Raising the order of multivariate approximation schemes using supplementary derivative data

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SWLIM VII, Teplá Monastery, June 2010

Context

- Functional approximation schemes

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  - Interpolation
  - Quasi-interpolation
  - (Moving) least-squares approximation
  - ...

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  - Interpolation
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- Minimal requirement: exact to some polynomial degree
- Omnipresent in computational sciences
  (and in observational / experimental sciences)

Motivation and preview

- Suppose available: supplementary derivative data

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- How to utilize?
- Full accommodation (fitting) often impossible or cumbersome
- Can we find a way to improve exisiting approximation schemes?
- Preview: *Yes we can!*
  Using n supplementary orders of derivatives the approximation order can be raised by n.

Approximator

Definition: Approximator

An *approximator of order* m *at* $x \in X$ is a functional:

$$A_x^m : V \to \mathbb{R}$$

that satisfies

$$A_x^m[p] = p(x) \quad \forall \quad p \in P^m.$$

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- The approximator A_x^m reproduces al polynomials of maximum total degree m at x.
- Summarizes:
  - Sampling
  - Construction of approximant
  - Evaluation of approximant at x

Taylor expansion

Definition: Taylor expansion

n-th order truncated Taylor expansion of $f \in V$ around x:

$$\mathcal{T}_x^n : V \to P^n$$

$$\mathcal{T}_x^n[f] := \sum_{|\kappa| \leq n} \frac{1}{\kappa!} (. - x)^\kappa f^{(\kappa)}(x)$$

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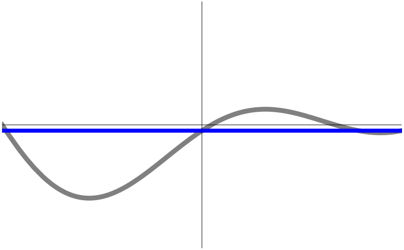
Taylor expansion comparison

(Primal) Taylor expansion

$$\mathcal{T}_x^n[f] := \sum_{|\kappa|\leq n} \frac{1}{\kappa!}(.-x)^\kappa f^{(\kappa)}(x)$$

$n = 0$



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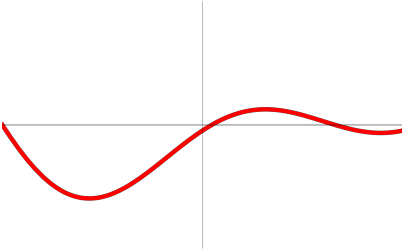

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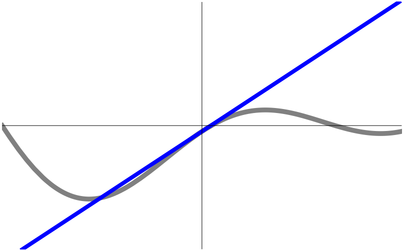

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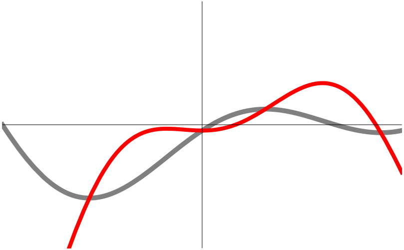

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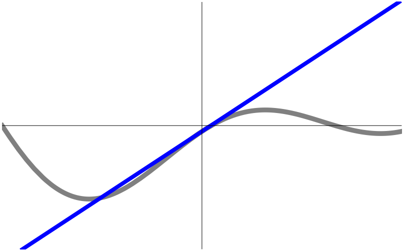

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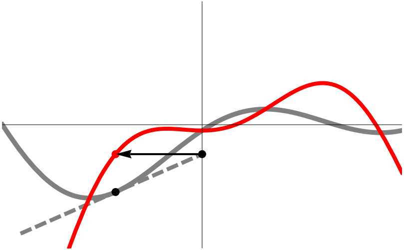

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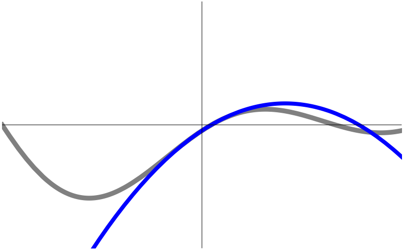

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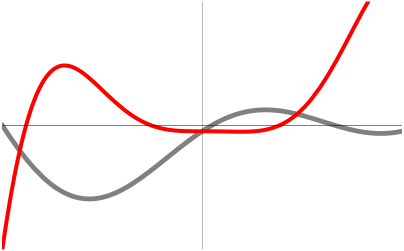

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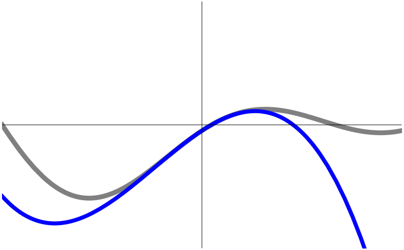

