Weak-contrast R/T coefficients in weakly anisotropic media: *P*-wave incidence

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Summary

Approximate *PP* and *PS* reflection/transmission coefficients for weak-contrast interfaces in general weakly anisotropic media are presented. The coefficients were obtained by applying the first-order perturbation theory and they are written in a compact and relatively simple form. The formulae can be used for arbitrary orientations of the incidence plane and the interface without transforming elasticity parameters to a local Cartesian coordinate system. The accuracy of the approximate formulae is illustrated on the *PS* reflection coefficient. Comparison of the exact and approximate coefficients shows that the approximate formulae are sufficiently accurate for a wide range of incidence angles and for all azimuths.

1 Introduction

Calculation of reflection/transmission coefficients of plane waves at a plane interface is more complex in anisotropic media than in isotropic media (see Keith & Crampin 1977). In isotropy, the problem can be separated to reflection/transmission of *P-SV* waves and independently to reflection/transmission of *SH*-waves. In anisotropy, the incident wave generates six waves: *P*, *S*1 and *S*2 reflected waves, and *P*, *S*1 and *S*2 transmitted waves, thus the problem is represented, in general, by six non-separable linear algebraic equations with six unknown R/T coefficients. An explicit analytic solution of such a system is cumbersome, even when a simple anisotropy like transverse isotropy is used (see Daley & Hron 1977). Therefore, the numerical solution of the system of equations is in common use (see Fryer & Frazer 1984; Gajewski & Pšenčík 1987). An alternative way is to use approximate analytical formulae derived for a weak-contrast interface in weakly anisotropic media. Since in many practical applications anisotropy and a contrast across an interface is not strong (Thomsen 1986), such formulae are often of a sufficient accuracy. Moreover, the formulae are relatively simple and comprehensible. The first attempt to find linearized R/T coefficients for weak anisotropic media was done by Thomsen (1993), who derived formulae for transversely isotropic halfspaces with axes of symmetry perpendicular to the interface. Rueger (1996) corrected Thomsen's results and found linearized R/T coefficients for transversely isotropic halfspaces with axes of symmetry parallel to the interface, but only in the symmetry plane. Haugen & Ursin (1996) derived *PP* reflection coefficients for another special orientation of symmetry axes of transversely isotropic halfspaces. Zillmer, Gajewski & Kashtan (1998) derived *PP* reflection coefficient for a horizontal interface in general weakly anisotropic media and *SVSV* and *SHSH* reflection coefficients for waves propagating in a symmetry plane. Pšenčík & Vavryčuk (1998) and Vavryčuk & Pšenčík (1998) derived *PP* displacement reflection as well as transmission coefficients for a horizontal interface in general weakly anisotropic media. In this paper, we extend the mentioned results by deriving formulae for *PP* as well as *PS* R/T

coefficients for arbitrary oriented weak-contrast interfaces in general weakly anisotropic media. We will present formulae for displacement as well as for energy coefficients.

2 Reflection/transmission of plane waves in anisotropic media

Let us consider two homogeneous anisotropic halfspaces separated by a plane interface Σ specified by a normal $\boldsymbol{\nu}$. We assume that the halfspaces are in a welded contact at interface Σ. The halfspace, into which the normal $\boldsymbol{\nu}$ points, will be referred to as the halfspace (1), the other halfspace will be referred to as the halfspace (2). The halfspaces are characterized by the densities $\rho^{(I)}$, I=1,2, and by the density normalized elasticity tensors $a_{ijkl}^{(I)}$, I=1,2. The incident harmonic plane wave propagates in halfspace (1) and it generates six plane harmonic waves, reflected P, $S1$ and $S2$, and transmitted P, $S1$ and $S2$. The displacement and traction vectors $\mathbf{u}(\mathbf{x},t)$ and $\mathbf{T}(\mathbf{x},t)$ of any of the waves can be expressed in the following way:

$$\mathbf{u}^{(N)}(\mathbf{x},t) = c^{(N)}\,\mathbf{g}^{(N)}\exp\left[i\omega\left(t-\mathbf{p}^{(N)}\cdot\mathbf{x}\right)\right],$$
$$\mathbf{T}^{(N)}(\mathbf{x},t) = c^{(N)}\boldsymbol{\sigma}^{(N)}\exp\left[i\omega\left(t-\mathbf{p}^{(N)}\cdot\mathbf{x}\right)\right], \tag{1}$$

where $c^{(N)}$ is the scalar amplitude, $\mathbf{g}^{(N)}$ is the unit polarization vector, $\mathbf{p}^{(N)}$ is the slowness vector, ω is the circular frequency, and t is the time. Superscript N denotes the type of the wave: N=0 corresponds to the incident wave, N=1,2 and 3 to the reflected P, $S1$ and $S2$; and N=4,5 and 6 to the transmitted P, $S1$ and $S2$, respectively. If the scalar amplitude of the incident wave $c^{(0)}$ is equal to 1 then the amplitude $c^{(N)}$ of the N-th scattered wave means its displacement R/T coefficient. $\boldsymbol{\sigma}^{(N)}$ is the amplitude-normalized traction vector of the N-th wave:

$$\sigma_i^{(N)} = \rho^{(I)} a_{ijkl}^{(I)} \nu_j g_k^{(N)} p_l^{(N)}. \tag{2}$$

Superscript I=1,2 identifies the halfspace, in which the wave propagates. Displacement and traction at the interface must satisfy boundary conditions requiring their continuity across the interface. If we introduce a 6-vector $\mathbf{d}^{(N)}$ for each wave

$$\mathbf{d}^{(N)} = \pm\begin{bmatrix}\mathbf{g}^{(N)}\\ \boldsymbol{\sigma}^{(N)}\end{bmatrix}, \tag{3}$$

where plus sign stands for reflected and minus sign for transmitted and incident waves, we can express boundary conditions by the following equation:

$$\mathbf{D}\mathbf{c} = \mathbf{d}^{(0)}. \tag{4}$$

Subsequently, we obtain

$$\mathbf{c} = \mathbf{D}^{-1}\mathbf{d}^{(0)}. \tag{5}$$

$\mathbf{D}$ is the 6 x 6 matrix called the displacement-stress matrix, $\mathbf{d}^{(N)}$ is the displacement-stress vector of the N-th wave, and $\mathbf{c}$ is the 6-vector of the displacement R/T coefficients:

$$\mathbf{D} = \begin{bmatrix}\mathbf{g}^{(1)} & \mathbf{g}^{(2)} & \mathbf{g}^{(3)} & -\mathbf{g}^{(4)} & -\mathbf{g}^{(5)} & -\mathbf{g}^{(6)}\\ \boldsymbol{\sigma}^{(1)} & \boldsymbol{\sigma}^{(2)} & \boldsymbol{\sigma}^{(3)} & -\boldsymbol{\sigma}^{(4)} & -\boldsymbol{\sigma}^{(5)} & -\boldsymbol{\sigma}^{(6)}\end{bmatrix} = \left[\mathbf{d}^{(1)} \quad \mathbf{d}^{(2)} \quad \mathbf{d}^{(3)} \quad \mathbf{d}^{(4)} \quad \mathbf{d}^{(5)} \quad \mathbf{d}^{(6)}\right],$$

$$\mathbf{c} = \left[c^{(1)} \quad c^{(2)} \quad c^{(3)} \quad c^{(4)} \quad c^{(5)} \quad c^{(6)}\right]^T = \left[R^P \quad R^{S1} \quad R^{S2} \quad T^P \quad T^{S1} \quad T^{S2}\right]^T.$$

