to be valid for asymptotic quantities. The main difference between the general perturbations and the perturbations for asymptotic quantities lies in treating with the perturbation of slowness vector $\Delta\mathrm{p}$. In general formulas, $\Delta\mathrm{p}$ is not apriori fixed but depends on the studied wave, characterized by its wave normal and wave inhomogeneity (Červený and Pšenčík, 2005). On the other hand, $\Delta\mathrm{p}$ is not arbitrary for asymptotic quantities, but takes just one specific value with no freedom in setting the wave normal or wave inhomogeneity. The value of $\Delta\mathrm{p}$ must ensure that the perturbed slowness vector p is taken at the stationary point on the complex-valued slowness surface. Taking the slowness vector at the stationary point implies that the complex energy velocity vector v is homogeneous and parallel to the fixed ray direction.

3.1. Perturbation $\Delta\mathrm{v}$

Since the ray direction N is fixed, the energy velocity vector v is parallel to the ray in the background as well as in the perturbed medium

$$\mathrm{v} = \mathrm{N}\,v\,, \quad \mathrm{v}^0 = \mathrm{N}\,v^0\,. \tag{24}$$

Consequently, velocity perturbation $\Delta\mathrm{v}$ is parallel to background velocity v^0,

$$\Delta\mathrm{v} = \mathrm{N}\,\Delta v\,. \tag{25}$$

Taking into account equation 16 and the fact that the eigenvalue G of Γ_{jk} for the studied wave is equal to 1 (see equation 9) in the background as well as in the perturbed medium, we get

$$v_i p_i = 1\,, \quad v_i^0 p_i^0 = 1\,, \tag{26}$$

hence

$$\Delta v_i p_i^0 + v_i^0 \Delta p_i = 0 \ . \tag{27}$$

Multiplying equation 17 by p_i^0 and taking into account equation 27 we obtain

$$\Delta v_i p_i^0 = \frac{1}{2} \Delta a_{ijkl} p_i^0 p_l^0 g_j^0 g_k^0 \ . \tag{28}$$

The left-hand side of equation 28 can further be simplified to read

$$\Delta v_i p_i^0 = \frac{\Delta v}{v^0}, \tag{29}$$

where we have used

$$\Delta v_i = \Delta v N_i \quad \text{and} \quad N_i p_i^0 = \frac{1}{v^0} \ . \tag{30}$$

Hence,

$$\Delta v = \frac{v^0}{2} \Delta a_{ijkl} p_i^0 p_l^0 g_j^0 g_k^0 \ , \tag{31}$$

and, finally,

$$\Delta v_i = \frac{v_i^0}{2} \Delta a_{mjkl} p_m^0 p_l^0 g_j^0 g_k^0 \ . \tag{32}$$

3.2. Perturbation Δp

Equations 18 and 32 yield

$$\Delta_a v_i + \Delta_p v_i = \frac{v_i^0}{2} \Delta a_{mjkl} p_m^0 p_l^0 g_j^0 g_k^0 \ , \tag{33}$$

where $\Delta_a \mathrm{v}$ and $\Delta_p \mathrm{v}$ are defined in equations 19 and 20. Equation 33 can be expressed as

$$H_{il}^0 \Delta p_l = \frac{v_i^0}{2} \Delta a_{mjkl} p_m^0 p_l^0 g_j^0 g_k^0 - \Delta_a v_i \,, \tag{34}$$

where H_{il}^0 is the wave metric tensor in the background medium (see Vavryčuk 2003, equation 15). The wave metric tensor H_{il} is defined as (Červený 2001, equation 2.2.65)

$$H_{il}(\mathrm{p}) = \frac{1}{2} \frac{\partial^2 G(\mathrm{p})}{\partial p_i \partial p_l}, \tag{35}$$

and is connected with the Gaussian curvature of the slowness surface (see Klimeš 2002). Taking into account equation 15 of Vavryčuk (2003) and equations 19 and 22, tensor H_{il}^0 and vector $\Delta_a \mathrm{v}$ in equation 34 are expressed for the P-wave, as follows:

$$H_{il}^{0(1)} = a_{ijkl}^0 g_j^{0(1)} g_k^{0(1)} + \frac{v_i^{0(12)} v_l^{0(12)}}{G^{0(1)} - G^{0(2)}} + \frac{v_i^{0(13)} v_l^{0(13)}}{G^{0(1)} - G^{0(3)}}, \tag{36}$$

$$\Delta_a v_i^{(1)} = \Delta a_{mjkl} p_l^{0(1)} g_j^{0(1)} \left[\delta_{im} g_k^{0(1)} + \frac{v_i^{0(12)} p_m^{0(1)} g_k^{0(2)}}{G^{0(1)} - G^{0(2)}} + \frac{v_i^{0(13)} p_m^{0(1)} g_k^{0(3)}}{G^{0(1)} - G^{0(3)}} \right], \tag{37}$$

where δ_{ij} is the Kronecker delta. Since all quantities in the background medium are real valued and Δa_{ijkl} is pure imaginary, equations 36 and 37 imply that H_{il}^0 is also real valued and $\Delta_a \mathrm{v}$ is pure imaginary.

Equation 34 represents a system of three linear equations in unknown perturbations Δp_1, Δp_2 and Δp_3, hence perturbation $\Delta \mathrm{p}$ finally comes out as

$$\Delta p_l = \left[H_{il}^0 \right]^{-1} \left(\frac{v_i^0}{2} \Delta a_{mjkl} p_m^0 p_l^0 g_j^0 g_k^0 - \Delta_a v_i \right). \tag{38}$$

Since all background quantities are real valued, and Δa_{ijkl} and $\Delta_a \mathrm{v}$ are pure imaginary, the perturbation $\Delta \mathrm{p}$ is pure imaginary. Obviously, this property is lost if applying higher-order perturbations.

Note that equation 38 is valid under several limitations. Firstly, viscous parameters must be small with respect to the elastic parameters (see equations 4 and 5). Secondly, the wave must not propagate in directions close to cusp edges on the wave surface. The cusp edges arise when the wavefront is multi-folded and correspond to parabolic lines on the slowness surface (Vavryčuk 2003). The inverse of the wave metric tensor $\left[H_{il}^{0}\right]^{-1}$ is large or diverges near parabolic lines, and thus the perturbation theory is inapplicable. Thirdly, the wave must not propagate near directions associated with singularitites (acoustic axes) on the slowness surface (Vavryčuk 2005). The Christoffel tensor is nearly degenerate in these directions and the standard perturbation formulas fail.