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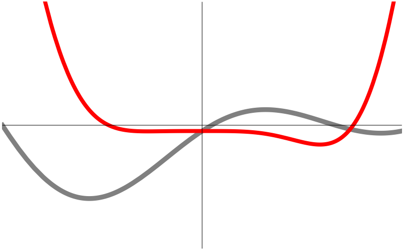

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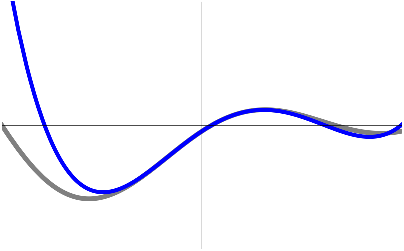

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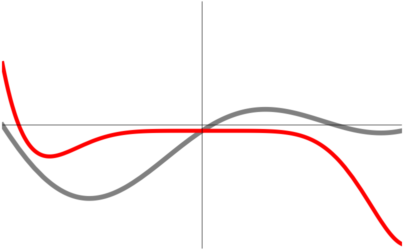

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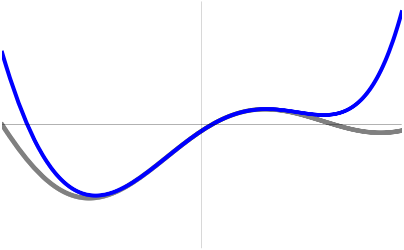

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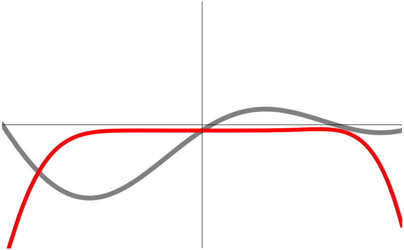

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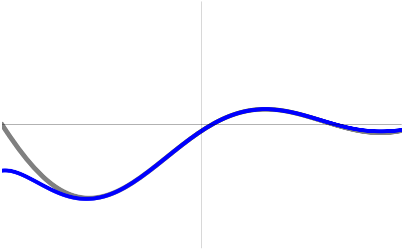

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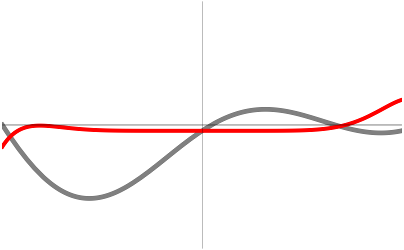

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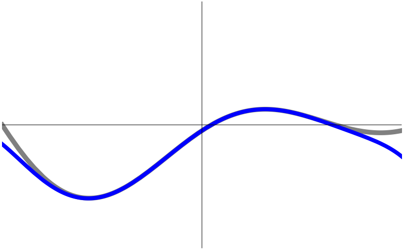

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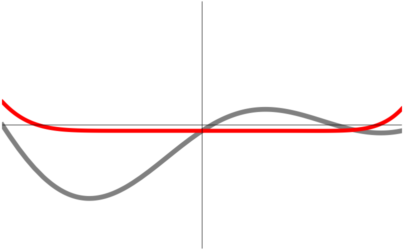

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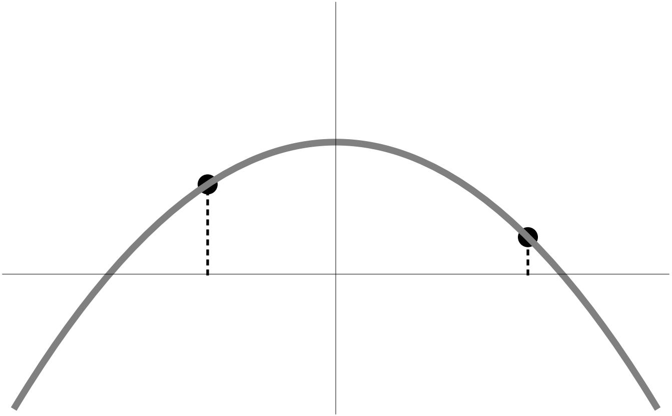
- How well does this work?