Instead of the *displacement R/T coefficients*, coefficients normalized to energy flux normal to Σ are also in use (Chapman 1994). We will call them the *energy R/T coefficients* and we will define them as a ratio of normal components of energy flux of the *N*-th scattered wave to the incident wave:

$$c_E^{(N)} = \left| \frac{\mathbf{w}^{(N)} \cdot \boldsymbol{\nu}}{\mathbf{w}^{(0)} \cdot \boldsymbol{\nu}} \right| , \tag{6}$$

where $\mathbf{w}^{(0)}$ and $\mathbf{w}^{(N)}$ are vectors of the energy flux of the incident and *N*-th scattered waves, and $\boldsymbol{\nu}$ is the normal to the interface. Subsequently, we can write

$$c_E^{(N)} = \left| \frac{\mathbf{T}^{(N)} \cdot \mathbf{u}^{(N)}}{\mathbf{T}^{(0)} \cdot \mathbf{u}^{(0)}} \right| = \left(c^{(N)}\right)^2 \left| \frac{\boldsymbol{\sigma}^{(N)} \cdot \mathbf{g}^{(N)}}{\boldsymbol{\sigma}^{(0)} \cdot \mathbf{g}^{(0)}} \right| = \left(c^{(N)}\right)^2 \left| \frac{\mathbf{v}_g^{(N)} \cdot \boldsymbol{\nu}}{\mathbf{v}_g^{(0)} \cdot \boldsymbol{\nu}} \right| , \tag{7}$$

with no summation over *N*. Vectors $\mathbf{v}_g^{(0)}$ and $\mathbf{v}_g^{(N)}$ are the group velocity vectors of the incident and *N*-th scattered waves.

3 *P*-wave incidence at a weak contrast interface between arbitrary anisotropic media

We now consider *P* wave incident at the interface Σ. Moreover, we assume that the interface Σ separates two halfspaces with very similar anisotropy. In this case, we can express elasticity parameters $a_{ijkl}^{(I)}$ and density $\rho^{(I)}$ in both halfspaces in terms of elasticity parameters a_{ijkl}^{0} and density ρ^{0} of the *background anisotropic medium*, which is the same for both halfspaces:

$$a_{ijkl}^{(I)} = a_{ijkl}^{0} + \Delta a_{ijkl}^{(I)} \ , \ \rho^{(I)} = \rho^{0} + \Delta\rho^{(I)} \ , \ I=1,2 \ . \tag{8}$$

The symbols $\Delta a_{ijkl}^{(I)}$ and $\Delta\rho^{(I)}$ denote deviations from parameters of the background medium. We will require that these deviations or equivalently *elasticity contrasts* $\Delta a_{ijkl} = a_{ijkl}^{(2)} - a_{ijkl}^{(1)}$ and *density contrast* $\Delta\rho = \rho^{(2)} - \rho^{(1)}$ at the interface must be small.

Under the weak-contrast condition, we can linearize the vector of R/T coefficients

$$\mathbf{c} = \left(R^{PP} \quad R^{PS1} \quad R^{PS2} \quad T^{PP} \quad T^{PS1} \quad T^{PS2} \right)^T$$

as follows:

$$\mathbf{c} = \mathbf{c}^0 + \Delta\mathbf{c} \ , \tag{9}$$

where $\mathbf{c}^0$ is the vector of R/T coefficients in the background medium and $\Delta\mathbf{c}$ is its small perturbation. Since the background medium is homogeneous, the vector $\mathbf{c}^0$ attains the following simple form

$$\mathbf{c}^0 = \left(0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \right)^T , \tag{10}$$

expressing an obvious fact that the incident *P* wave is traversing the fictitious (non-existent) interface in the background medium unaffected. For perturbation

$$\Delta\mathbf{c} = \left(R^{PP} \quad R^{PS1} \quad R^{PS2} \quad T^{PP} - 1 \quad T^{PS1} \quad T^{PS2} \right)^T$$

Eq. (5) yields (Thomsen 1993, Eq. A-18a; Vavryčuk & Pšenčík 1998, Eqs 18 and 28)

$$\Delta\mathbf{c} = \left(\mathbf{D}^0\right)^{-1}\left(\Delta\mathbf{d}^{(0)} - \Delta\mathbf{D}\mathbf{c}^0\right) = -\left(\mathbf{D}^0\right)^{-1}\left(\Delta\mathbf{d}^{(4)} - \Delta\mathbf{d}^{(0)}\right). \tag{11}$$

The inverse matrix $\left(\mathbf{D}^0\right)^{-1}$ can be expressed explicitly (Fryer & Frazer 1984, Eq. 3.20; Chapman 1994, Eq. 22), thus for the perturbation of the R/T coefficient of the *N*-th wave we obtain finally:

$$\boxed{\Delta c^{(N)} = -\frac{\mathbf{d}^{0(N)} \cdot \Delta\mathbf{s}}{\mathbf{d}^{0(N)} \cdot \mathbf{s}^{0(N)}} = -\frac{\mathbf{s}^{0(N)} \cdot \Delta\mathbf{d}}{\mathbf{d}^{0(N)} \cdot \mathbf{s}^{0(N)}}}, \text{ with no summation over } N. \tag{12}$$

Vectors $\mathbf{d}^{0(N)}$ and $\mathbf{s}^{0(N)}$ are the displacement-stress and stress-displacement vectors of the *N*-th scattered wave in the background medium, and $\Delta\mathbf{s}$ and $\Delta\mathbf{d}$ denote the differences between the stress-displacement and displacement-stress vectors of the transmitted and incident *P* waves across the interface:

$$\Delta\mathbf{s} = -\begin{bmatrix}\Delta\boldsymbol{\sigma}\\ \Delta\mathbf{g}\end{bmatrix},\ \Delta\mathbf{d} = -\begin{bmatrix}\Delta\mathbf{g}\\ \Delta\boldsymbol{\sigma}\end{bmatrix},\ \Delta\mathbf{g} = \Delta\mathbf{g}^{(4)} - \Delta\mathbf{g}^{(0)},\ \Delta\boldsymbol{\sigma} = \Delta\boldsymbol{\sigma}^{(4)} - \Delta\boldsymbol{\sigma}^{(0)}. \tag{13}$$

In analogy to terms elasticity and density contrasts, we will call quantities $\Delta\mathbf{s}$, $\Delta\mathbf{d}$, $\Delta\mathbf{g}$ and $\Delta\boldsymbol{\sigma}$ as *stress-displacement, displacement-stress, polarization* and *traction contrasts*.

