Kirchhoff integrals and Fresnel zones

Luděk Klimeš

Department of Geophysics, Charles University, Ke Karlovu 3, 121 16 Praha 2, Czech Republic, Fax: +42-2-299272, E-mail: psencik@earn.cvut.cz

Summary

The accuracy and efficiency of the numerical quadrature of the Kirchhoff integrals is studied. The weighting function localizing the Kirchhoff integral is derived. The weighting function minimizes the number of discrete values necessary for the numerical quadrature. The Fresnel zones are then derived as the minimum areas of integration with the specified accuracy. The new, explicit, and very general definition of Fresnel zones is purely local, independent of the reference travel times. The definition of Fresnel zones is expressed in terms of the first and second derivatives of the sum of travel times along the surface of integration. As far as the author can appreciate, the new, quantitative definition of Fresnel zones is conformal with other, usually qualitatively or empirically derived definitions, within the regions of their applicability.

Keywords

Kirchhoff integral, Fresnel zone, travel time.

1 Introduction

The Fresnel zone, corresponding to the couple of travel–time fields $\tau^{(1)} = \tau^{(1)}(\mathbf{x})$ and $\tau^{(2)} = \tau^{(2)}(\mathbf{x})$, is usually defined in terms of the travel–time deviations from the reference travel time τ^{ref} corresponding to the reference ray. The n^{th} Fresnel zone is then often defined as the set of points, along given surface S, at which

$$\frac{n-1}{2}\,T \leq |\tau^{(1)}(\mathbf{x}) + \tau^{(2)}(\mathbf{x}) - \tau^{\mathrm{ref}}| \leq \frac{n}{2}\,T \quad , \tag{1}$$

with T being the period of a monochromatic wave, see Kravtsov & Orlov (1980), Eq. (10.5). This definition is formal and, especially in the vicinity of rays corresponding to saddle (i.e. hyperbolic) stationary points of the sum $\tau(\mathbf{x}) = \tau^{(1)}(\mathbf{x}) + \tau^{(2)}(\mathbf{x})$ of travel times along the surface of integration, does not conform to the regions of major contributions to Kirchhoff integrals. The problem of the hyperbolic Fresnel zones around the saddle stationary points was solved by Kravtsov & Orlov (1990), Eq. (2.10.13), in terms of the paraxial ray approximation. However, the paraxial ray approximation of the Fresnel zones is not sufficiently accurate, especially at caustic points, and breaks down for the Fresnel zones corresponding to the pair of caustic points. Moreover, the recent definitions do not specify how to restrict the Kirchhoff integral onto the Fresnel zones and what error is caused by such a restriction.

Thus, we shall study the spatial distribution of the relative amplitude of contributions to Kirchhoff integrals to derive an accurate and efficient method for the discretization of the Kirchhoff integrals. Then the new, more physically meaningful definition of Fresnel zones will be based on the magnitude of these contributions. In this way, the Fresnel zones will directly correspond to the regions over which the Kirchhoff integrals have to be enumerated. Moreover, the windowing functions for the enumeration, smoothly tapering at the boundaries of the Fresnel zones, will be derived, and also the error corresponding to the limitation of the Kirchhoff integrals to the Fresnel zones will be estimated.

The resulting definition of Fresnel zones is independent of the reference rays and reference travel times. It is expressed in terms of the local properties of the travel–time fields. In this way, it applies simultaneously to all Fresnel zones of all rays corresponding to the travel–time fields, even if the number of rays or Fresnel zones is not a priori known. It is thus applicable everywhere, and may define Fresnel zones even in the regions influencing the Kirchhoff integral without the existence of stationary points.

On the other hand, the proposed definition of Fresnel zones depends on the surface at which it is situated. In this way, Fresnel zones and Fresnel volumes are mutually independent physical objects. A Fresnel zone does not coincide with the intersection of the Fresnel volume with the surface at which it is situated. This is, however, reasonable. For example, the Fresnel zone at a reflector, corresponding to the wavefields incident from the source and the receiver, locally depends on the slope and curvature of the reflector, not only on the reflector position with respect to the Fresnel volume.