4. PHASE VELOCITY, ATTENUATION AND QUALITY FACTOR

4.1. Definitions

Phase quantities describe propagation of a plane tangential to the wave front. By decomposing the complex-valued slowness vector p we obtain (see Vavryčuk 2007b, equation 2),

$$\mathrm{p} = \left[\left(V^{\mathrm{phase}}\right)^{-1} + iA^{\mathrm{phase}}\right]\mathrm{s} + iD^{\mathrm{phase}}\,\mathrm{t}\,, \tag{39}$$

where V^{phase}, A^{phase} and D^{phase} are the real-valued phase velocity, phase attenuation and phase inhomogeneity. Vectors s and t are real-valued, mutually perpendicular unit vectors, s is normal to the wave front (called the wave normal), and t lies in the wave front (called the wave tangent). Hence, the phase velocity and phase attenuation are calculated from p as

$$V^{\mathrm{phase}} = \frac{1}{\left|\mathrm{p}^{R}\right|}, \tag{40}$$

$$A^{\text{phase}} = \mathrm{p}^I \cdot \mathrm{s}, \ , \tag{41}$$

$$D^{\text{phase}} = \mathrm{p}^I \cdot \mathrm{t}\,, \tag{42}$$

where

$$\mathrm{s} = \frac{\mathrm{p}^R}{\left|\mathrm{p}^R\right|} \ , \ \mathrm{t} = \frac{\mathrm{p}^I - \left(\mathrm{p}^I \cdot \mathrm{s}\right)\mathrm{s}}{\left|\mathrm{p}^I - \left(\mathrm{p}^I \cdot \mathrm{s}\right)\mathrm{s}\right|}\,, \tag{43}$$

and symbol $|\mathrm{a}| = \sqrt{\mathrm{a} \cdot \mathrm{a}} = \sqrt{a_j a_j}$ denotes the magnitude of real-valued vector a.

The phase quality factor (Q^{phase}-factor) is defined as (Carcione 2000, equation 14; Carcione 2001, equation 4.92)

$$Q^{\text{phase}} = -\frac{\left(c^2\right)^R}{\left(c^2\right)^I} \ , \tag{44}$$

where c is the complex-valued phase velocity, $c = 1/\sqrt{p_i p_i}$.

In elastic media, the stationary slowness vector p is homogeneous. Consequently, velocity c becomes real valued, the Q^{phase}-factor is infinite, and the velocity c equals the phase velocity V^{phase}.

4.2. Perturbations

Taking into account that

$$\mathrm{p} = \mathrm{p}^0 + \Delta\mathrm{p}\,, \tag{45}$$

where p^0 is a real-valued vector and perturbation $\Delta\mathrm{p}$ is a pure imaginary vector (see equation 38),

$$\mathrm{p}^0 = \mathrm{p}^{0R}, \ \Delta\mathrm{p} = i\Delta\mathrm{p}^I\,, \tag{46}$$

we readily obtain

$$V^{\text{phase}} = \left(V^{\text{phase}}\right)^0, \tag{47}$$

$$A^{\text{phase}} = \Delta \mathrm{p}^I \cdot \mathrm{s}^0 \ , \tag{48}$$

$$D^{\text{phase}} = \Delta \mathrm{p}^I \cdot \mathrm{t}^0 \ . \tag{49}$$

Using the following approximate formula

$$c^2 = \frac{1}{\left(p_i^0 + \Delta p_i\right)\left(p_i^0 + \Delta p_i\right)} = \frac{1}{p_i^0 p_i^0 + 2 p_i^0 \Delta p_i} = \frac{1}{p_j^0 p_j^0}\left(1 - 2\frac{\Delta p_i p_i^0}{p_j^0 p_j^0}\right), \tag{50}$$

the inverse of the phase quality factor comes out as

$$\left[Q^{\text{phase}}\right]^{-1} = 2\frac{\Delta p_i^I p_i^{0R}}{p_j^{0R} p_j^{0R}} = 2c^0 A^{\text{phase}} . \tag{51}$$

Similarly as the phase velocity, the wave normal s and the wave tangent t remain unchanged being the same for the background and the perturbed medium.

5. RAY VELOCITY, ATTENUATION AND QUALITY FACTOR

5.1. Definitions

Ray quantities describe the propagation of waves along a ray. The ray velocity V^{ray} (see Vavryčuk 2007b, equation 21)

$$V^{\text{ray}} = \frac{v v^*}{v^R} = \frac{v^R v^R + v^I v^I}{v^R} \tag{52}$$

controls the propagation velocity along a ray, and the ray attenuation A^{ray} (see Vavryčuk 2007b, equation 22)

$$A^{\text{ray}} = -\frac{v^I}{vv^*} = -\frac{v^I}{v^R v^R + v^I v^I} \tag{53}$$

controls the amplitude decay along a ray. The ray velocity and ray attenuation are real-valued and can be observed and measured in wave fields along a ray. In analogy to the phase quality factor defined in equation 44, we can also introduce the ray quality factor (see Vavryčuk 2007b, equation 24)

$$Q^{\text{ray}} = -\frac{\left(v^2\right)^R}{\left(v^2\right)^I} \ . \tag{54}$$

Introducing ray inhomogeneity D^{ray} has no sense because the complex energy velocity vector v is homogeneous, hence D^{ray} is identically zero.

5.2. Perturbations

Substituting the following equations

$$\mathrm{v}^R = \mathrm{v}^{0R} \text{ and } \Delta\mathrm{v} = i\mathrm{v}^I, \tag{55}$$

into equations 52 and 53, taking into account equation 32 and retaining the first-order perturbations, we get

$$V^{\text{ray}} = v^{0R} = v^0, \tag{56}$$

$$A^{\text{ray}} = \frac{i\Delta v}{v^{0R} v^{0R}} = -\frac{1}{2v^0} \Delta a^I_{ijkl} p^0_i p^0_l g^0_j g^0_k \ . \tag{57}$$

Finally, using equation 31 and the following approximate formula

$$v^2 = \left(v^0 + \Delta v\right)^2 = \left(v^0\right)^2 + 2v^0 \Delta v \ , \tag{58}$$

the inverse of the ray quality factor comes out as

$$\left[Q^{\text{ray}}\right]^{-1} = -\Delta a^I_{ijkl} p^0_i p^0_l g^0_j g^0_k = 2v^0 A^{\text{ray}} . \tag{59}$$

The ray direction N is the same for the background and the perturbed media, because it was fixed when deriving perturbations. Similar formulas to formulas 57 and 59 are valid also for the ray attenuation and ray quality factor of weakly inhomogeneous plane waves (see Červený and Pšenčík 2007). Note that equation 59 has also been derived by Gajewski and Pšenčík (1992) who assumed weakly attenuating medium described by the Futterman's dispersion relation (Futterman 1962). Here, I have shown that equation 59 is valid independently of any type of the dispersion.

6. PHASE AND RAY VELOCITIES WITH IMPROVED ACCURACY

The phase and ray velocities in the perturbed viscoelastic medium have the same value as in the elastic background medium (see equations 47 and 56). This implies that the first-order perturbations do not reflect the velocity shift owing to attenuation of the medium. If such approximation is not sufficient and more accurate formulas are needed, we can combine perturbations with exact calculations. In this way, the perturbations are used only for determining the stationary slowness vector that is the crucial and most complicated step in the calculations. All other quantities are computed using exact formulas. Specifically, we apply the following procedure. First, the slowness vector p is calculated using equations 38 and 45. Second, the slowness vector p is normalized to obtain the slowness direction $\mathrm{n} = \mathrm{p} / \sqrt{p_j p_j}$. Third, the complex phase velocity c and real phase velocity V^{phase} are calculated using equations 8 and 40. Finally, the complex energy velocity vector v and the ray velocity V^{ray} are calculated using equations 16 and 52.