Simple example: 1D, two samples



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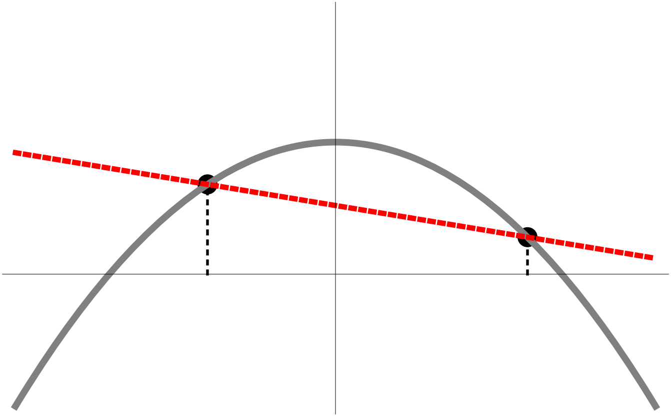

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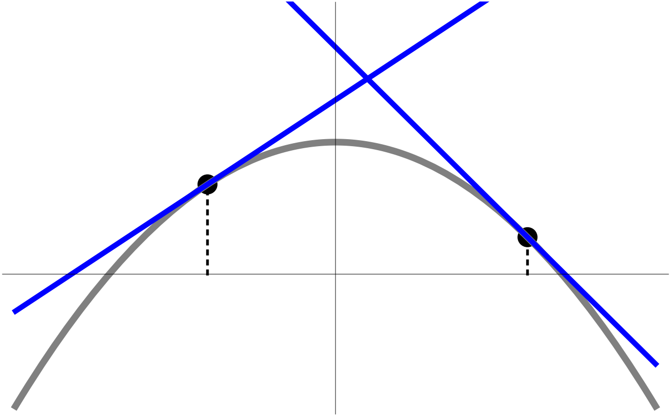

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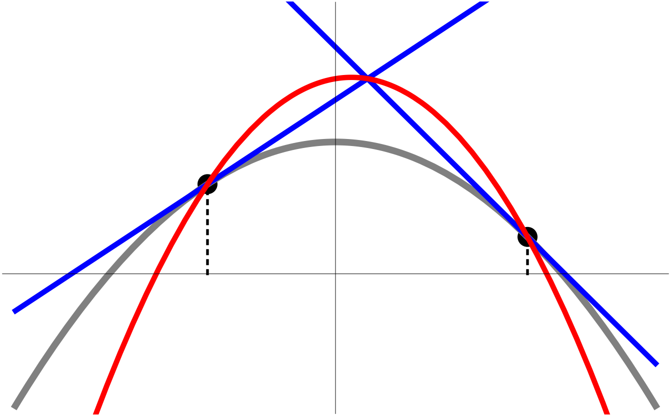

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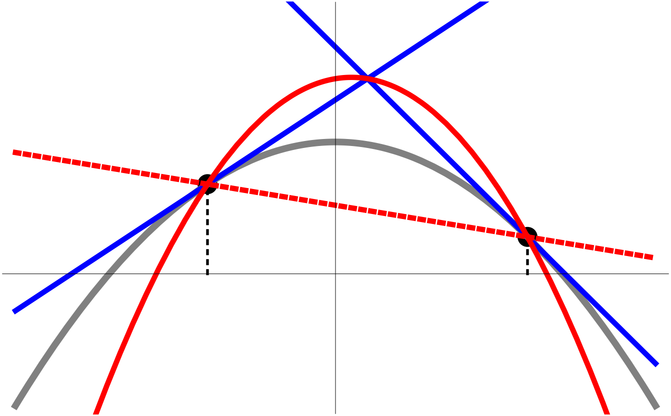

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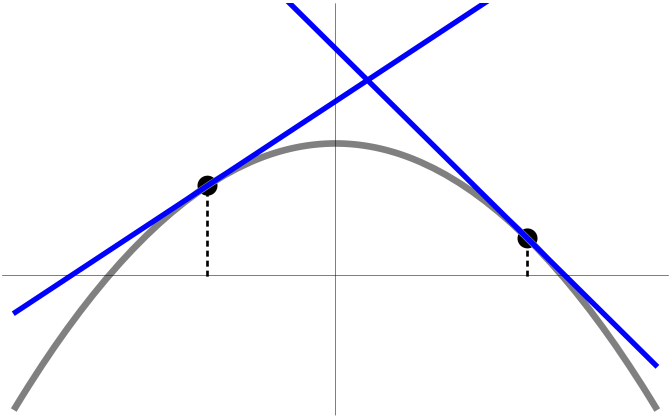

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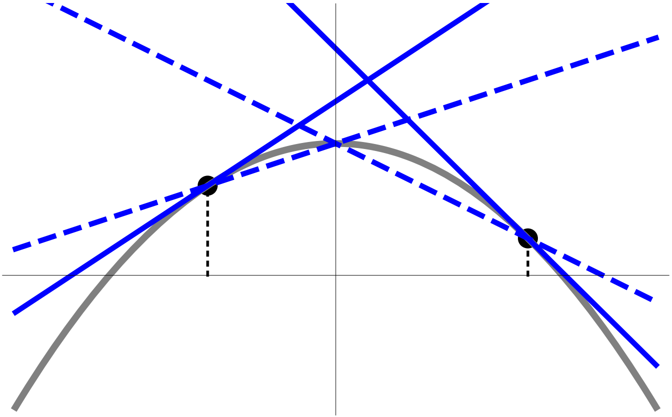