2 Kirchhoff integral

Assume the Kirchhoff integral along the surface S parametrized by coordinates $\mathbf{x} = (x_1, x_2)$, in the frequency domain, expressed in the form of

$$\mathbf{K} = \int_S \mathbf{a}(\mathbf{x}) \exp\{\mathrm{i}\omega\tau(\mathbf{x})\} \, \mathrm{d}^2\mathbf{x} \quad , \tag{2}$$

where $\mathbf{a} = \mathbf{a}(\mathbf{x})$ is some slowly varying amplitude of the integrand, $\omega = 2\pi/T$ is an angular frequency, and $\tau(\mathbf{x}) = \tau^{(1)}(\mathbf{x}) + \tau^{(2)}(\mathbf{x})$ is sum of two travel times.

To study locally the Kirchhoff integral (2), we introduce, along surface S, local weighting functions $g(\mathbf{x},\mathbf{x}')$ concentrated at points $\mathbf{x}'$, and normalize them in order to form the decomposition of the unit function into local functions $w(\mathbf{x},\mathbf{x}')$,

$$w(\mathbf{x},\mathbf{x}') = g(\mathbf{x},\mathbf{x}') \left[\int_S g(\mathbf{x},\mathbf{x}'') \mathrm{d}^2\mathbf{x}'' \right]^{-1} \quad . \tag{3}$$

Then the Kirchhoff integral (2) may be decomposed,

$$\mathbf{K} = \int_S \mathbf{A}(\mathbf{x}') \mathrm{d}^2\mathbf{x}' \quad , \tag{4}$$

into the local contributions

$$\mathbf{A}(\mathbf{x}') = \int_S w(\mathbf{x},\mathbf{x}') \, \mathbf{a}(\mathbf{x}) \exp\{\mathrm{i}\omega\tau(\mathbf{x})\} \, \mathrm{d}^2\mathbf{x} \quad . \tag{5}$$

In this paper, we shall choose the weighting functions $g(\mathbf{x},\mathbf{x}')$ to be Gaussian,

$$g(\mathbf{x},\mathbf{x}') = \exp\left\{\mathrm{i}\omega\tfrac{1}{2}[\mathbf{x}-\mathbf{x}']^{\mathrm{T}} \mathbf{W}(\mathbf{x}')[\mathbf{x}-\mathbf{x}']\right\} \quad , \tag{6}$$

where complex–valued symmetric matrix $\mathbf{W}(\mathbf{x}) = \mathbf{R}(\mathbf{x}) + \mathrm{i}\mathbf{Y}(\mathbf{x})$, with positive–definite imaginary part $\mathbf{Y}(\mathbf{x})$, controls the width of the local weighting functions.

Since, for rapidly oscillating functions (i.e. for high frequencies ω), the direct numerical quadrature of (2) or (5) is very time consuming, the integrals (2) or (5) are sometimes approximated using local high–frequency approximations, see also Červený, Pleinerová, Klimeš & Pšenčík (1987), Sec. 5.3. Hereinafter, we shall employ the same method to estimate the magnitude of the local contributions to the Kirchhoff integral.

3 Local high–frequency approximation

If the local weighting functions are chosen sufficiently narrow, the integrals in (3) and (5) may be estimated analytically, with a sufficient accuracy. Then, substituting (6) into (3) and assuming $\mathbf{W}(\mathbf{x})$ approximately locally constant, we arrive at

$$w(\mathbf{x},\mathbf{x}') \simeq \frac{\omega}{2\pi}\sqrt{\det[-\mathrm{i}\mathbf{W}(\mathbf{x}')]}\exp\{\mathrm{i}\omega\tfrac{1}{2}[\mathbf{x}-\mathbf{x}']^{\mathrm{T}}\mathbf{W}(\mathbf{x}')[\mathbf{x}-\mathbf{x}']\} \quad . \tag{7}$$

Let us assume that the sum τ of the travel times may locally be approximated by the Taylor expansion up to the second order

$$\tau(\mathbf{x}) \simeq \tau(\mathbf{x}') + [\mathbf{x}-\mathbf{x}']^{\mathrm{T}}\mathbf{p}(x') + \tfrac{1}{2}[\mathbf{x}-\mathbf{x}']^{\mathrm{T}}\mathbf{M}(\mathbf{x}')[\mathbf{x}-\mathbf{x}'] \quad , \tag{8}$$

where vector $\mathbf{p}$ and matrix $\mathbf{M}$ contain the spatial derivatives