Formula (12) is valid *for strongly as well as for weakly anisotropic background medium or even for isotropic background medium*. Similar formulae to (12) for *P*-wave incidence can also be written for *S*1- or *S*2-wave incidences. However, anisotropy of the background medium must be sufficiently strong in this case. For *S* waves incident at a weak contrast interface in weakly anisotropic medium, our approach cannot be applied without a modification. The reason is a degeneration of *S* waves in isotropic media (see Jech & Pšenčík 1989).

For energy R/T coefficients, Eq. (7) yields immediately

$$\boxed{\Delta c_E^{(N)} = \left(\Delta c^{(N)}\right)^2 \left|\frac{\mathbf{v}_g^{0(N)} \cdot \boldsymbol{\nu}}{\mathbf{v}_g^{0(0)} \cdot \boldsymbol{\nu}}\right|}, \text{ for } N=1,2,3,5,6,\ N \neq 4, \tag{14}$$

where $\mathbf{v}_g^{0(0)}$ and $\mathbf{v}_g^{0(N)}$ are group velocities of the incident and of the *N*-th scattered waves in the background medium. Formula for the perturbation of the energy coefficient of the transmitted *P* wave (*N*=4) is more complicated and it is more convenient to calculate this coefficient from energy conservation law for scattered waves

$$\Delta c_E^{(4)} = 1 - c_E^{(1)} - c_E^{(2)} - c_E^{(3)} - c_E^{(5)} - c_E^{(6)}. \tag{15}$$

Obviously, formulae for energy coefficients are of the order of second perturbations.

4 *P*-wave incidence at a weak contrast interface in weakly anisotropic media

In the previous section, we presented a formula for R/T coefficients at weak contrast interfaces in general anisotropic media including strong anisotropy. In this section, we will confine ourselves to only *weakly anisotropic media* by assuming that the background medium is *isotropic*. Under this assumption, we can express formulae for contrasts $\Delta\mathbf{g}$ and $\Delta\boldsymbol{\sigma}$, and vectors $\mathbf{g}^{0(N)}$ and $\boldsymbol{\sigma}^{0(N)}$ for all scattered waves that are necessary for the determination of R/T coefficients by Eq (12). In this section, we will show that contrasts $\Delta\mathbf{g}$ and $\Delta\boldsymbol{\sigma}$ are linear functions of elasticity and density contrasts Δa_{ijkl} and $\Delta\rho$ in the first-order perturbation

theory. Obviously, the polarization and traction vectors $\mathbf{g}^{0(N)}$ and $\boldsymbol{\sigma}^{0(N)}$ for unconverted waves do not depend on elasticity or density contrasts at all. However, the problem is more complicated for converted waves, because polarization and traction vectors $\mathbf{g}^{0(N)}$ and $\boldsymbol{\sigma}^{0(N)}$ are functions of elasticity contrasts Δa_{ijkl} in this case. Although these vectors are vectors specified for the isotropic background medium, they depend on properties of a perturbed medium (see Jech & Pšenčík 1989). Moreover, this dependence is even *non-linear*. Consequently, also R/T coefficients for converted waves are *non-linear* functions of elasticity contrasts Δa_{ijkl} at the interface even in the first-order perturbation theory. In order to avoid this difficulty we introduce new quantities called *vector R/T coefficients of S waves* which are defined as follows:

$$\mathbf{R}^{PS} = c^{(2)}\mathbf{g}^{0(2)} + c^{(3)}\mathbf{g}^{0(3)}, \ \mathbf{T}^{PS} = c^{(5)}\mathbf{g}^{0(5)} + c^{(6)}\mathbf{g}^{0(6)}. \tag{16}$$

$\mathbf{R}^{PS}$ will be called the *vector reflection coefficient*, and $\mathbf{T}^{PS}$ the *vector transmission coefficient of S waves*. Obviously, the scalar R/T coefficient is obtained by projecting the vector coefficient into the direction of polarization of the scattered wave. Later we will show that likewise R/T coefficients for *P* waves, these coefficients are linear functions of contrasts Δa_{ijkl}.

In this section we will present formulae for contrasts $\Delta\mathbf{g}$ and $\Delta\boldsymbol{\sigma}$, and vectors $\mathbf{g}^{0(N)}$ and $\boldsymbol{\sigma}^{0(N)}$, in the next section we will present final formulae for R/T coefficients.

4.1. Polarization and traction contrasts $\Delta\mathbf{g}$ and $\Delta\boldsymbol{\sigma}$

If we perturb polarization vectors of the incident and transmitted *P* waves, we get the contrast $\Delta\mathbf{g}$ in the following form:

$$\Delta\mathbf{g} = \Delta\mathbf{g}_1 + \Delta\mathbf{g}_2 \ ,$$

$$\Delta g_{1m} = \frac{\Delta V}{\alpha}\left(N_m^P - \frac{1}{\cos\theta^P}\nu_m \right), \ \Delta V = \frac{1}{2\alpha}\Delta a_{ijkl} N_i^P N_j^P N_k^P N_l^P \ , \tag{17}$$

$$\Delta g_{2m} = \frac{\Delta\Gamma_{kl}}{\alpha^2 - \beta^2} N_k^P \left(\delta_{lm} - N_l^P N_m^P \right) \ , \ \Delta\Gamma_{kl} = \Delta a_{ijkl} N_i^P N_j^P \ ,$$

$$\cos\theta^P = \mathbf{N}^P \cdot \boldsymbol{\nu} \ ,$$

where α and β denote the *P*- and *S*-wave velocities in the background isotropic medium, $\mathbf{N}^P$ is the phase normal of the incident and transmitted *P* waves in the isotropic background medium, and $\boldsymbol{\nu}$ is the normal to the interface. Quantity ΔV is the *P-wave phase-velocity contrast* defined as

$$\Delta V = V^{(4)} - V^{(0)} = \Delta V^{(4)} - \Delta V^{(0)}, \tag{18}$$

where $V^{(0)}$ and $V^{(4)}$ are phase velocities of the incident and transmitted *P* waves. The quantity $\Delta\mathbf{g}_1$ denotes the polarization contrast due to a contrast of the phase velocity at the interface and it is determined by the Snell's law. For no velocity contrast at the interface in weakly anisotropic media, $\Delta\mathbf{g}_1$ is equal to zero. The quantity $\Delta\mathbf{g}_2$ denotes the polarization contrast due to weak anisotropy. For a weak contrast interface in isotropic media, $\Delta\mathbf{g}_2$ is equal to zero.