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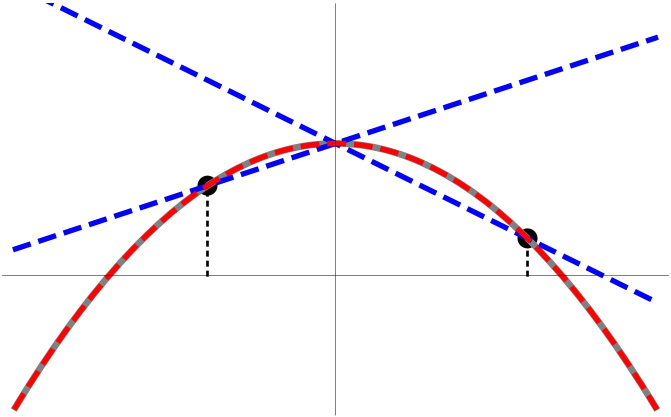

Simple example: 1D, two samples

- Solution: replace

$$\widetilde{\mathcal{T}}_x^1[f] = f(.) + (x - .)f^{(1)}(.)$$

by

$$\widetilde{\mathcal{D}}_x^{11}[f] = f(.) + \frac{1}{2}(x - .)f^{(1)}(.)$$

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- Modifying coefficient raises approximation order
- Is this possible more generally?

Reduced dual Taylor expansion

Definition: Reduced dual Taylor expansion

n-th order reduced dual Taylor expansion of the m-th kind:

$$\widetilde{\mathcal{D}}_x^{mn} : V \to V$$

$$\widetilde{\mathcal{D}}_x^{mn}[f] := \sum_{|\kappa| \leq n} \frac{1}{\kappa!} C_{|\kappa|}^{mn} (x - .)^{\kappa} f^{(\kappa)}(.)$$

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- Compare to: $\widetilde{\mathcal{T}}_x^{n}[f] := \sum_{|\kappa| \leq n} \frac{1}{\kappa!} (x - .)^{\kappa} f^{(\kappa)}(.)$

Reduced dual Taylor expansion

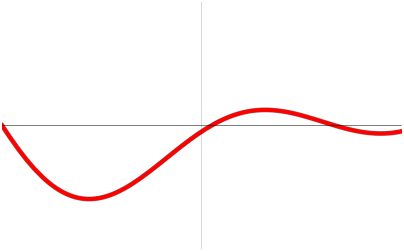
$$C_{|\kappa|}^{mn} := \binom{m+n}{m}^{-1} \binom{m+n-|\kappa|}{m}$$

Coefficients "reduced": $C_{|\kappa|}^{mn} \leq 1$

- $C_k^{11} = \{1, \frac{1}{2}\}$
- $C_k^{12} = \{1, \frac{2}{3}, \frac{1}{3}\}$
- $C_k^{13} = \{1, \frac{3}{4}, \frac{2}{4}, \frac{1}{4}\}$
- $C_k^{22} = \{1, \frac{3}{6}, \frac{1}{6}\}$

Reduced dual Taylor expansions

$m = 1, n = 0$



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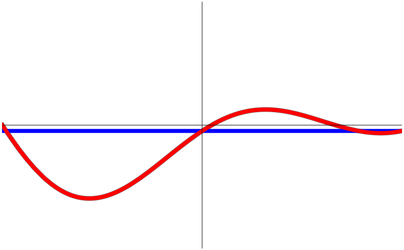
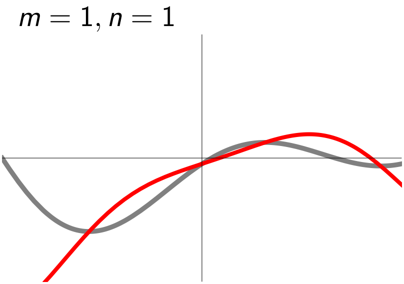


(Dual) Taylor expansion

Reduced dual Taylor expansions



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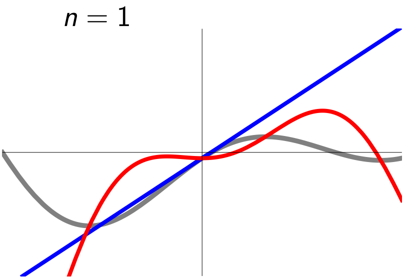
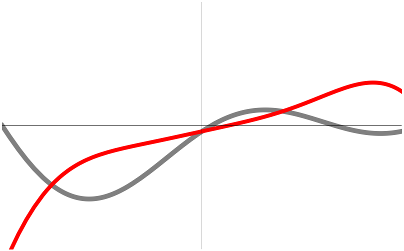