$$p_I = \frac{\partial\tau}{\partial x_I} \quad , \qquad M_{IJ} = \frac{\partial^2\tau}{\partial x_I \partial x_J} \tag{9}$$

of the sum τ of the travel times. Similarly, we approximate the amplitude of the integrand, within the effective region of the weighting function, by a constant,

$$\mathbf{a}(\mathbf{x}) \simeq \mathbf{a}(\mathbf{x}') \quad . \tag{10}$$

Substituting (7), (8) and (10) into (5),

$$\mathbf{A}(\mathbf{x}') \simeq \int_S \frac{\omega}{2\pi}\sqrt{\det[-\mathrm{i}\mathbf{W}(\mathbf{x}')]}\,\mathbf{a}(\mathbf{x}')$$
$$\times\exp\{\mathrm{i}\omega\big[\tau(\mathbf{x}') + [\mathbf{x}-\mathbf{x}']^{\mathrm{T}}\mathbf{p}(\mathbf{x}') + \tfrac{1}{2}[\mathbf{x}-\mathbf{x}']^{\mathrm{T}}[\mathbf{W}(\mathbf{x}')+\mathbf{M}(\mathbf{x}')][\mathbf{x}-\mathbf{x}']\big]\}\,\mathrm{d}^2\mathbf{x} \ , \tag{11}$$

and evaluating the integral, we arrive at

$$\mathbf{A} \simeq \mathbf{a}\sqrt{\frac{\det[-\mathrm{i}\mathbf{W}]}{\det[-\mathrm{i}(\mathbf{W}+\mathbf{M})]}}\,\exp\{\mathrm{i}\omega[\tau - \tfrac{1}{2}\mathbf{p}^{\mathrm{T}}(\mathbf{W}+\mathbf{M})^{-1}\mathbf{p}]\} \quad , \tag{12}$$

where all the quantities are taken at the same point $\mathbf{x}'$.

Substituting (12) into (4) we arrive at the computationally very efficient approximation of (2),

$$\mathbf{K} \simeq \int_S \mathbf{a}(\mathbf{x})\exp\{\mathrm{i}\omega\tau(\mathbf{x})\}\,W(\mathbf{x})\,\mathrm{d}^2\mathbf{x} \quad , \tag{13}$$

with

$$W \simeq \sqrt{\frac{\det[-\mathrm{i}\mathbf{W}]}{\det[-\mathrm{i}(\mathbf{W}+\mathbf{M})]}}\,\exp\{-\mathrm{i}\omega\tfrac{1}{2}\mathbf{p}^{\mathrm{T}}(\mathbf{W}+\mathbf{M})^{-1}\mathbf{p}\} \quad . \tag{14}$$

Equation (13) is nearly the same as Kirchhoff integral (2), only the weighting function (14) is applied to localize the integral. Accuracy of approximation (13) depends on the widths of the weighting functions (7). The widths are, at the fixed frequency, controlled by imaginary part $\mathbf{Y}$ of matrix $\mathbf{W}$. For instance, for infinitely narrow weighting functions (7), the weighting function (14) is a unit function and approximation (13) becomes identical to Kirchhoff integral (2).

For regular $\mathbf{M}$ and fixed $\mathbf{W}$, equation (13) is sufficiently accurate approximation of (2) for sufficiently high frequencies ω. For low frequencies ω, equation (13) is sufficiently accurate only if reasonably large imaginary part $\mathbf{Y}$ of matrix $\mathbf{W}$ is chosen.

As we will see below, weighting (13) enables: (a) to accurately and efficiently discretize the Kirchhoff integral; (b) to limit the Kirchhoff integral to small areas called Fresnel zones.

4 Discretization of the Kirchhoff integral

Let us select a "reasonably regular" grid of points $\mathbf{x}_\alpha$ and discretize integral (13),

$$\mathbf{K} \simeq \sum_\alpha \mathbf{a}(\mathbf{x}_\alpha)\,\exp\{\mathrm{i}\omega\tau(\mathbf{x}_\alpha)\}\,W(\mathbf{x}_\alpha)\,\Delta^2\mathbf{x}_\alpha \quad , \tag{15}$$

where $\Delta^2\mathbf{x}_\alpha$ is the area of surface S, corresponding to point $\mathbf{x}_\alpha$.