7. NUMERICAL EXAMPLES

In this section, I demonstrate the accuracy of the perturbation formulas using numerical examples performed on the P-wave. I adopted eight test viscoelastic models displaying transverse isotropy with various strength of anisotropy and attenuation.. The anisotropy strength is chosen in the range of 10-20% to be close to observations typical for sedimentary rocks (Thomsen 1986; Vernik and Liu 1997; Jakobsen and Johansen 2000). The Q-factors range over a broad interval of values: from moderate ($Q = 40 - 60$) to very strong attenuation ($Q = 5 - 7$). The very strong attenuation is rather extreme and not typically measured (see Shankland et al. 1993; Liu et al. 1997; Best et al. 2007) or predicted from theoretical models of sedimentary rocks (Carcione 2000; Carcione et al. 2003). Here it is used just to demonstrate limits of applicability of perturbations. Although, the presented models are artificial and do not cover all possible variations of anisotropy and attenuation in realistic rocks, they should give a sufficient insight into the efficiency of perturbation formulas derived.

The viscoelastic parameters of the models are summarized in Tables 1 and 2. Table 1 lists the parameters in the Voigt notation, and Table 2 in the Thomsen-style notation. The eight models combine two models of velocity anisotropy (Models A and B) and four levels of attenuation (Models A1-A4 and B1-B4). The P-wave anisotropy is about 23% for Models A1-A4, and about 10% for Models B1-B4 (see Table 3). The strongest attenuation is in Models A1 and B1, the mean Q^{ray}-factor being about 5.5. The weakest attenuation is in Models A4 and B4, the mean Q^{ray}-factor being about 43.5. The attenuation anisotropy is 65% for Models A1-A4, and 55% for Models B1-B4. The Q-factor anisotropy is 45.5% for all eight models. The frequency of the signal is assumed to be 30 Hz.

Figure 1 shows polar plots of the P- and SV-wave group velocities for the elastic background of Models A and B. The SV-wave group velocity displays a triplication. The triplication causes a non-unique relation between the phase and ray directions. For some ray directions, three different wave normals can be observed (see Figure 2b, 2d). As mentioned above, the triplication prevents the perturbation formulas being applicable, hence the calculations are performed only for the P-wave.

Table 1. Viscoelastic parameters.

Model	Elastic parameters				Attenuation parameters			
	A_{11}^R (km^2/s^2)	A_{13}^R (km^2/s^2)	A_{33}^R (km^2/s^2)	A_{44}^R (km^2/s^2)	Q_{11}	Q_{13}	Q_{33}	Q_{44}
A1	14.4	4.50	9.00	2.25	7.5	4	5	4
A2	14.4	4.50	9.00	2.25	15	8	10	8
A3	14.4	4.50	9.00	2.25	30	16	20	16
A4	14.4	4.50	9.00	2.25	60	32	40	32
B1	10.8	3.53	9.00	2.25	7.5	4	5	4
B2	10.8	3.53	9.00	2.25	15	8	10	8
B3	10.8	3.53	9.00	2.25	30	16	20	16
B4	10.8	3.53	9.00	2.25	60	32	40	32

The two-index Voigt notation is used for density-normalized elastic parameters a_{ijkl}^R and quality parameters q_{ijkl}. Paramaters A_{66}^R and Q_{66} are not listed because the P-wave is not sensitive to them.

Table 2. Viscoelastic parameters in the Thomsen-style notation.

Model	Elastic parameters				Attenuation parameters			
	V_{P0} [km/s]	V_{S0} [km/s]	ε	δ	A_{P0} [10^{-2}]	A_{S0} [10^{-2}]	ε_Q	δ_Q
A1	3.00	1.50	0.30	0.00	9.90	12.31	-0.333	0.500
A2	3.00	1.50	0.30	0.00	4.99	6.23	-0.333	0.500
A3	3.00	1.50	0.30	0.00	2.50	3.12	-0.333	0.500
A4	3.00	1.50	0.30	0.00	1.25	1.56	-0.333	0.500
B1	3.00	1.50	0.10	-0.10	9.90	12.31	-0.333	0.383
B2	3.00	1.50	0.10	-0.10	4.99	6.23	-0.333	0.383
B3	3.00	1.50	0.10	-0.10	2.50	3.12	-0.333	0.383
B4	3.00	1.50	0.10	-0.10	1.25	1.56	-0.333	0.383

For the definition of parameters in the Thomsen-style notation, see Thomsen (1986) and Zhu & Tsvankin (2006). Parameters γ and γ_Q are not listed, because the P-wave is not sensitive to them.

Table 3. P-wave velocity and attenuation anisotropy.

Model	$\overline{V}^{ray}$ [km/s]	a_V^{ray} [%]	$\overline{A}^{ray}$ [s/km]	a_A^{ray} [%]	$\overline{Q}^{ray}$	a_Q^{ray} [%]
A1	3.32	22.6	$28.8 \cdot 10^{-3}$	64.3	5.4	45.4
A2	3.28	23.2	$14.7 \cdot 10^{-3}$	65.4	10.8	45.4
A3	3.27	23.3	$7.4 \cdot 10^{-3}$	65.6	21.5	45.4
A4	3.27	23.4	$3.7 \cdot 10^{-3}$	65.7	43.0	45.4
B1	3.10	9.5	$30.2 \cdot 10^{-3}$	53.9	5.4	45.6
B2	3.07	10.3	$15.4 \cdot 10^{-3}$	55.1	10.9	45.6
B3	3.06	10.5	$7.7 \cdot 10^{-3}$	55.4	21.7	45.6
B4	3.06	10.5	$3.9 \cdot 10^{-3}$	55.4	43.5	45.6

$\overline{V}^{ray}$, $\overline{A}^{ray}$ and $\overline{Q}^{ray}$ are the mean P-wave ray velocity, attenuation and Q-factor; a_V^{ray}, a_A^{ray} and a_Q^{ray} are the P-wave ray velocity anisotropy, attenuation anisotropy and Q-factor anisotropy. The anisotropy is calculated as $a = 200\left(U^{\max} - U^{\min}\right)/\left(U^{\max} + U^{\min}\right)$, where $U^{\max}$ and $U^{\min}$ are the maximum and minimum values of the respective quantity.

Figure 3 shows the directional variations of real and imaginary parts of the slowness, the directional variations of the phase and ray velocities, attenuations and Q-factors for Model A1. The angles range from 0° to 90°. The slowness, velocities, attenuations and Q-factors are calculated using two approaches: first, using asymptotic formulas described in Vavryčuk (2007b); and second, using the first-order perturbations derived in this paper. The figure shows that the perturbations yield biased results with errors of about 2-4%. The best approximated is the Q-factor having errors less than 0.5%. Since Models A1 displays the strongest attenuation among Models A1-A4, the perturbation formulas work with the lowest accuracy. For models of weaker attenuation, the accuracy increases. For Models A2, A3 and A4, the errors are less than 1%, 0.3% and 0.1%, respectively (see Table 4). Figure 4 shows a comparison of the asymptotic solution and the first-order perturbations for Model A3 proving that the differences between the both solutions are almost invisible. Similar values of errors are obtained also for Models B1-B4 (see Table 4).