(Dual) Taylor expansion

Reduced dual Taylor expansions

$m = 1, n = 2$



Reduced dual Taylor expansion

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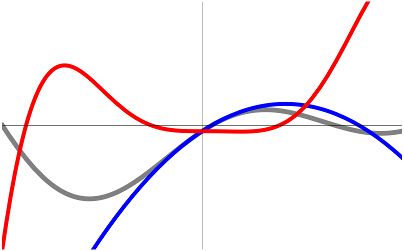

(Dual) Taylor expansion

Reduced dual Taylor expansions

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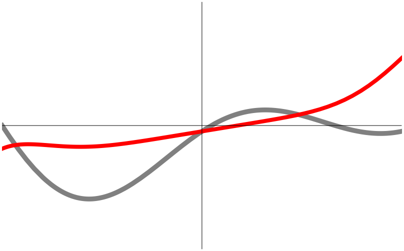

Reduced dual Taylor expansion

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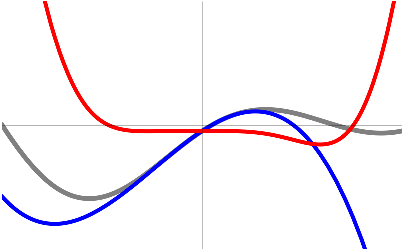

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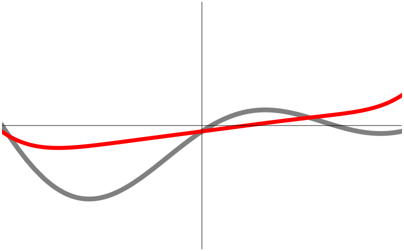
Reduced dual Taylor expansions

$m = 1, n = 4$



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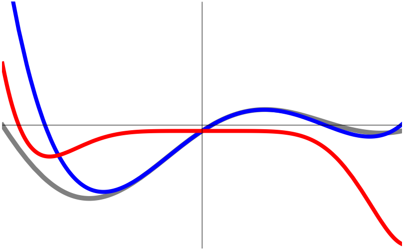
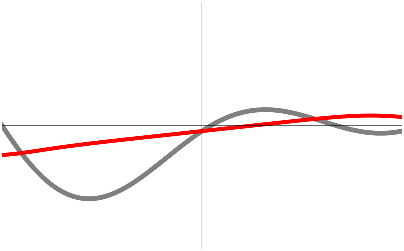


(Dual) Taylor expansion

Reduced dual Taylor expansions

$m = 1, n = 5$



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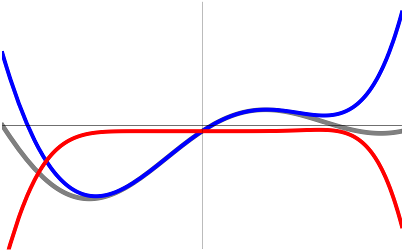

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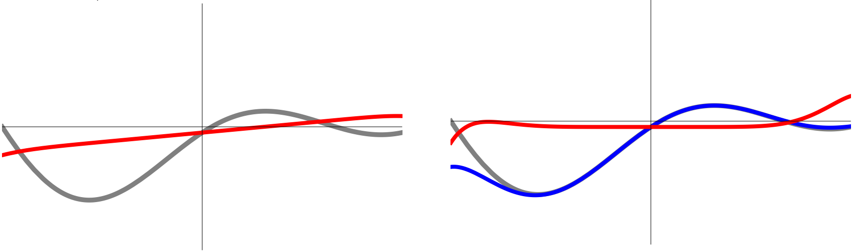
Reduced dual Taylor expansions

$m = 1, n = 6$ $n = 6$



Reduced dual Taylor expansion

(Dual) Taylor expansion

Reduced dual Taylor expansions

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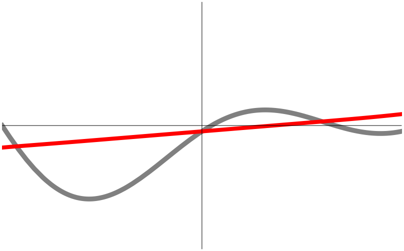

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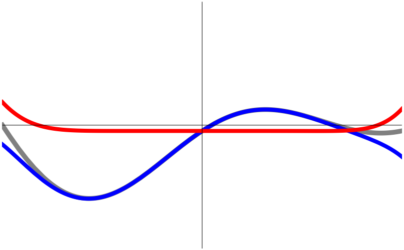

(Dual) Taylor expansion

Main theorem

Theorem

The combination of reduced dual Taylor expansion $\widetilde{\mathcal{D}}_x^{mn}$ and approximator A_x^m yields an effective approximation order of $m+n$:

$$A_x^m[\widetilde{\mathcal{D}}_x^{mn}[p]] = p(x) \quad \forall \quad p \in P^{m+n}.$$

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- Proof: several binomial identities

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- Compare to: $A_x^m[p] = p(x) \quad \forall \quad p \in P^m$
- Proof: several binomial identities
- General procedure for arbitrary A_x^m!