Under a "reasonably regular" grid we understand the set of points which may be covered by a system of roughly "parallel" smooth curves in such a way, that the distances of the points along the curves vary sufficiently slowly and, simultaneously, the distances between the curves vary sufficiently slowly along the surface S.

The discretization error is then governed by covariant metric matrix

$$\mathbf{G} = -\left[\mathrm{Im}(\mathbf{W}^{-1})\right]^{-1} = \mathbf{Y} + \mathbf{R}\mathbf{Y}^{-1}\mathbf{R} \quad . \tag{16}$$

The distances h along surface S are then measured with respect to metric G,

$$h^2 = \Delta\mathbf{x}^{\mathrm{T}}\,\mathbf{G}\,\Delta\mathbf{x} \quad . \tag{17}$$

If the points can be arranged into a system of roughly parallel curves in such a way that both the distances of the points along the curves and the distances between the curves, measured with respect to $\mathbf{G}$, do not exceed

$$\sqrt{\frac{T}{n}} \quad , \tag{18}$$

see r.h.s. of (1), the discretization error does not exceed

$$\delta_{\max} = 4\exp(-\pi n) \quad , \tag{19}$$

see Klimeš (1986), Eq. (144) with $\kappa^2 = \pi/n$.

To keep the discretization error within this limit, the density of reasonably distributed points (i.e. number of points / area) should be approximately

$$\frac{n}{T}\sqrt{\det\mathbf{G}} \quad . \tag{20}$$

5 Cut–off error

Now we estimate the error due to limiting integral (13) and sum (15) to the vicinity of the stationary point $\mathbf{x}''$, $\mathbf{p}(\mathbf{x}'') = \mathbf{0}$. Since the error, unlike the value of the integral, may be estimated only roughly, we replace here the quantities in integral (13) by the paraxial approximation from the stationary point $\mathbf{x}''$,

$$\begin{aligned}
\mathbf{a}(\mathbf{x}) &\approx \mathbf{a}(\mathbf{x}'') \quad , \\
\tau(\mathbf{x}) &\approx \tau(\mathbf{x}'') + \tfrac{1}{2}[\mathbf{x}-\mathbf{x}'']^{\mathrm{T}}\mathbf{M}(\mathbf{x}'')[\mathbf{x}-\mathbf{x}''] \quad , \\
\mathbf{p}(\mathbf{x}) &\approx \mathbf{M}(\mathbf{x}'')[\mathbf{x}-\mathbf{x}''] \quad , \\
\mathbf{M}(\mathbf{x}) &\approx \mathbf{M}(\mathbf{x}'') \quad , \\
\mathbf{W}(\mathbf{x}) &\approx \mathbf{W}(\mathbf{x}'') \quad .
\end{aligned} \tag{21}$$

The boundary of the area of integration may be expressed in polar coordinates r, φ, centred at the stationary point $\mathbf{x}''$, in terms of dependence $r = r(\varphi)$,

$$\mathbf{x} - \mathbf{x}'' = r(\varphi)\,\mathbf{e}(\varphi) \quad , \tag{22}$$

with $\mathbf{e}(\varphi)$ being the unit directional vector.

Substituting (21) and (22) into (13),

$$K \simeq \mathbf{a}(x'')\exp\{\mathrm{i}\omega\tau(\mathbf{x}'')\}\sqrt{\frac{\det\{-\mathrm{i}\mathbf{W}(x'')\}}{\det\{-\mathrm{i}[\mathbf{W}(x'')+\mathbf{M}(x'')]\}}} \times \int_0^{2\pi}\mathrm{d}\varphi\int_0^{r(\varphi)}\mathrm{d}r\; r\exp\{-\tfrac{1}{2}m(\varphi)r^2\} \ , \tag{23}$$

with

$$m(\varphi) = \mathrm{i}\,\omega\,\mathbf{e}(\varphi)^{\mathrm{T}}\left\{\mathbf{M}(x'')[\mathbf{W}(x'')+\mathbf{M}(x'')]^{-1}\mathbf{M}(x'') - \mathbf{M}(x'')\right\}\mathbf{e}(\varphi) \quad , \tag{24}$$

and integrating first with respect to r, and then with respect to φ, we see that the relative error

$$\delta \simeq \frac{\int_0^{2\pi}\mathrm{d}\varphi\,[m(\varphi)]^{-1}\exp\{-\frac{1}{2}m(\varphi)[r(\varphi)]^2\}}{\int_0^{2\pi}\mathrm{d}\varphi\,[m(\varphi)]^{-1}} \tag{25}$$

due to the cut–off approximately equals the weighted average value of the integrand along the cut–off line divided by the value of the integrand at the stationary point. Consequently, if everywhere outside the region of integration the integrand is smaller, in its absolute value, than ϵ–times its stationary value, relative error δ due to the cut–off is unlike to exceed $\delta_{\max} = \epsilon$.