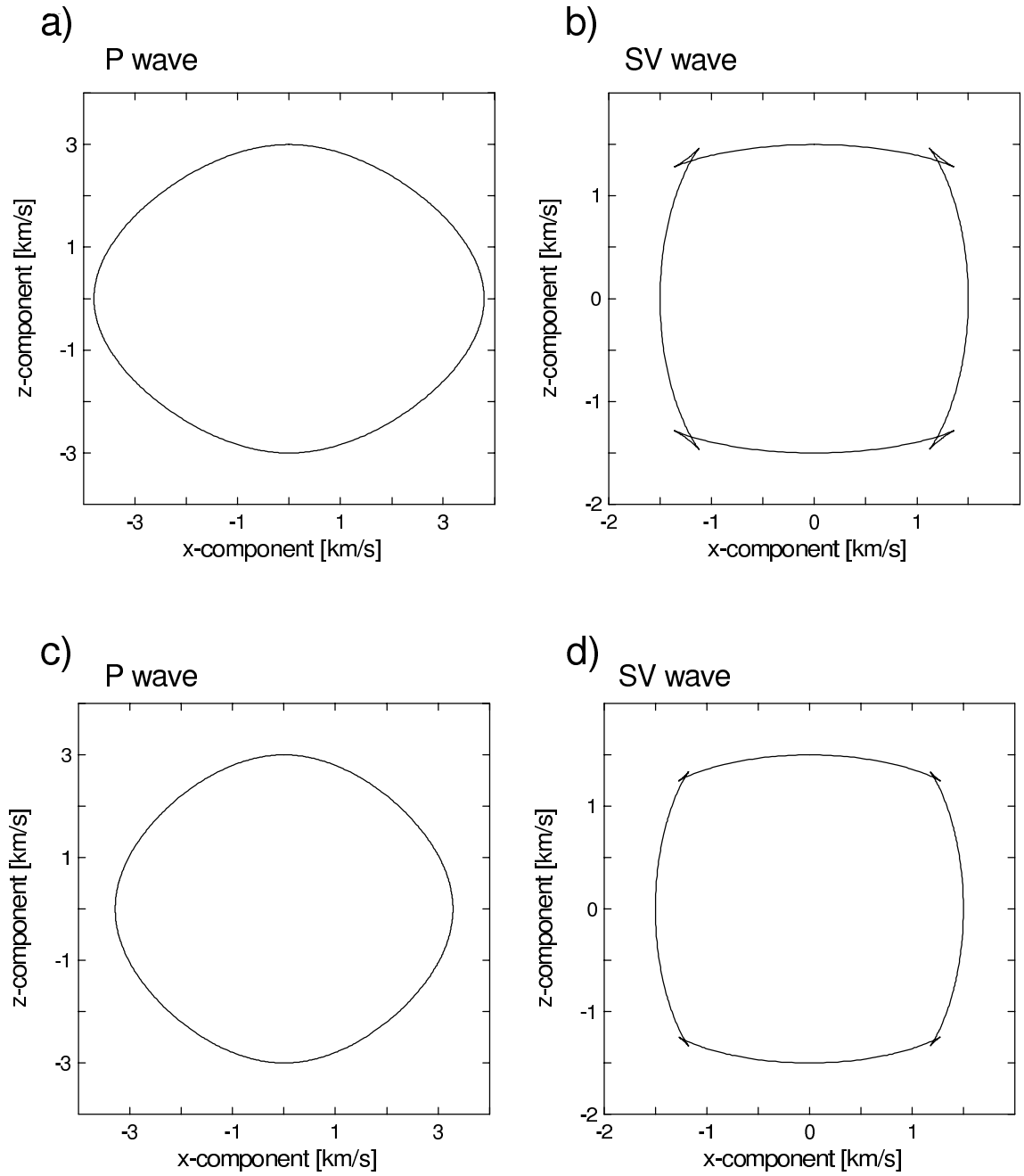

Figure 1. Polar plots of the P- and SV-wave group velocities for (a-b) Model A and (c-d) Model B. (a) P-wave velocity in Model A, (b) SV-wave velocity in Model A, (c) P-wave velocity in Model B, and (d) SV-wave velocity in Model B. Only parameters of the elastic background medium are considered. For parameters of the models, see Tables 1 and 2.

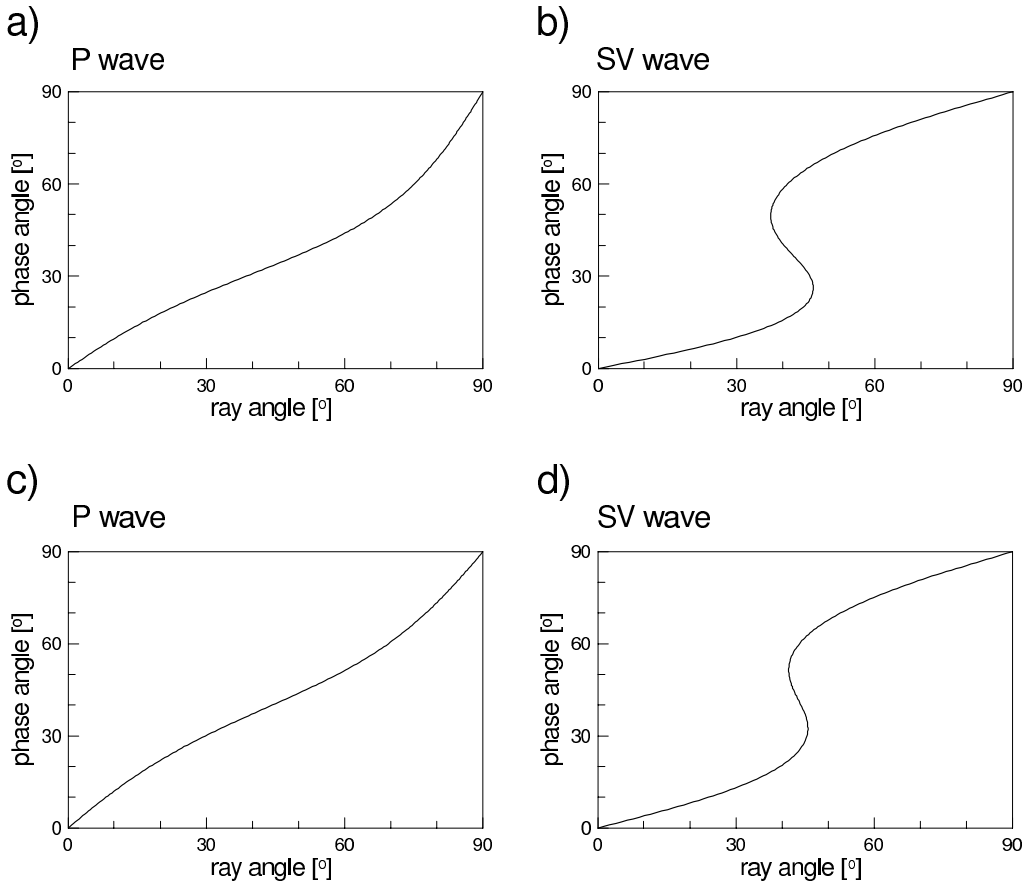

Figure 2. P- and SV-wave phase angles as a function of the ray direction for (a-b) Model A and (c-d) Model B. (a) P-wave phase angle in Model A, (b) SV-wave phase angle in Model A, (c) P-wave phase angle Model B, and (d) SV-wave phase angle in Model B. Only parameters of the elastic background medium are considered. For parameters of the models, see Tables 1 and 2.