Error expression

$$f(x) - A_x^m[\widetilde{\mathcal{D}}_x^{mn}[f]] = A_x^m \left[\int_0^1 \frac{(-t)^m(1-t)^n}{(m+n)!} \frac{d^{m+n+1}}{dt^{m+n+1}} f(. + t(x - .))dt \right].$$

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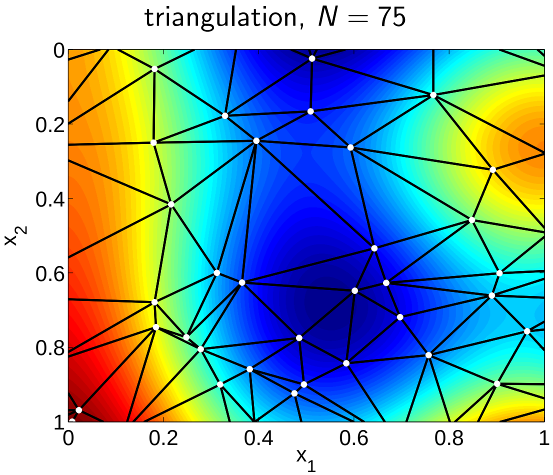
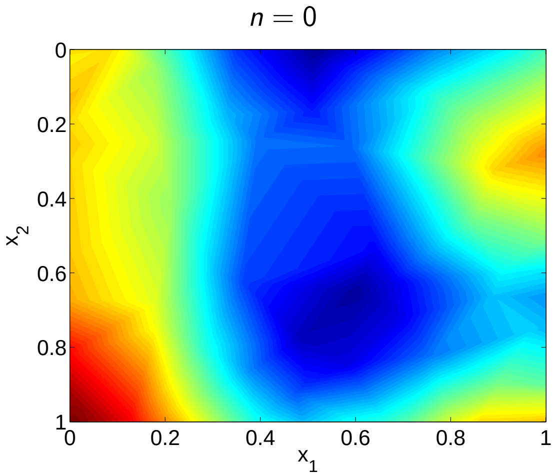
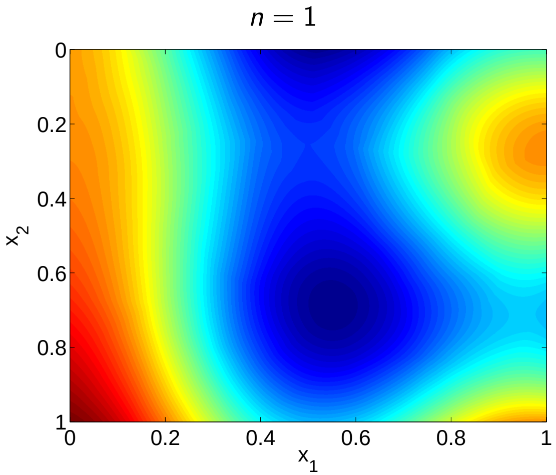
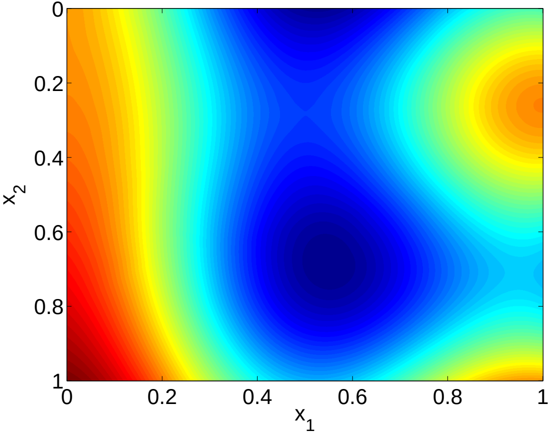
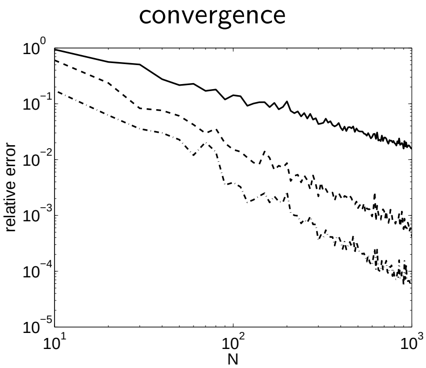

triangulation, $N = 75$
0
0.2
0.4
0.6
0.8
1
0
0.2
0.4
0.6
0.8
1
x_2
x_1


n = 0
0
0.2
0.4
0.6
0.8
1
x2
0
0.2
0.4
0.6
0.8
1
x1


n = 1
0
0.2
0.4
0.6
0.8
1
x2
0
0.2
0.4
0.6
0.8
1
x1

$n = 2$




convergence
relative error
N
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{1}
10^{2}
10^{3}