Analogously to determination of $\Delta\mathbf{g}$, we can perturb traction vectors of the incident and transmitted *P* waves to obtain contrast $\Delta\boldsymbol{\sigma}$:

$$\Delta\boldsymbol{\sigma} = \Delta\boldsymbol{\sigma}_1 + \Delta\boldsymbol{\sigma}_2 + \Delta\boldsymbol{\sigma}_3,$$

$$\Delta\sigma_{1m} = \frac{\Delta\rho}{\rho}\sigma_m^{0(0)} = \Delta\rho\left[\frac{\alpha^2 - 2\beta^2}{\alpha}\nu_m + 2\frac{\beta^2}{\alpha}\cos\theta^P N_m^P\right],$$

$$\Delta\sigma_{2m} = -\Delta V\frac{\rho}{\cos\theta^P}\left\{2\frac{\beta^2}{\alpha^2}\left[\frac{\alpha^2+\beta^2}{\alpha^2-\beta^2}\cos^2\theta^P + 1\right]N_m^P + \cos\theta^P\nu_m\right\}, \tag{19}$$

$$\Delta\sigma_{3m} = \Delta\Gamma_{kl}\frac{\rho}{\alpha}\left[\nu_k\delta_{lm} + \frac{\beta^2}{\alpha^2-\beta^2}N_k^P\left(\cos\theta^P\delta_{lm} + \nu_l N_m^P\right)\right].$$

$\Delta\boldsymbol{\sigma}_1$ denotes the perturbation due to the density contrast, $\Delta\boldsymbol{\sigma}_2$ is the perturbation due to the *P*-wave phase velocity contrast ΔV at the interface, and $\Delta\boldsymbol{\sigma}_3$ is that part of $\Delta\boldsymbol{\sigma}$ which is non-zero even for no density and no velocity contrasts $\Delta\rho$ and ΔV at the interface.

4.2 Polarization and traction vectors $\mathbf{g}^{0(N)}$ and $\boldsymbol{\sigma}^{0(N)}$

For unconverted waves, we obtain

$$g_m^{0(1)} = N_m^{RP},\ g_m^{0(4)} = N_m^P,$$

$$\sigma_m^{0(1)} = \frac{\rho}{\alpha}\left\{\left(\alpha^2 - 2\beta^2\right)\nu_m + 2\beta^2\cos\theta^{RP}N_m^{RP}\right\}, \sigma_m^{0(4)} = \frac{\rho}{\alpha}\left[\left(\alpha^2 - 2\beta^2\right)\nu_m + 2\beta^2\cos\theta^P N_m^P\right], \tag{20}$$

where $\mathbf{N}^{RP}$ is the phase normal of the reflected *P* wave in the isotropic background.

As we mentioned before, the problem for converted waves is more complicated than for unconverted waves. Both polarization and traction vectors $\mathbf{g}^{0(N)}$ and $\boldsymbol{\sigma}^{0(N)}$ are non-linear functions of elasticity contrast Δa_{ijkl}. However, if we try to determine the vector R/T coefficients of *S* waves by (16), we need to specify in (12) dyadics $\mathbf{g}^{0(2)}\mathbf{g}^{0(2)} + \mathbf{g}^{0(3)}\mathbf{g}^{0(3)}$ and $\boldsymbol{\sigma}^{0(2)}\mathbf{g}^{0(2)} + \boldsymbol{\sigma}^{0(3)}\mathbf{g}^{0(3)}$ instead of vectors $\mathbf{g}^{0(N)}$ and $\boldsymbol{\sigma}^{0(N)}$. For reflected *S* waves we obtain:

$$g_k^{0(2)}g_l^{0(2)} + g_k^{0(3)}g_l^{0(3)} = \delta_{kl} - N_k^{RS}N_l^{RS},$$

$$\sigma_k^{0(2)}g_l^{0(2)} + \sigma_k^{0(3)}g_l^{0(3)} = \rho\beta\left(\cos\theta^{RS}\delta_{kl} + N_k^{RS}\nu_l - 2\cos\theta^{RS}N_k^{RS}N_l^{RS}\right), \tag{21}$$

where

$$\cos\theta^{RS} = \mathbf{N}^{RS}\cdot\boldsymbol{\nu}.$$

$\mathbf{N}^{RS}$ is the phase normal of the reflected *S* waves in the isotropic background. Analogous formulae can also be expressed for transmitted *S* waves. Notice that these formulae are no more functions of polarization vectors of *S* waves, thus they do not depend on elasticity contrast Δa_{ijkl} at all.

5 R/T coefficients in weakly anisotropic media

Likewise the stress contrast $\Delta\boldsymbol{\sigma}$ we will divide the perturbation of R/T coefficients $\Delta\mathbf{c}$ into three parts:

$$\Delta\mathbf{c} = \Delta\mathbf{c}_1 + \Delta\mathbf{c}_2 + \Delta\mathbf{c}_3. \tag{22}$$

The first part, $\Delta\mathbf{c}_1$, is the perturbation due to the density contrast $\Delta\rho$, the second part, $\Delta\mathbf{c}_2$, will be caused by the phase-velocity contrast ΔV, and the last part, $\Delta\mathbf{c}_3$, will be that part of $\Delta\mathbf{c}$ which can be non-zero even for no density and no phase-velocity contrasts at the interface.

5.1 *PP* coefficients

Inserting Eqs (17), (19) and (20) into (12) we obtain for *PP* coefficients

$$R^{PP} = R_1^{PP} + R_2^{PP} + R_3^{PP}\ ,\ T^{PP} = 1 + T_1^{PP} + T_2^{PP} + T_3^{PP}\ , \tag{23}$$

where

$$R_1^{PP} = \frac{1}{2}\frac{\Delta\rho}{\rho}\left[1 - 4\frac{\beta^2}{\alpha^2}\sin^2\theta^P\right],\ R_2^{PP} = \frac{1}{2\cos^2\theta^P}\frac{\Delta V}{\alpha}\ ,\ R_3^{PP} = \frac{\Delta\Gamma_{kl}}{\alpha^2}\nu_k\left[\nu_l - \frac{1}{\cos\theta^P}N_l^P\right], \tag{24}$$

$$T_1^{PP} = -\frac{1}{2}\frac{\Delta\rho}{\rho}\ ,\ T_2^{PP} = \frac{1}{2\cos^2\theta^P}\frac{\Delta V}{\alpha}\left[1 + 2\cos^2\theta^P\right],\ T_3^{PP} = -\frac{\Delta\Gamma_{kl}}{\alpha^2}\frac{\nu_k N_l^P}{\cos\theta^P}\ , \tag{25}$$

and

$$\Delta V = \frac{1}{2\alpha}\Delta a_{ijkl}N_i^P N_j^P N_k^P N_l^P\ ,\ \Delta\Gamma_{kl} = \Delta a_{ijkl}N_i^P N_j^P,\ \cos\theta^P = \mathbf{N}^P\cdot\boldsymbol{\nu} < 0\,.$$

$\mathbf{N}^P$ is the phase normal of the incident wave in the background medium, $\boldsymbol{\nu}$ is the normal to the interface, θ^P is the angle between the normal $\boldsymbol{\nu}$ and the phase normal $\mathbf{N}^P$. α, β and ρ are *P*- and *S*-wave velocities and the density of the isotropic background medium. Δa_{ijkl} and $\Delta\rho$ are elasticity and density contrasts at the interface. Note that our definition of orientations of $\boldsymbol{\nu}$ and $\mathbf{N}^P$ implicates that $\cos\theta^P$ is negative.