Finally, Figures 5 and 6 show a comparison of the correct asymptotics, the first-order perturbations, and the perturbations with improved accuracy for Models A1 and B1. The errors are summarized in Table 5. The comparison indicates that the accuracy of the first-order perturbations can remarkably be improved if following the procedure described in Section 6. For Models A1 and B1, which have the strongest attenuation, the refined formulas yield the errors in the phase and ray velocities less than 0.5%. This documents that the perturbation approach is sufficiently accurate not only for media with weak attenuation but for the whole range of attenuative media which might be observed in seismics.

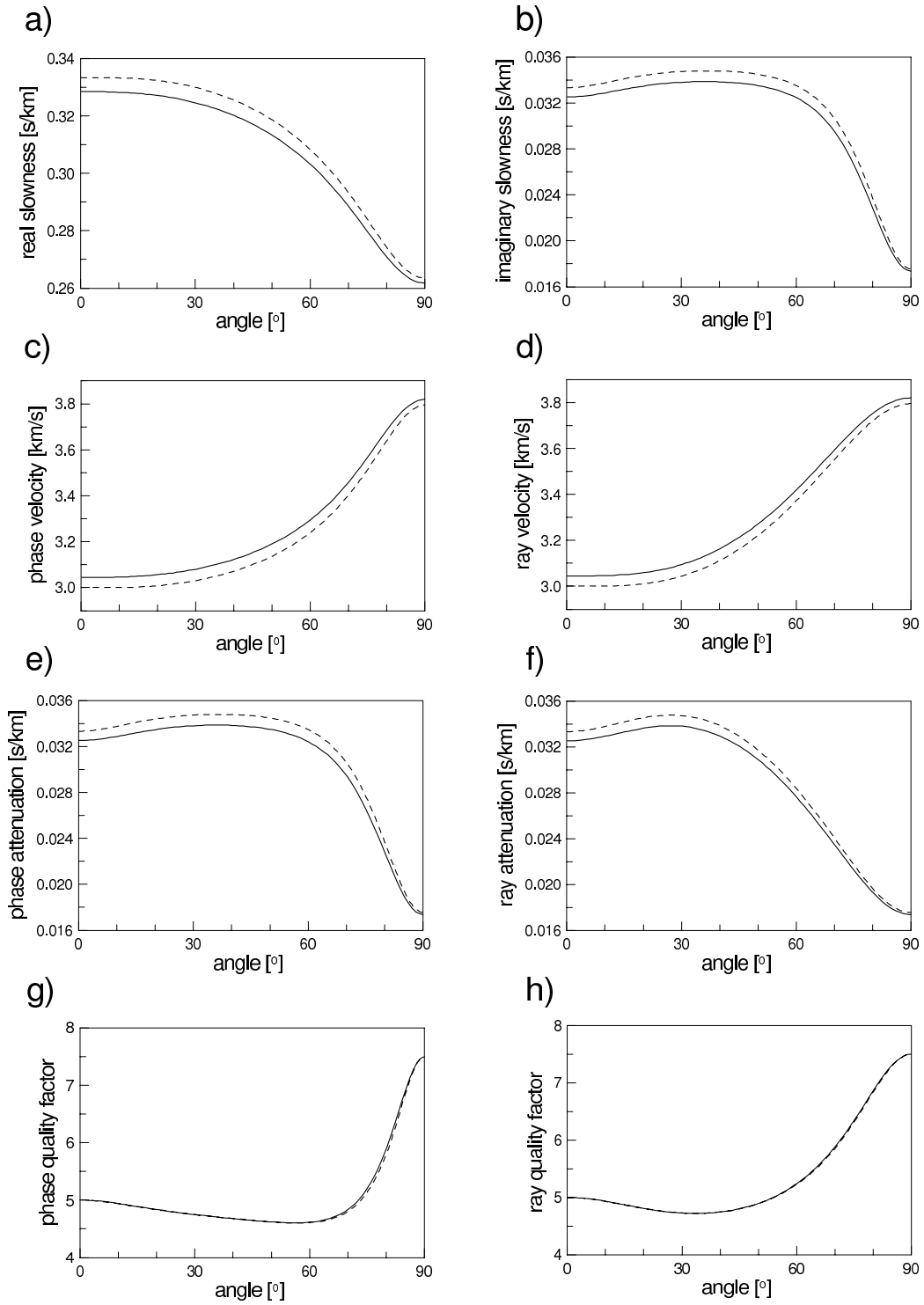

Figure 3. Plots of real and imaginary parts of the P-wave slowness (a-b), P-wave phase and ray velocities (c-d), attenuations (e-f) and quality factors (g-h) in Model A1. Solid line – exact formulas, dashed line – first-order perturbations. The angle represents the deviation of a ray from the vertical axis.