$-n = 0$, - -$n = 1$, -.$n = 2$

Moving least-squares approximation

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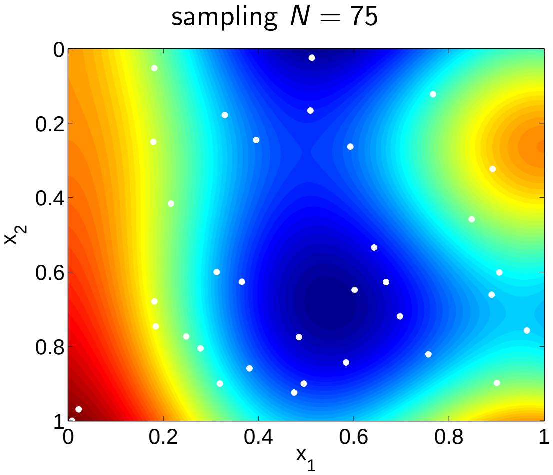
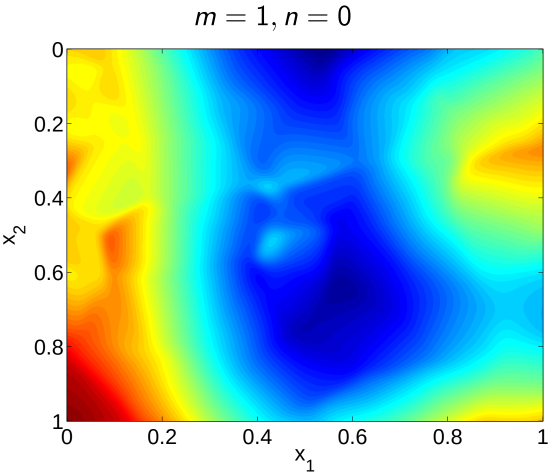
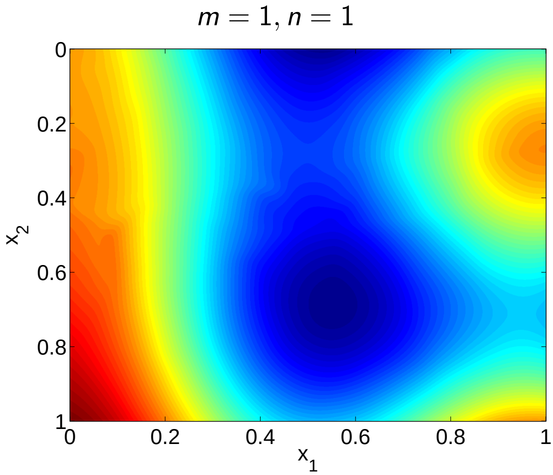