5.2 *PS* coefficients

Reflection coefficients of converted waves can be calculated from the vector coefficient $\mathbf{R}^{PS}$ in the following way:

$$\Delta c^{(2)} = \mathbf{R}^{PS}\cdot\mathbf{g}^{0(2)},\ \Delta c^{(3)} = \mathbf{R}^{PS}\cdot\mathbf{g}^{0(3)}\,. \tag{26}$$

Inserting Eqs (17), (19), (21) and (12) into (16) we can express the vector coefficient $\mathbf{R}^{PS}$ as follows:

$$\mathbf{R}^{PS} = \mathbf{R}_1^{PS} + \mathbf{R}_2^{PS} + \mathbf{R}_3^{PS}\ ,$$

$$R_{1m}^{PS} = \frac{\Delta\rho}{\rho}\frac{1}{2\cos\theta^{RS}}\left\{\frac{\alpha^2 - 2\beta^2}{\alpha\beta}\nu_m + 2\frac{\beta}{\alpha}\cos\theta^P N_m^P\right\},$$

$$R_{2m}^{PS} = -\frac{\Delta V}{\beta}\frac{1}{2\cos\theta^P\cos\theta^{RS}}\left\{\left[2\frac{\beta^2}{\alpha^2} + \frac{\beta}{\alpha}\frac{\alpha^2+\beta^2}{\alpha^2-\beta^2}\cos\theta^P\left(2\frac{\beta}{\alpha}\cos\theta^P + \cos\theta^{RS}\right)\right]N_m^P\right.$$

$$\left. + \left[\cos\theta^P + \frac{\beta}{\alpha}\left(\frac{\alpha^2+\beta^2}{\alpha^2-\beta^2}\cos\theta^P\cos\theta^{PRS} + 2\cos\theta^{RS}\right)\right]\nu_m\right\}, \tag{27}$$

$$R_{3m}^{PS}=\frac{\Delta\Gamma_{kl}}{\alpha^2-\beta^2}\frac{1}{2\cos\theta^{RS}}\left\{\frac{\alpha^2-\beta^2}{\alpha\beta}\nu_k\delta_{lm}+N_k^P\left[\left(\frac{\beta}{\alpha}\cos\theta^P+\cos\theta^{RS}\right)\delta_{lm}+\frac{\beta}{\alpha}\nu_l N_m^P+N_l^{RS}\nu_m\right]\right\},$$

where

$\cos\theta^P=\mathbf{N}^P\cdot\boldsymbol{\nu}<0$, $\cos\theta^{RS}=\mathbf{N}^{RS}\cdot\boldsymbol{\nu}>0$, and $\cos\theta^{PRS}=\mathbf{N}^P\cdot\mathbf{N}^{RS}$.

$\mathbf{N}^{RS}$ is the phase normal of the reflected *S* waves in the background medium, θ^{RS} is the angle between the phase normal $\mathbf{N}^{RS}$ and the normal of the interface $\boldsymbol{\nu}$, and θ^{PRS} is the angle between the phase normals $\mathbf{N}^P$ and $\mathbf{N}^{RS}$. Since the phase normals $\mathbf{N}^P$ and $\mathbf{N}^{RS}$ are considered for the background medium, angles θ^P and θ^{RS} are related by Snell's law:

$$\sin\theta^{RS}=\frac{\beta}{\alpha}\sin\theta^P.$$

In order to keep formula for $\mathbf{R}^{PS}$ reasonably short, we present in Eq. (27) only a simplified formula for $\mathbf{R}^{PS}$. This formula is considerably shorter than the complete formula, because we omitted in (27) all terms, which were explicitly directed in the phase normal direction $\mathbf{N}^{RS}$. As a consequence, $\mathbf{R}^{PS}$ in (27) is no more a vector lying in the plane perpendicular to the phase normal $\mathbf{N}^{RS}$. Nevertheless its scalar product with the polarization vectors $\mathbf{g}^{0(2)}$ and $\mathbf{g}^{0(3)}$ is unchanged, thus calculating scalar coefficients by Eqs (26) and (27) we obtain correct results. For determination of polarization vectors of reflected *S* waves $\mathbf{g}^{0(2)}$ and $\mathbf{g}^{0(3)}$ see Jech & Pšenčík (1989).

Coefficients of transmitted converted waves can be calculated in a similar way as for the reflected waves. Formula for $\mathbf{T}^{PS}$ can be obtained from the formula for $\mathbf{R}^{PS}$, if we multiply this formula by the minus sign and if we substitute $\mathbf{N}^{RS}$, θ^{RS} and θ^{PRS} by $\mathbf{N}^{TS}$, θ^{TS} and θ^{PTS}.

5.3 Energy coefficients

Taking into account Eqs (14) and (15) we can write readily

$$R_E^{PP}=\left(R^{PP}\right)^2,\ T_E^{PP}=1-R_E^{PP}-R_E^{PS1}-R_E^{PS2}-T_E^{PS1}-T_E^{PS2}$$

$$R_E^{PS1}=\frac{\beta}{\alpha}\left|\frac{\cos\theta^S}{\cos\theta^P}\right|\left(R^{PS1}\right)^2,\ R_E^{PS2}=\frac{\beta}{\alpha}\left|\frac{\cos\theta^S}{\cos\theta^P}\right|\left(R^{PS2}\right)^2, \tag{28}$$

$$T_E^{PS1}=\frac{\beta}{\alpha}\left|\frac{\cos\theta^S}{\cos\theta^P}\right|\left(T^{PS1}\right)^2,\ T_E^{PS2}=\frac{\beta}{\alpha}\left|\frac{\cos\theta^S}{\cos\theta^P}\right|\left(T^{PS2}\right)^2.$$

Obviously,

$$R_E^{PS1}+R_E^{PS2}=\frac{\beta}{\alpha}\left|\frac{\cos\theta^S}{\cos\theta^P}\right|\left|\mathbf{R}^{PS}\right|^2,\ T_E^{PS1}+T_E^{PS2}=\frac{\beta}{\alpha}\left|\frac{\cos\theta^S}{\cos\theta^P}\right|\left|\mathbf{T}^{PS}\right|^2. \tag{29}$$

5.4 Vertical *P*-wave incidence

For strictly vertical *P*-wave incidence at the horizontal interface we obtain

$$R^{PP}=\frac{1}{2}\frac{\Delta\rho}{\rho}+\frac{1}{2}\frac{\Delta V}{\alpha},\ R_1^{PS}=\frac{1}{2\beta}\frac{\Delta a_{35}}{\alpha+\beta},\ R_2^{PS}=\frac{1}{2\beta}\frac{\Delta a_{34}}{\alpha+\beta},\ R_3^{PS}=0,$$

$$T^{PP} = 1 - R^{PP},\ T_1^{PS} = \frac{1}{2\beta}\frac{\Delta a_{35}}{\alpha-\beta},\ T_2^{PS} = \frac{1}{2\beta}\frac{\Delta a_{34}}{\alpha-\beta},\ T_3^{PS} = 0\ , \tag{30}$$

where $\Delta V = \frac{1}{2\alpha}\Delta a_{33}$.