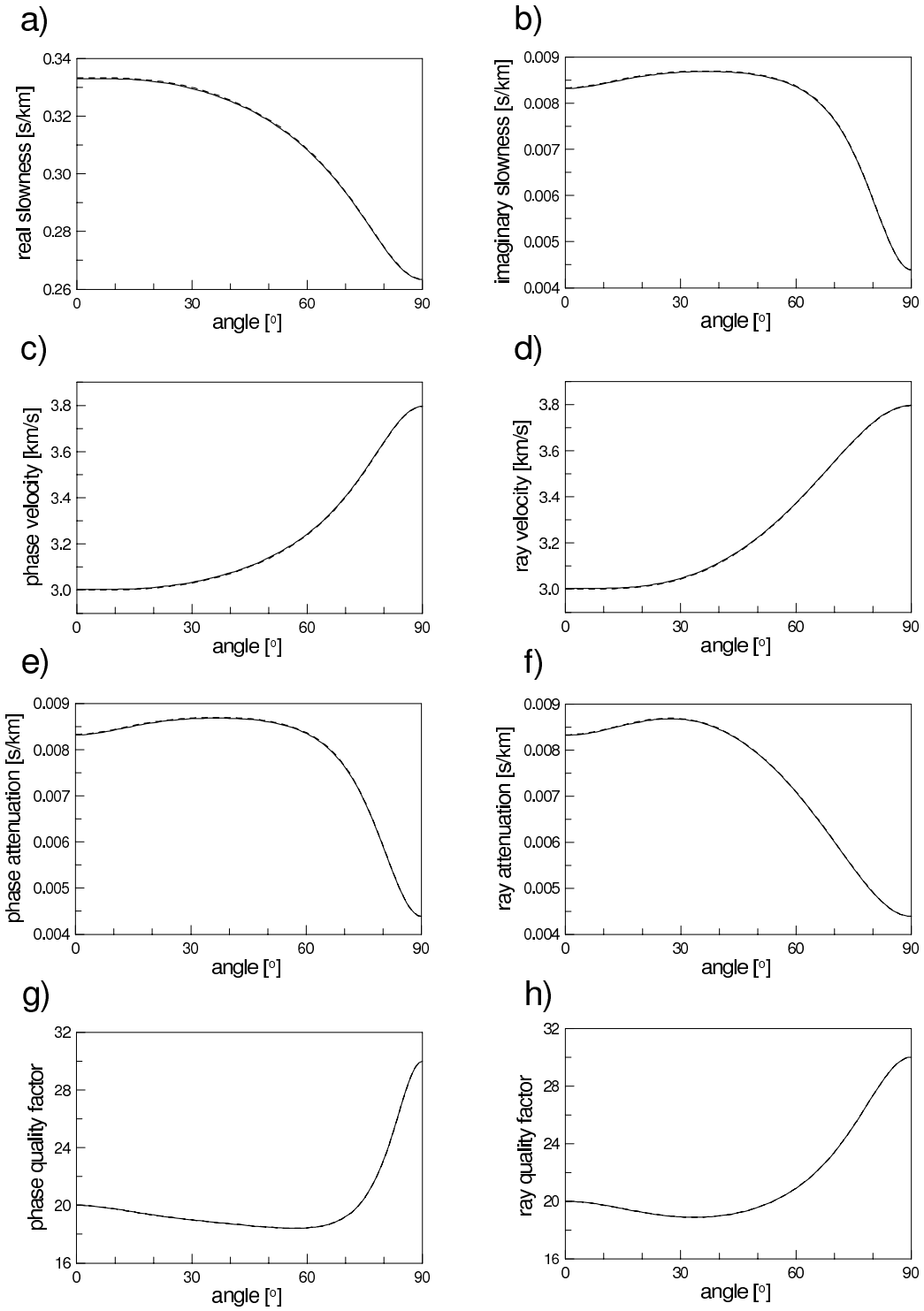

Figure 4. Plots of real and imaginary parts of the P-wave slowness (a-b), P-wave phase and ray velocities (c-d), attenuations (e-f) and quality factors (g-h) in Model A3. Solid line – exact formulas, dashed line – first-order perturbations. The angle represents the deviation of the ray from the vertical axis.

Table 4. Maximum errors of the first-order perturbations.

Model	Error p^R (%)	Error p^i (%)	Error V^{phase} (%)	Error V^{ray} (%)	Error A^{phase} (%)	Error A^{ray} (%)	Error Q^{phase} (%)	Error Q^{ray} (%)
A1	1.72	4.25	1.63	1.63	2.88	2.79	0.13	0.50
A2	0.44	1.08	0.42	0.42	0.73	0.70	0.10	0.13
A3	0.11	0.29	0.11	0.11	0.19	0.18	0.10	0.04
A4	0.03	0.10	0.03	0.03	0.10	0.05	0.10	0.03
B1	1.66	3.84	1.63	1.63	2.79	2.80	0.10	0.50
B2	0.42	0.97	0.42	0.42	0.70	0.71	0.05	0.13
B3	0.11	0.25	0.11	0.11	0.18	0.18	0.05	0.04
B4	0.03	0.07	0.03	0.03	0.05	0.05	0.05	0.02

The error for a particular ray is calculated as $E = 100 \left| U^{\text{exact}} - U^{\text{aprox}} \right| / U^{\text{exact}}$, where U^{exact} and U^{aprox} are the exact and approximate values of the respective quantity. The presented values are the maxima over all rays.

Table 5. Maximum errors of the improved perturbations.

Model	Error p^R (10^{-1}%)	Error p^i (10^{-1}%)	Error V^{phase} (10^{-1}%)	Error V^{ray} (10^{-1}%)
A1	4.60	23.62	4.68	3.72
A2	1.25	6.22	1.11	1.02
A3	0.33	1.70	0.29	0.26
A4	0.13	1.17	0.13	0.07
B1	2.36	18.11	2.41	2.16
B2	0.67	4.77	0.72	0.62
B3	0.18	1.27	0.20	0.16
B4	0.05	0.54	0.06	0.04

8. CONCLUSIONS

For calculating asymptotic wave quantities in media with attenuation, it is advantageous to apply the first-order perturbation theory. The elastic medium is considered as a background medium and the attenuation effects are incorporated as perturbations. Interestingly, some quantities are unaffected by the first-order perturbations being the same for perturbed as well as unperturbed media. This concerns the phase and ray velocities and their directions. The perturbations of the slowness vector, polarization vector and of the other quantities under study are non-zero.

The most complicated task is to calculate perturbation of the stationary slowness vector, $\Delta\mathbf{p}$. Perturbation $\Delta\mathbf{p}$ is calculated by solving a system of three linear equations and involves inverting the wave metric tensor of the background medium H_{il}^0. Equation 38 for calculating $\Delta\mathbf{p}$ is the key result of the paper. Once $\Delta\mathbf{p}$ is evaluated, calculating the phase attenuation A^{phase}, phase quality factor Q^{phase} and the phase inhomogeneity D^{phase} is straightforward (see equations 48, 49 and 51). Calculating the ray quantities A^{ray} and Q^{ray} is even simpler than calculating the corresponding phase quantities. The ray quantities can be expressed just in terms of medium perturbations Δa_{ijkl} without the necessity to calculate perturbation $\Delta\mathbf{p}$ (see equations 57 and 59). This finding is important because it can greatly facilitate the inversion for quality parameters from observations of wave attenuation in anisotropic media.

The derived formulas are valid under the following limitations. First, they describe high-frequency wave fields at distances far from the source. Second, the wave must not propagate in directions close to cusp edges on the wave surface, because the standard asymptotics fails in these directions. And third, the wave must not propagate near directions associated with singularities (acoustic axes) on the slowness surface, because the Christoffel tensor becomes nearly degenerate and the standard perturbation formulas fail.

Numerical modeling documents that the first-order perturbations are highly accurate. They yield a reasonable accuracy even for extremely strong attenuation. The accuracy is 4% or less for media with the Q-factor of about 5, and 0.3% or less for media with the Q-factor of 20 or higher. Since the majority of rocks in the Earth's crust and in the mantle is characterized by Q-factors higher than 20, the applicability area of the first-order

perturbations is very broad covering applications from global earthquake seismology as well as from seismic exploration. Moreover, the accuracy of perturbations can further be enhanced. If perturbations are used just for calculating the complex direction of the stationary slowness vector and all other computations are performed exactly, the achieved accuracy is remarkably higher. This costs almost no additional computer time and results in errors of the phase and ray velocities less than 0.5% in media with Q-factor of 5, and less than 0.03% in media with Q-factor of 20 or higher.

