Fermat's principle in inhomogeneous anisotropic media [1]

Ivan Pšenčík

Geophysical Institute, Acad.Sci.of Czech Republic, Boční II, Praha 4, Czech Republic; e-mail: ip@ig.cas.cz

Summary It is shown that the ray tracing equations for inhomogeneous anisotropic media are equations of extremals of the Fermat functional for such media.

Key words: Fermat's principle, Fermat's functional, extremals, rays.

Introduction

Kline and Kay (1965) were probably the first who proved that rays of electromagnetic waves in inhomogeneous anisotropic media are extremals of the corresponding Fermat's functional. Recently, Hanyga (1997) and Stephen (1997) studied a similar problem for elastic media. Hanyga (1997) uses Hamiltonian formalism and shows that application of Fermat's principle to rays associated with non-convex slowness surfaces in anisotropic media must be done with care. Here an alternative derivation to the two above mentioned papers is presented. It is shown that the ray tracing equations for inhomogeneous anisotropic media are equivalent to equations of extremals of the Fermat functional in inhomogeneous anisotropic media. For details concerning the Fermat principle, see for example, Born and Wolf (1959), Kline and Kay (1965), Červený (1997).

Proof

The curve $x_i(s)$, which renders the integral – Fermat's functional

$$\int_A^B F(x_k, \dot{x}_k) ds, \quad \dot{x}_k = \frac{dx_k}{ds} \tag{1}$$

stationary is called an extremal (Kline and Kay, 1965) and it satisfies the Euler eqautions

$$\frac{d}{ds}\left(\frac{\partial F}{\partial \dot{x}_i}\right) - \frac{\partial F}{\partial x_i} = 0. \tag{2}$$

Here, $F(x_k, \dot{x}_k)$ is a continuous function with continuous partial derivatives up to the second order in all its variables (Born and Wolf, 1959). It is natural to expect independency of (1) on the choice of a parameter along the curve x_i. This implies that F must be a homogeneous function of the first order with respect to $\dot{x}_i$,

$$F(x_k, \lambda \dot{x}_k) = \lambda F(x_k, \dot{x}_k). \tag{3}$$

[1] Seismic Waves in Complex 3-D Structures, Report 6 (Department of Geophysics, Charles University, Prague 1997)

We prove now that the ray tracing system for inhomogeneous anisotropic media

$$\frac{dx_i}{d\tau} = \frac{1}{2}\frac{\partial G}{\partial p_i}, \qquad \frac{dp_i}{d\tau} = -\frac{1}{2}\frac{\partial G}{\partial x_i} \tag{4}$$

can be rewritten into a form equivalent to the Euler equations (2) for the extremals of the Fermat's functional. In (4),

$$G(x_i, p_i) = a_{ijkl}p_i p_l g_j g_k = v_i p_i = 1 \tag{5}$$

is one of the three eigenvalues of the Christoffel matrix $\Gamma_{jk} = a_{ijkl}p_i p_l$, $v_i = \frac{dx_i}{d\tau}$ is the ray (group) velocity. It easily follows from (5) that G is a homogeneous function of the second order with respect to p_i,

$$p_i\frac{\partial G}{\partial p_i} = 2G = 2. \tag{6}$$

Hanyga (1997) proved that the first set of equations in (4) can be locally used for finding p_i for a given v_i. We can thus, formally, write

$$G(x_i, p_i) = G(x_i, p_i(v_k)) = \bar{G}(x_i, v_i). \tag{7}$$

Using the first set of equations in (4) and Eq.(6), we can find that

$$\frac{\partial G}{\partial v_i} = \frac{\partial G}{\partial p_k}\frac{\partial p_k}{\partial v_i} = 2v_k\frac{\partial p_k}{\partial v_i} = 2\frac{\partial G}{\partial v_i} - 2p_i \quad \Rightarrow \quad p_i = \frac{1}{2}\frac{\partial G}{\partial v_i} \tag{8}$$

We also get from (6):

$$p_i\frac{\partial G}{\partial p_i} = 2G \quad \Rightarrow \quad p_k\frac{\partial^2 G}{\partial x_i \partial p_k} = 2\frac{\partial G}{\partial x_i}), \qquad p_k\frac{\partial^2 G}{\partial p_i \partial p_k} = \frac{\partial G}{\partial p_i}. \tag{9}$$