6 Fresnel zones

Let us restrict integral (13) and sum (15) onto the region where the relative amplitude

$$\exp\{-\mathrm{i}\omega\tfrac{1}{2}\mathbf{p}^{\mathrm{T}}\mathrm{Im}[(\mathbf{W}+\mathbf{M})^{-1}]\,\mathbf{p}\} \tag{26}$$

of (14) is not less than

$$\delta_{\max} = \exp\{-\pi n\} \quad . \tag{27}$$

This region may be defined by means of inequality

$$\mathbf{p}^{\mathrm{T}}\tilde{\mathbf{G}}^{-1}\mathbf{p} \le n\,T \quad , \tag{28}$$

with

$$\tilde{\mathbf{G}} = \{\mathrm{Im}[(\mathbf{W}+\mathbf{M})^{-1}]\}^{-1} = \mathbf{Y} + (\mathbf{R}+\mathbf{M})\mathbf{Y}^{-1}(\mathbf{R}+\mathbf{M}) \quad , \tag{29}$$

and we shall call it the union of the first n *Fresnel zones.*

Let us now vindicate the usage of name "Fresnel zone". Since we have not yet restricted the choice of matrix $\mathbf{W}$, the zones defined by inequality (28) are not, in fact, the Fresnel zones as being presently understood in common. On the other hand, if the choice of matrix $\mathbf{W}$ is optimized, zones (28) will become identical with the Fresnel zones being used to in practice. Thus the reader may consider inequality (28) to be either (a) a considerably generalized definition of the Fresnel zones, or (b) the common definition of the Fresnel zones with the restriction that matrix $\mathbf{W}$ and, consequently, $\tilde{\mathbf{G}}$ are optimized. Now we describe the high–frequency optimization.

The area of Fresnel zones (28) is approximately

$$\pi\,n\,T\,\sqrt{\det(\mathbf{M}^{-1}\tilde{\mathbf{G}}\mathbf{M}^{-1})} \quad . \tag{30}$$

The number of points per the Fresnel zones is then approximately, see (20) and (30),

$$N = \pi\,n^2\,\sqrt{\det(\mathbf{G}\mathbf{M}^{-1}\tilde{\mathbf{G}}\mathbf{M}^{-1})} \quad . \tag{31}$$

Taking into account (16) and (29), for any choice of $\mathbf{Y}$, the number N of points is minimized by

$$\mathbf{R} = -\frac{\mathbf{M}}{2} \quad . \tag{32}$$

Then

$$\mathbf{G} = \tilde{\mathbf{G}} = \mathbf{Y} + \frac{\mathbf{M}}{2}\mathbf{Y}^{-1}\frac{\mathbf{M}}{2} \quad . \tag{33}$$

Within the scope of this equality, matrix $\tilde{\mathbf{G}}^{-1}$ in (28) is the contravariant form of the covariant metric matrix (16), independently of the choice of $\mathbf{Y}$.

For low frequencies, positively definite symmetric matrix $\mathbf{Y}$ has to be chosen reasonably great, but for regular $\mathbf{M}$ and sufficiently *high frequencies*, matrix $\mathbf{Y}$ may be optimized: minimum number N of points, see (31), is reached for

$$\mathbf{Y} = \frac{|\mathbf{M}|}{2} \quad . \tag{34}$$

Here $|\mathbf{M}|$ denotes the absolute value of matrix $\mathbf{M}$, i.e. the symmetric positively–definite matrix $|\mathbf{M}|$ such that

$$|\mathbf{M}|^2 = \mathbf{M}^2 \quad . \tag{35}$$

Then

$$\mathbf{G} = \tilde{\mathbf{G}} = |\mathbf{M}| \tag{36}$$

in (17), (20), (30), and (31), and the high–frequency Fresnel zones are defined by (28) with (36).