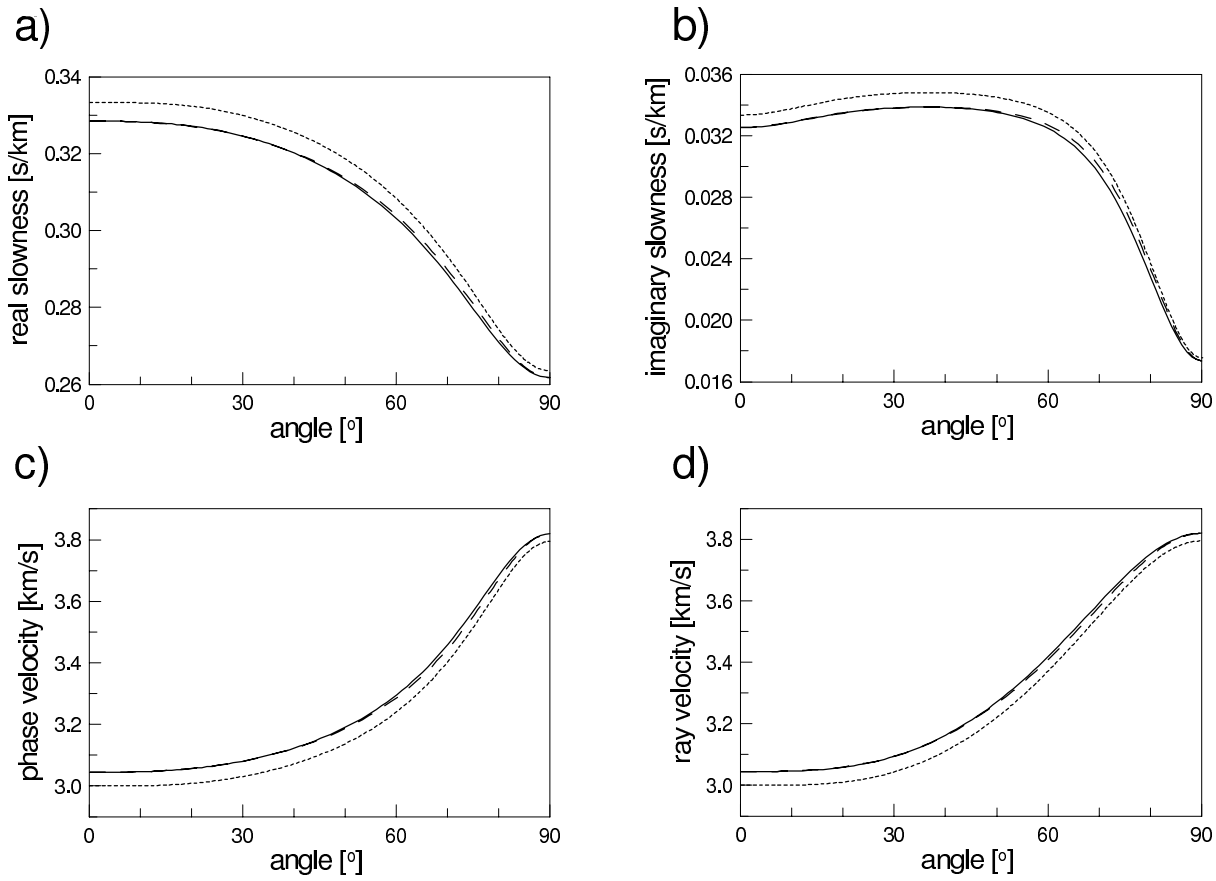

Figure 5. Plots of the P-wave slowness (a-b) and velocities (c-d) in Model A1. Solid line – exact formulas, dotted line - first-order perturbations, dashed line – perturbations with improved accuracy. The angle represents the deviation of the ray from the vertical axis.

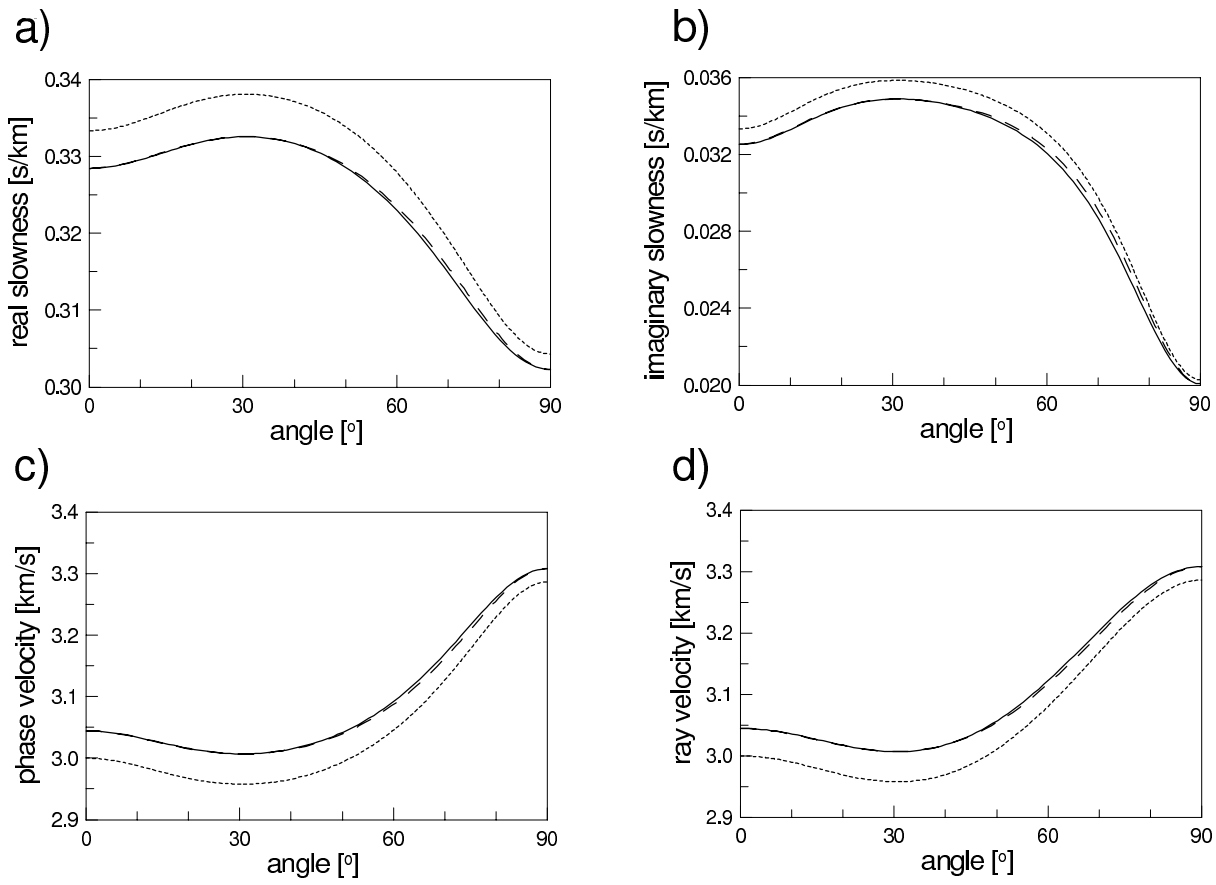

Figure 6. Plots of the P-wave slowness (a-b), velocities (c-d) in Model B1. Solid line – exact formulas, dotted line - first-order perturbations, dashed line – perturbations with improved accuracy. The angle represents the deviation of the ray from the vertical axis.