Notice that for no density contrast and for no *P*-wave phase velocity contrast, we observe no reflected *P* wave, but we can observe reflected *S* waves. The reflected *S* waves will be observed even in the case, when the halfspace in which the incident *P* wave propagates is isotropic. This is of particular interest, because such phenomenon is unknown for interfaces between two isotropic media. For example, *detection of reflected S waves for vertical incident P wave can serve as a clear evidence for anisotropic structures beneath isotropic subsurface layers*, because no contrast between two horizontal isotropic layers can generate reflected *S* wave in this case

6 Numerical test

Accuracy of perturbation formulae for *PP* displacement R/T coefficients in weakly anisotropic media is extensively tested by Pšenčík & Vavryčuk (1998) and by Vavryčuk & Pšenčík (1998). It is shown in those papers that perturbation formulae for *PP* R/T coefficients are of reasonable accuracy not only for weak anisotropy and weak contrast interfaces, but also in cases when contrast at the interface and/or anisotropy of halfspaces are rather strong (nearly 20%). Therefore, we will not repeat numerical tests for *PP* waves here, but confine ourselves to numerical tests of approximate formulae for *PS* coefficients only.

We will examine behaviour of *PS* reflection coefficient for two models that have been used by Vavryčuk & Pšenčík (1998). The models consist of homogeneous halfspaces separated by a horizontal interface. The incident *P* wave propagates in the upper halfspace that is isotropic. The *P*- and *S*-wave velocities and the density of the halfspace are: $V_{1P} = 4.00$ km/s, $V_{1S} = 2.31$ km/s and $\rho_1 = 2.65$ g/cm^3. The lower halfspace is transversely isotropic with a horizontal axis of symmetry. Anisotropy of the lower halfspace is assumed to be caused by a system of vertical parallel penny-shaped dry cracks (Hudson 1981). The *P*- and *S*-wave velocities and density of the host rock are: $V_{P-ROCK} = 4.00$ km/s, $V_{S-ROCK} = 2.31$ km/s and $\rho_2 = 2.60$ g/cm^3. The crack normal is parallel to the *x*-axis, the crack density is alternatively $e = 0.05$ (model A/C) or $e = 0.1$ (model A/D). The corresponding density-normalized elastic parameters in Voigt notation are: a_{11}=11.96, a_{13}=3.99, a_{33}=15.55, a_{44}=5.33, and a_{66}=4.76 for crack density e=0.05; and a_{11}=9.43, a_{13}=3.14, a_{33}=15.27, a_{44}=5.33, and a_{66}=4.25 for crack density e=0.1 . Phase velocities in the *x*-*z* plane as a function of the incidence angle for both models are shown in Fig. 1. For calculation of approximate reflection coefficients we used the following parameters of the isotropic background: α=3.97 km/s, β=2.25 km/s and ρ=2.63 g/cm^3 for model A/C, and α=3.95 km/s, β=2.19 km/s and ρ=2.63 g/cm^3 for model A/D. These values were obtained by averaging densities and vertical velocities in the both halfspaces (see Vavryčuk & Pšenčík 1998).

Similarly to reflection coefficients of *P* waves (Vavryčuk & Pšenčík 1998), approximate reflection coefficients for *S* waves start to deviate from exact coefficients considerably for greater values of angles of incidence, thus we consider the angles of incidence only for a limited interval of values (0°, 40°). The reflection coefficients for *S* waves are displayed for each model in the form of four plots (see Figs 2 and 3). In all the

plots, the horizontal axis corresponds to the angle of incidence θ^P, the vertical axis corresponds to the azimuth φ. Azimuth φ=0° corresponds to the profile in the *x*-*z* plane. The upper-left and upper-right plots show the exact R^{PSV} reflection coefficient and relative errors of the approximate formula. The lower-left and lower-right plots show exact values and relative errors for the R^{PSH} reflection coefficient.

From Figs 2 and 3 it follows that relative errors of the approximate reflection coefficient for *SV* waves behave in a similar way as errors of the reflection coefficient for *P* waves (see Vavryčuk & Pšenčík 1998, Figs 2 and 3). The errors are very similar in their directional dependence as well as in their magnitude. The relative error is always less than 8% for angles of incidence up to 30° for the A/C model and less than 13% for the A/D model. Also approximate reflection coefficient for *SH* waves displays similar accuracy as for *SV* waves, although their directional dependence is quite different. Only for azimuths between 80° and 90° the formula for *SH* waves works slightly worse, but still the accuracy is acceptable. Thus we can conclude that the numerical test proved a good performance of the approximate formulae.

7 Conclusion

Approximate formulae for R/T coefficients of *PP* and *PS* plane waves at weak-contrast interfaces in weakly anisotropic media are derived by the first-order perturbation theory. Formulae are written in a coordinate system, which is not, in general, connected with the interface and the incidence plane. Therefore, the formulae can be used for arbitrary orientations of the incidence plane and an interface without necessity to transform elasticity parameters to a local Cartesian coordinate system. Formulae for *PP* R/T coefficients are linear functions of elasticity and density contrasts Δa_{ijkl} and $\Delta\rho$ at the interface. Formulae for *PS* R/T coefficients are, however, non-linear functions of contrasts Δa_{ijkl}. Therefore, new quantities called *vector R/T coefficients of S waves* or *vector PS R/T coefficients* are introduced. These coefficients are linear functions of contrasts Δa_{ijkl} and $\Delta\rho$ similarly to R/T coefficients of *PP* waves. From vector R/T coefficients and polarization directions of scattered *S* waves, the scalar *PS* R/T coefficients can easily be calculated. The accuracy of the derived approximate formulae is numerically tested on reflection coefficients of *S* waves for models containing an horizontal interface between an isotropic halfspace and a transversely isotropic halfspace with the symmetry axis lying in the interface. The transverse isotropy is caused by presence of cracks with crack density *e*=0.05 or *e*=0.1. Comparison of exact and approximate coefficients shows that the approximate formulae are sufficiently accurate for all azimuths and for incidence angles up to 35°.

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References

Chapman, C. H., 1994. Reflection/transmission coefficient reciprocities in anisotropic media, *Geophys. J. Int.*, **116**, 498-501.

Daley, P. F. & Hron, F., 1977. Reflection and transmission coefficients for transversely isotropic media, *Bull. Seism. Soc. Am.*, **67**, 661-675.

Fryer, G. J. & Frazer L. N., 1984. Seismic waves in stratified anisotropic media, *Geophys. J. R. astr. Soc.*, **78**, 691-710.

Gajewski, D. & Pšenčík, I., 1987. Computation of high-frequency seismic wavefields in 3-D laterally inhomogeneous anisotropic media, *Geophys. J. Roy. Astr. Soc.*, **91**, 383-411.

Haugen, G. U. & Ursin, H., 1996. AVO-A analysis of vertically fractured reservoir underlaying shale, 66th Ann. Int. Mtg. of the SEG, Denver, Expanded Abstracts, 1826-1829.