Moving least-squares approximation

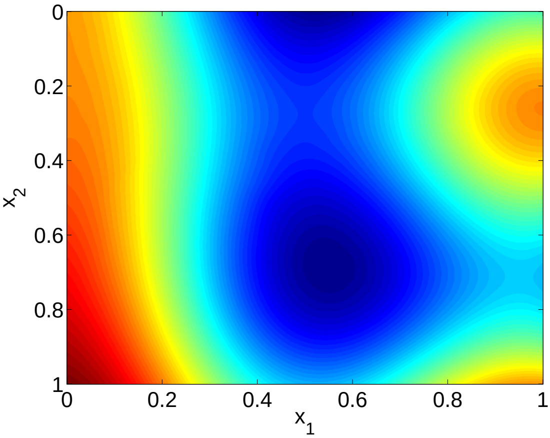


Moving least-squares approximation

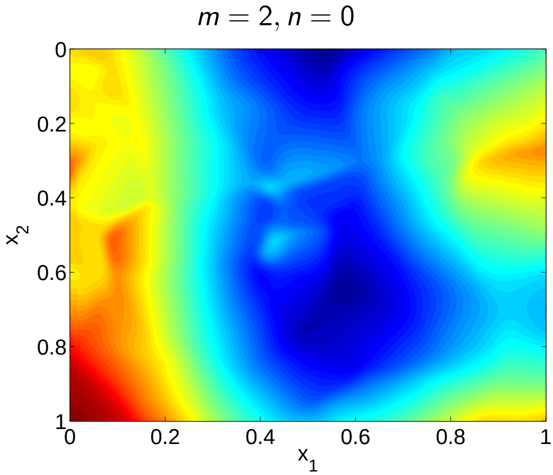


Moving least-squares approximation

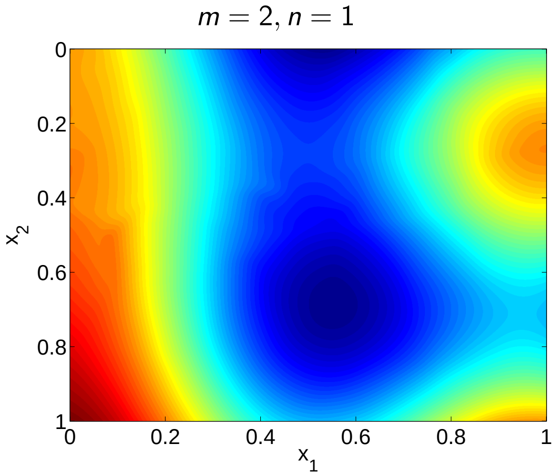
$m = 1, n = 2$



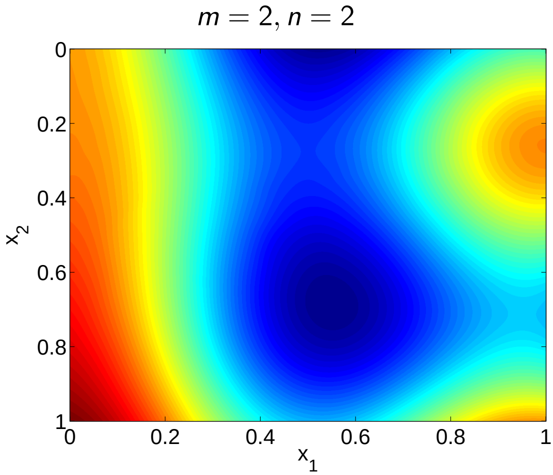
Moving least-squares approximation



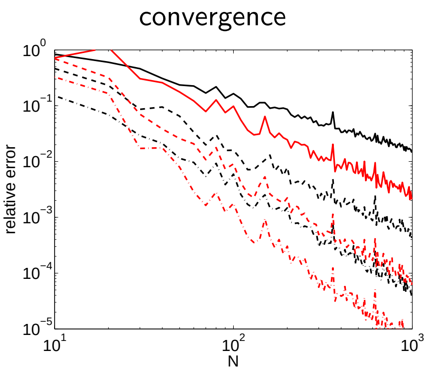
Moving least-squares approximation



Moving least-squares approximation



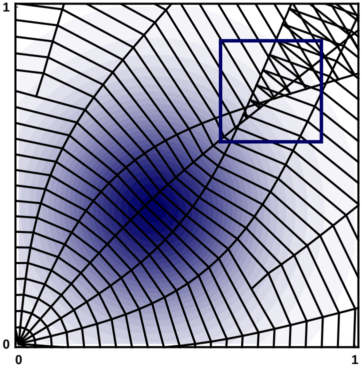
Moving least-squares approximation



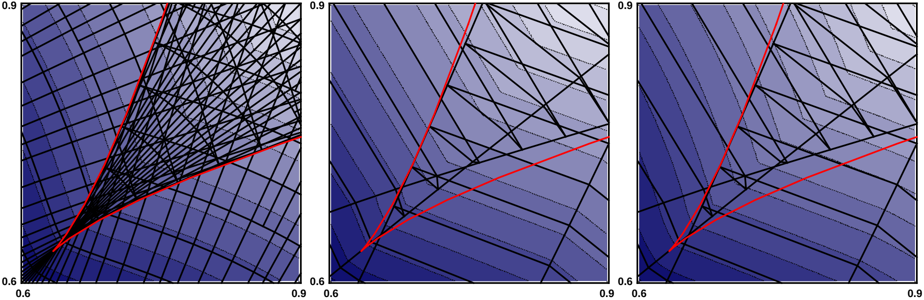
m\n	0	1	2
1	—	- -	-.
2	—	- -	-.

Ray tracing

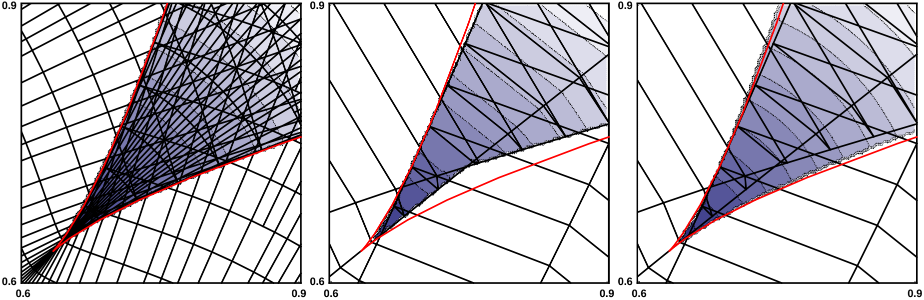
- Gaussian low velocity anomaly
- Triplication
- Sparse ray field



Ray tracing



Ray tracing



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- Used to raise the order of large class of approximation schemes using supplementary derivative data
- The procedure is very easy to implement, even for ‘black box’ codes
- Expected to be useful in a wide range of applications

Acknowledgements

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- You. Thanks for listening!