ACKNOWLEDGEMENTS

I thank four anonymous reviewers for their helpful comments. The work was supported by the Grant Agency of the Czech Republic, Grant No. 205/05/2182, by the Grant Agency of the Academy of Sciences of the Czech Republic, Grant No. IAA300120801, by the Consortium Project "Seismic Waves in Complex 3-D Structures", and by the EU Consortium Project IMAGES "Induced Microseismics Applications from Global Earthquake Studies", Contract No. MTKI-CT-2004-517242. A part of the work was done while the author was visiting researcher at the Schlumberger Cambridge Research, Cambridge, U.K.

REFERENCES

Best, A.I., J. Sothcott, and C. McCann, 2007, A laboratory study of seismic velocity and attenuation anisotropy in near-surface sedimentary rocks: Geophysical Prospecting, 55, 609-625, doi:10.1111/j.1365-2478.2007.00642.x.

Carcione, J.M., 2000, A model for seismic velocity and attenuation in petroleum source rocks: Geophysics, 65, 1080-1092.

Carcione, J.M., 2001, Wave Fields in Real Media: Wave Propagation in Anisotropic, Anelastic and Porous Media: Pergamon, Amsterdam.

Carcione, J.M., K. Helbig, and H.B. Helle, 2003, Effects of pressure and saturating fluid on wave velocity and attenuation in anisotropic rocks, International Journal of Rock Mechanics and Mining Sciences, 40, 389-403, doi: 10.1016/S1365-1609(03)00016-9..

Červený, V., 2001, *Seismic Ray Theory*: Cambridge University Press, Cambridge, U.K.

Červený, V., and I. Pšenčík, 2005, Plane waves in viscoelastic anisotropic media – I. Theory: Geophysical Journal International, 161, 197-212; http//dx.doi.org/10.1111/j.1365-246X.2005.02589.x.

Červený, V., and I. Pšenčík, 2007, Quality factor Q in dissipative anisotropic media: *Seismic Waves in Complex 3-D Structures*, Report 17, pp. 173-194, Dept. of Geophysics, Charles University Prague.

Chichinina, T., V. Sabinin, and G. Ronquillo-Jarillo, 2006, QVOA analysis: *P*-wave attenuation anisotropy for fracture characterization: Geophysics, 71, No. 3, C37-C48, doi: 10.1190/1.2194531.

Farra, V., 2001, High-order perturbations of the phase velocity and polarization of qP and qS waves in anisotropic media: Geophysical Journal International, 147, 93-104.

Farra, V., 2004, Improved First-Order Approximation of Group Velocities in Weakly Anisotropic Media: Studia Geophysica et Geodaetica, 48, 199-213, doi: 10.1023/B:SGEG.0000015592.36894.3b.

Futterman, W., 1962, Dispersive body waves: Journal of the Geophysical Research, 67, 5279-5291.

Gajewski, D., and I. Pšenčík, 1992, Vector wavefields for weakly attenuating anisotropic media by the ray method: Geophysics, 57, 27-38.

Jakobsen, M., and T. A. Johansen, 2000, Anisotropic approximations for mudrocks: A seismic laboratory study: Geophysics, 65, 1711-1725.

Jech, J., and I. Pšenčík, 1989, First-order perturbation method for anisotropic media: Geophysical Journal International, 99, 369-376.

Klimeš, L., 2002, Relation of the wave-propagation metric tensor to the curvatures of the slowness and ray-velocity surfaces: Studia Geophysica et Geodaetica, 46, 589–597, doi: 10.1023/A:1019551320867.

Korn G.A., and T.M. Korn, 2000, Mathematical Handbook for Scientists and Engineers: Dover Publications, New York.

Liu, B., H. Kern, and T. Popp, 1997, Attenuation and velocities of P- and S-waves in dry and saturated crystalline and sedimentary rocks at ultrasonic frequencies: Physics and Chemistry of the Earth, 22, 75-79.

Pšenčík, I., and V. Vavryčuk, 2002, Approximate relation between the ray vector and wave normal in weakly anisotropic media: Studia Geophysica et Geodaetica, 46, 793-807, doi: 10.1023/A:1021189724526.

Shankland, T.J., P.A. Johnson, and T.M. Hopson, 1993, Elastic wave attenuation and velocity of Berea sandstone measured in the frequency domain: Geophysical Research Letters, 20, 391–394.

Song, L.P., A.G. Every, and C. Wright, 2001, Linearized approximations for phase velocities of elastic waves in weakly anisotropic media: Journal of Physics D - Applied Physics, 34, 2052-2062, doi:10.1088/0022-3727/34/13/316.

Thomsen, L., 1986, Weak elastic anisotropy: Geophysics, 51, 1954-1966.

Tsvankin, I., and L. Thomsen, 1994, Nonhyperbolic reflection moveout in anisotropic media: Geophysics, 59, 1290−1304.

Vavryčuk, V., 1997, Elastodynamic and elastostatic Green tensors for homogeneous weak transversely isotropic media: Geophysical Journal International, 130, 786-800.

Vavryčuk, V., 2003, Parabolic lines and caustics in homogeneous weakly anisotropic solids: Geophysical Journal International, 152, 318-334. doi:10.1046/j.1365-246X.2003.01845.x.

Vavryčuk, V., 2005, Acoustic axes in triclinic anisotropy: Journal of Acoustical Society of America, 118, 647-653, doi: 10.1121/1.1954587.

Vavryčuk, V., 2006, Calculation of the slowness vector from the ray vector in anisotropic media: Proceedings of the Royal Society, A 462, 883-896, doi: 10.1098/rspa.2005.1605.

Vavryčuk, V., 2007a, Asymptotic Green's function in homogeneous anisotropic viscoelastic media: Proceedings of the Royal Society, A 463, 2689-2707, doi:10.1098/rspa.2007.1862.

Vavryčuk, V., 2007b, Ray velocity and ray attenuation in homogeneous anisotropic viscoelastic media: Geophysics, 72, D119-D127, doi:10.1190/1.2768402.

Vernik, L., and X. Liu, 1997, Velocity anisotropy in shales: A petrophysical study: Geophysics, 62, 521-532.

Zhu, Y., and I. Tsvankin, 2006, Plane-wave propagation in attenuative transversely isotropic media: Geophysics, 71, No. 2, T17-T30, doi:10.1190/1.2187792.

Zhu, Y., and I. Tsvankin, 2007, Plane-wave attenuation anisotropy in orthorhombic media: Geophysics, 72, No. 1, D9-D19, doi:10.1190/1.2387137.