7 Conclusions

For regular matrix $\mathbf{M}$ of second derivatives of the sum of travel times along the surface of integration, and for sufficiently high frequencies, the optimum weighting function to be applied to the Kirchhoff integral during the numerical quadrature is

$$\begin{aligned} W &= \sqrt{\frac{\det(|\mathbf{M}|+\mathrm{i}\mathbf{M})}{\det(|\mathbf{M}|-\mathrm{i}\mathbf{M})}} \exp\{-\mathrm{i}\omega\mathbf{p}^{\mathrm{T}}(\mathrm{i}|\mathbf{M}|+\mathbf{M})^{-1}\mathbf{p}]\} \\ &= \sqrt{\frac{\det(|\mathbf{M}|+\mathrm{i}\mathbf{M})}{\det(|\mathbf{M}|-\mathrm{i}\mathbf{M})}} \exp\{-\omega\tfrac{1}{2}\mathbf{p}^{\mathrm{T}}(|\mathbf{M}|^{-1}+\mathrm{i}\mathbf{M}^{-1})\,\mathbf{p}]\} \quad , \end{aligned} \tag{37}$$

see (14), (32), and (34). To keep the relative error less than

$$\delta_{\max} = 5\exp(-\pi\, n) \quad , \tag{38}$$

see (12) and (27), the numerical quadrature should be carried out over the first n *Fresnel zones*, defined by inequality

$$\mathbf{p}^{\mathrm{T}}|\mathbf{M}|^{-1}\mathbf{p} \leq n\,T \quad . \tag{39}$$

The grid of points for the discrete quadrature should be reasonably regular and the grid intervals should satisfy the inequality

$$\Delta\mathbf{x}^{\mathrm{T}}|\mathbf{M}|\,\Delta\mathbf{x} \leq \frac{T}{n} \quad . \tag{40}$$

The average number of discrete points necessary for the numerical quadrature is then about

$$N = \pi\, n^2 \tag{41}$$

for a single frequency, and

$$N = \pi\, n^2\,\frac{T_{\max}}{T_{\min}} \tag{42}$$

for periods from the range $< T_{\min}, T_{\max} >$.

The new definition of Fresnel zones expressed by inequality (39) requires the knowledge of the first and second derivatives $\mathbf{p}$ and $\mathbf{M}$ of the sum of travel times along the surface of integration. On the other hand, the definition is purely local, independent of the reference travel times. It is thus applicable everywhere, and may define Fresnel zones even in the regions influencing the Kirchhoff integral without the existence of stationary points. Let us recall that exact values $\mathbf{p} = \mathbf{p}(\mathbf{x})$, $\mathbf{M} = \mathbf{M}(\mathbf{x})$ are considered at each point $\mathbf{x}$ along the surface of integration, and no paraxial approximation is included in (39).

As far as the author can appreciate, the new, quantitative definition of Fresnel zones is conformal with other, usually qualitatively or empirically derived definitions, within the regions of their applicability. For example, if the paraxial ray approximation at stationary point $\mathbf{x}''$ is applicable over the Fresnel zones, new definition (39) is equivalent to Eq. (2.10.13) by Kravtsov & Orlov (1990),

$$[\mathbf{x}-\mathbf{x}'']^{\mathrm{T}}|\mathbf{M}(\mathbf{x}'')|\,[\mathbf{x}-\mathbf{x}''] \le \frac{T}{n} \quad , \tag{43}$$

see also Červený & Soares (1992).

Acknowledgements

The author is indebted to Professor Vlastislav Červený and Dr. Michal Kvasnička for advising him to think about Fresnel zones.

References

Červený, V., Pleinerová, J., Klimeš, L. & Pšenčík, I. (1987): High–frequency radiation from earthquake sources in laterally varying layered structures. Geophys. J. R. astr. Soc., **88**, 43–79.

Červený, V. & Soares, J.E.P. (1992): Fresnel volume ray tracing. Geophysics, **57**, 902–915.

Klimeš, L. (1986): Discretization error for the superposition of Gaussian beams. Geophys. J. R. astr. Soc., **86**, 531–551.

Kravtsov, Yu.A. & Orlov, Yu.I. (1980): Geometrical Optics of Inhomogeneous Media (in Russian). Nauka, Moscow.

Kravtsov, Yu.A. & Orlov, Yu.I. (1990): Geometrical Optics of Inhomogeneous Media. Springer–Verlag, Heidelberg.