Because $G = 1$ and, of course, also $\bar{G} = 1$ holds along the ray, we have

$$\frac{dG}{d\tau} = \frac{d\bar{G}}{d\tau} = 0 = \frac{\partial G}{\partial x_i}\frac{dx_i}{d\tau} + \frac{\partial G}{\partial p_i}\frac{dp_i}{d\tau} = \frac{\partial \bar{G}}{\partial x_i}\frac{dx_i}{d\tau} + \frac{\partial \bar{G}}{\partial v_i}\frac{dv_i}{d\tau}. \tag{10}$$

Using successively (10), (8), (4), (9) and again (10), we get

$$\frac{\partial \bar{G}}{\partial x_i}\frac{dx_i}{d\tau} = -\frac{\partial \bar{G}}{\partial v_i}\frac{dv_i}{d\tau} = -p_i(\frac{\partial^2 G}{\partial x_k \partial p_i}\frac{dx_k}{d\tau} + \frac{\partial^2 G}{\partial p_k \partial p_i}\frac{dp_k}{d\tau})$$

$$= -2\frac{\partial G}{\partial x_k}\frac{dx_k}{d\tau} - \frac{\partial G}{\partial p_k}\frac{dp_k}{d\tau} = -\frac{\partial G}{\partial x_k}\frac{dx_k}{d\tau} \quad \Rightarrow \quad \frac{\partial G}{\partial x_k} = -\frac{\partial \bar{G}}{\partial x_k}. \tag{11}$$

From Eq.(8), from the second set of equations in (4) and from (11), we get

$$\frac{dp_i}{d\tau} = \frac{1}{2}\frac{d}{d\tau}(\frac{\partial \bar{G}}{\partial v_i}) = -\frac{1}{2}\frac{\partial G}{\partial x_i} = \frac{1}{2}\frac{\partial \bar{G}}{\partial x_i} \Rightarrow \quad \frac{1}{2}\frac{d}{d\tau}(\frac{\partial \bar{G}}{\partial v_i}) - \frac{1}{2}\frac{\partial \bar{G}}{\partial x_i} = 0. \tag{12}$$

Equation (12) can be rewritten into the form

$$\frac{d}{d\tau}(\frac{\partial \bar{G}^{1/2}}{\partial v_i}) - \frac{\partial \bar{G}^{1/2}}{\partial x_i} = 0. \tag{13}$$

Substitution of the parameter τ in (13) by the parameter s, $\frac{ds}{d\tau} = v$ leads to

$$\frac{d}{ds}(\frac{\partial \bar{G}^{1/2}}{\partial v_i}) - \frac{\partial \bar{G}^{1/2}}{\partial x_i} = 0. \tag{14}$$

Eqs.(14) represent a system of differential equations equivalent to the ray tracing system (4). Let us note that the transformation from (13) to (14) was possible because of the first order homogeneity of the function $\bar{G}^{1/2}$. If we take into account that

$$\bar{G}(x_k, v_k) = v^2 \bar{G}(x_k, \dot{x}_k), \tag{15}$$

we can identify Eqs.(14) as the Euler equations (2), in which $F = \bar{G}^{1/2}$ and which render the functional

$$\int_A^B \bar{G}^{1/2}(x_k, \dot{x}_k) ds = \int_A^B \bar{G}^{1/2}(x_k, v_k) \frac{ds}{v} = \int_A^B \frac{ds}{v} \tag{16}$$

stationary.

References

Born,M and Wolf,E., 1959. Principles of optics, Pergamon Press, London.

Červený, V. 1997. Seismic ray theory. In: Seismic Waves in Complex 3-D Structures, Report 5, Department of Geophysics, Charles University, Prague.

Hanyga, A., 1997. Fermat's principle in anisotropic elasticity. special SEG volume on Seismic Anisotropy, submitted.

Kline, M. and Kay, I. W., 1965. Electromagnetic theory and geometrical optics, Interscience, New York.

Stephen, R., 1997. Fermat's principle for anisotropic elastic media. special SEG volume on Seismic Anisotropy, submitted.