Hudson, J. A., 1981. Wave speeds and attenuation of elastic waves in material containing cracks, *Geophys. J. Roy. Astr. Soc.*, **64**, 133-150.

Jech, J. & Pšenčík, I., 1989. First-order perturbation method for anisotropic media, *Geophys. J. Int.*, **99**, 369-376.

Keith, C. M. & Crampin, S., 1977. Seismic body waves in anisotropic media: Reflection and refraction at a plane interface, *Geophys. J. Roy. Astr. Soc.*, **49**, 181-0208.

Pšenčík, I. & Vavryčuk, V., 1998. Weak contrast *PP* wave displacement R/T coefficients in weakly anisotropic elastic media, *Pageoph ,* **151***,* 699-718.

Rueger, A., 1996. Analytic description of reflection coefficients in anisotropic media. AEGE Extended Abstracts, Vol. 1, Amsterdam, P026.

Thomsen, L., 1986. Weak elastic anisotropy, *Geophysics*, **51**, 1954-1966.

Thomsen, L., 1993. Weak anisotropic reflections, in: Offset-Dependent Reflectivity - Theory and Practice of AVO Analysis, Castagna & Backus, eds, SEG Investigations in Geophysics, Vol. 8, 103-111.

Vavryčuk, V. & Pšenčík, I., 1998. *PP* wave reflection coefficients in weakly anisotropic elastic media, *Geophysics*, accepted.

Zillmer M., Gajewski, D. & Kashtan B.,M., 1998. Anisotropic reflection coefficients for a weak contrast interface, *Geophys. J. Int*., 132, 159-166.

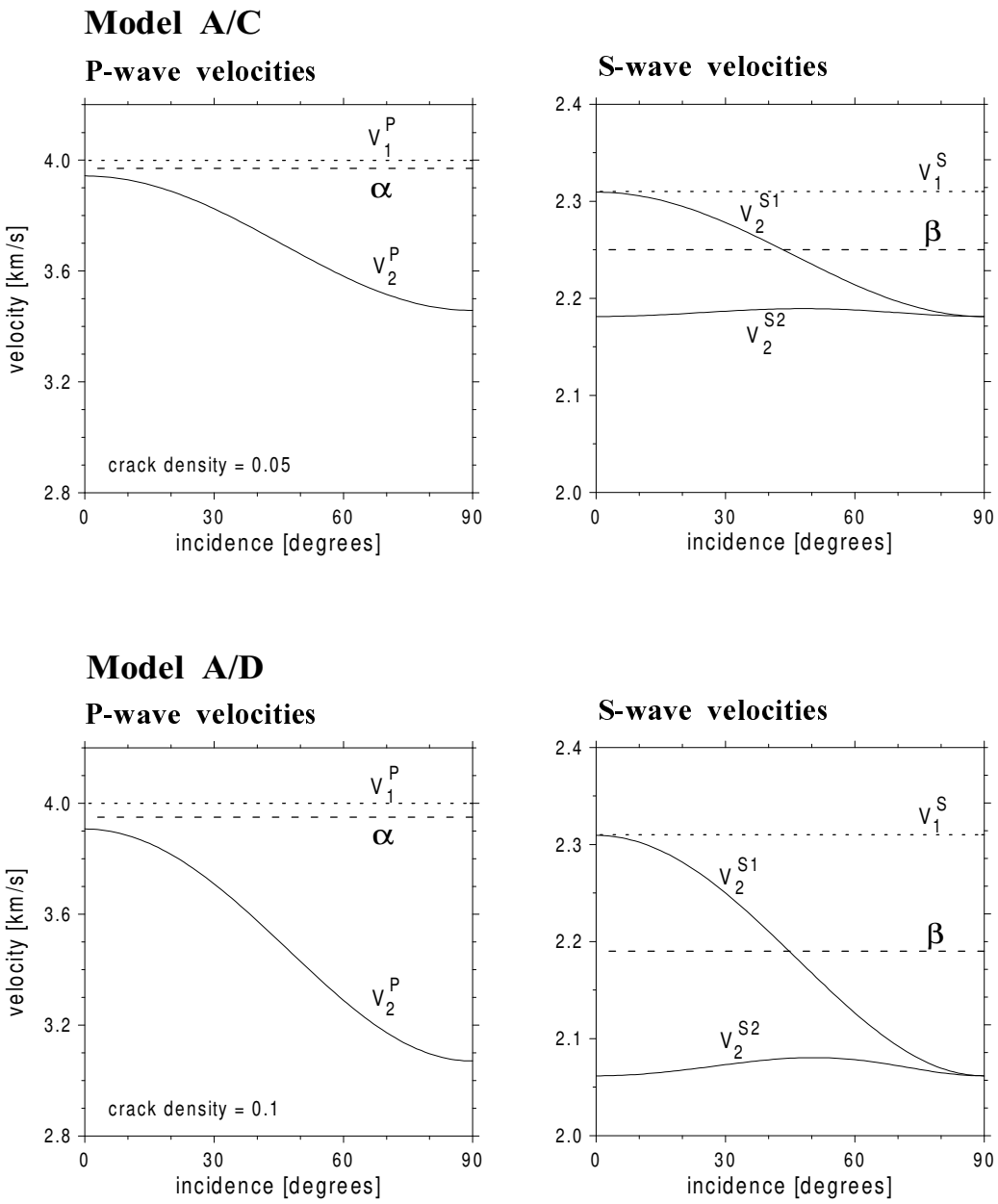

Figure 1.

P- and *S*-wave phase velocities in the *x*-*z* plane. The incidence angle is defined as an angle between the phase normal of the wave and the z-axis. α and β are *P*- and *S*-wave velocities of an isotropic background (for values see the text). v_1^P and v_1^S are *P*- and *S*-wave velocities in the upper isotropic halfspace, v_2^P, v_2^{S1} and v_2^{S2} are *P*- and *S*-wave velocities in the lower anisotropic halfspace.

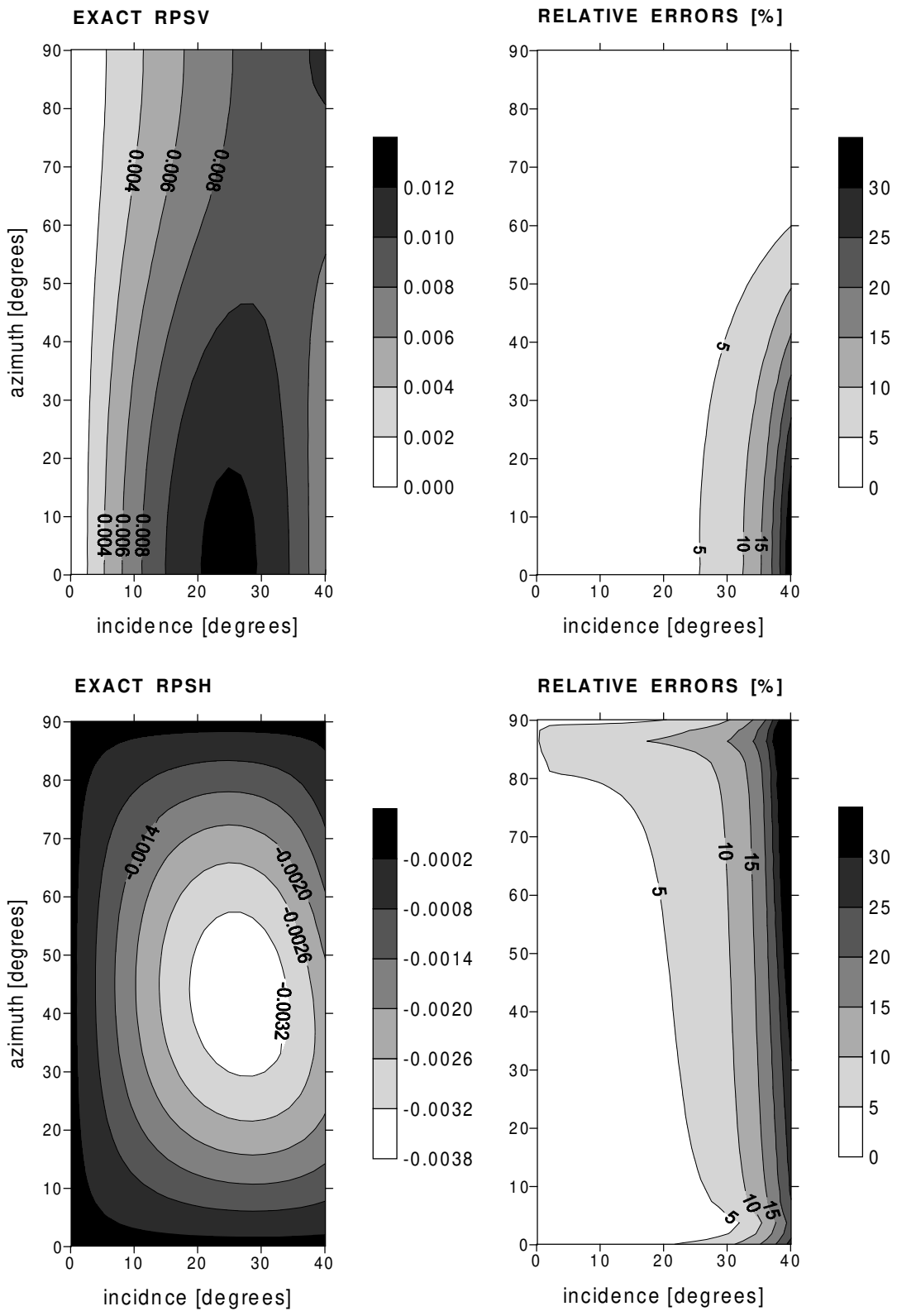

Figure 2

Exact reflection coefficients for *SV* and *SH* waves together with relative errors of approximate formulae for model A/C.

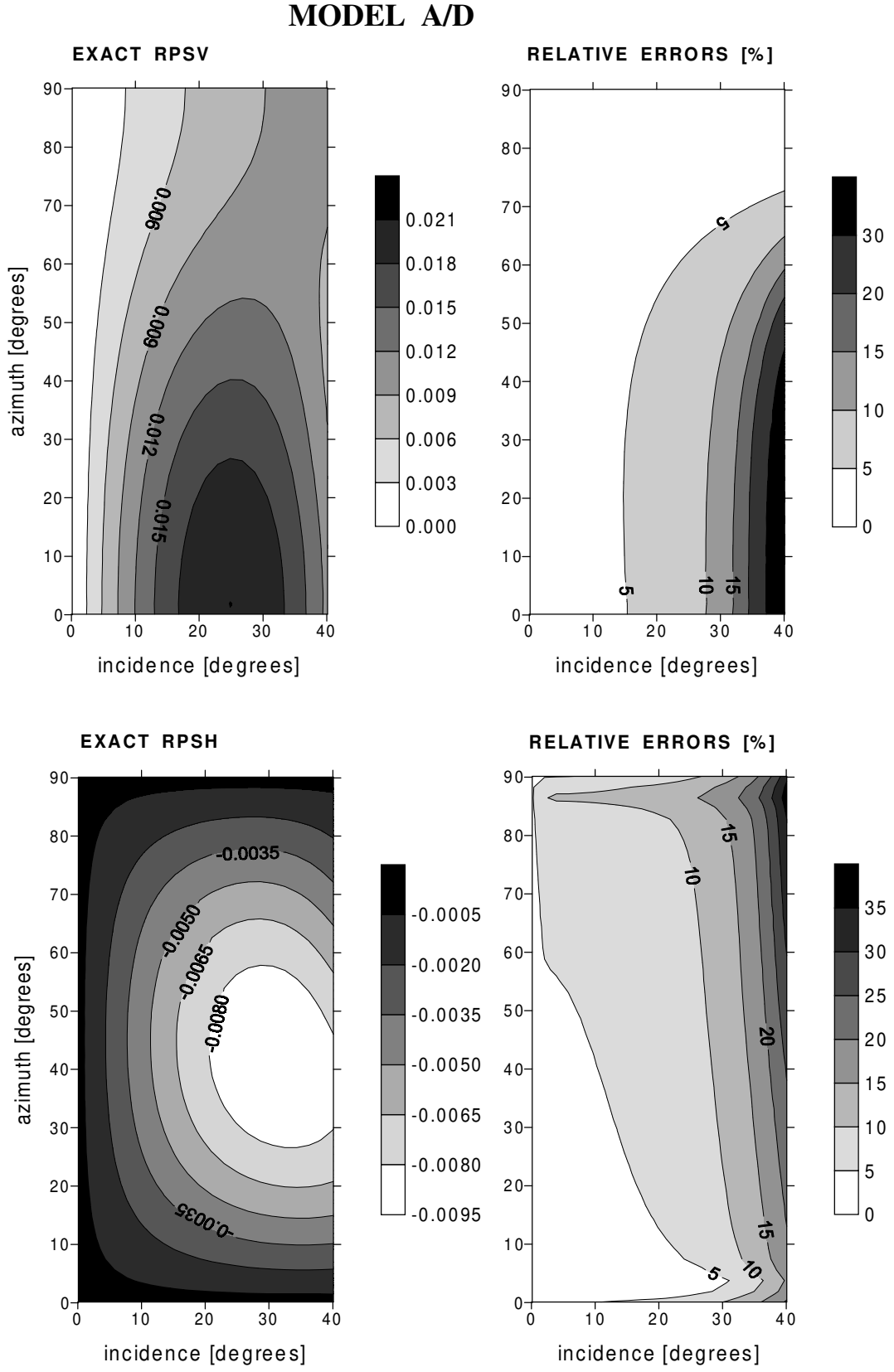

Figure 3

Exact reflection coefficients for *SV* and *SH* waves together with relative errors of approximate formulae for model A/D.